- <u>Fourier Series Representation of DT Periodic Signals</u> $x[n] = x[n+N] \quad \forall n$

  N = integer

- Recall features of DT sinewaves $e^{j\omega_0 n}$
  - only periodic if $\frac{\omega_0}{2\pi} = \frac{m}{N}$ = rational

    m, N integers

  - integer

    $$e^{j(\omega_0 + l 2\pi)n} = e^{j\omega_0 n}$$

    $\Rightarrow$ DT frequencies are only unique ~~to~~ over a 2π interval, e.g. $[0, 2\pi)$ or $[-\pi, \pi)$

- as a result, for $x[n] = x[n+N]$ only need to sum N DT sinewaves at N frequencies <u>equi-spaced</u> over some 2π interval

  $\Rightarrow$ we'll use $[0, 2\pi)$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{+j 2\pi \frac{k}{N} n}$$

- The N DT sinewaves $s_k[n] = e^{j 2\pi \frac{k}{N} n}$, $k = 0, 1, \ldots, N-1$

are orthogonal:

$$\boxed{\sum_{n=0}^{N-1} s_k[n]\, s_\ell^*[n] = N\,\delta[k-\ell] = \begin{cases} N, & \text{if } k = \ell \\ 0, & \text{if } k \neq \ell \end{cases} \quad k, \ell \in [0, N-1] \text{ integers}}$$

Proof:

$$\sum_{n=0}^{N-1} e^{j 2\pi \frac{k}{N} n} e^{-j 2\pi \frac{\ell}{N} n} = \sum_{n=0}^{N-1} \left(e^{+j 2\pi \frac{(k-\ell)}{N}}\right)^n$$

$$= \frac{1 - e^{j 2\pi \frac{(k-\ell)}{N} N}}{1 - e^{j 2\pi \frac{(k-\ell)}{N}}} = \frac{1 - \left(e^{j 2\pi}\right)^{k-\ell}}{1 - e^{j 2\pi \frac{(k-\ell)}{N}}}$$

$= 0$ if $k \neq \ell$ since $e^{j 2\pi} = 1$ Q.E.D.

- as a result of the orthogonality, the FS coefficients (complex amplitudes of the sinewaves) may be found as:

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, s_k^*[n] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j 2\pi \frac{k}{N} n}$$

$k = 0, 1, \dots, N-1$

- Compare to CT formula: $a_k = \frac{1}{T} \int_0^T x(t)\, s_k^*(t)\, dt$

$$= \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, e^{-j 2\pi \frac{k}{T} t}\, dt \qquad = \frac{1}{T} \int_0^T x(t)\, e^{-j 2\pi \frac{k}{T} t}\, dt$$

- Similarly, can sum over any period:

$$a_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n]\, e^{-j 2\pi \frac{k}{N} n}$$

- Basic DT FS property: time-shift

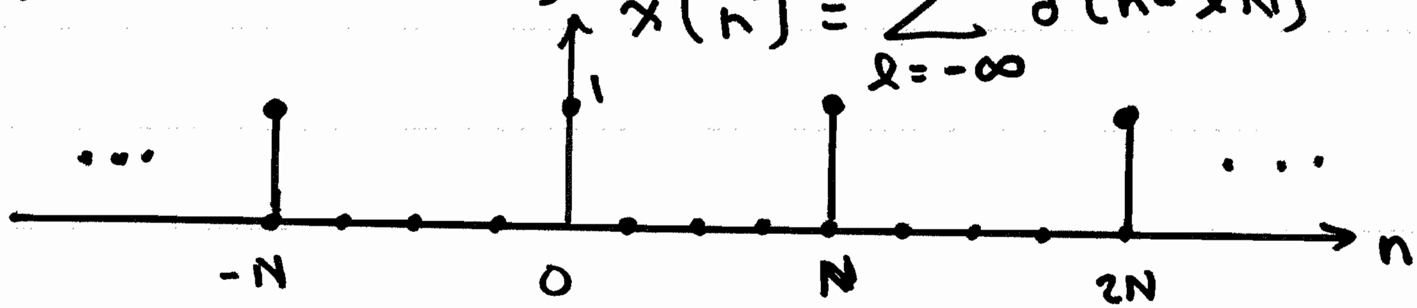
If: $x[n] = \sum_{k=0}^{N-1} a_k e^{+j 2\pi \frac{k}{N} n}$

Then: $y[n] = x[n-n_0] = \sum_{k=0}^{N-1} a_k e^{j 2\pi \frac{k}{N}(n-n_0)}$

$$= \sum_{k=0}^{N-1} \underbrace{\left(a_k e^{-j 2\pi \frac{k}{N} n_0}\right)}_{\text{FS coeffs. for } x[n-n_0]} e^{j 2\pi \frac{k}{N} n}$$

- Other properties of DT FS are listed in Table 3.7 on pg. 221

- Basic Fourier Series for DT Train of Kronecker Delta Functions $x[n] = \sum_{\ell=-\infty}^{\infty} \delta[n-\ell N]$



- FS coeffs. are: $a_k = \frac{1}{N} \quad \forall k \qquad k = 0, 1, \ldots, N-1$

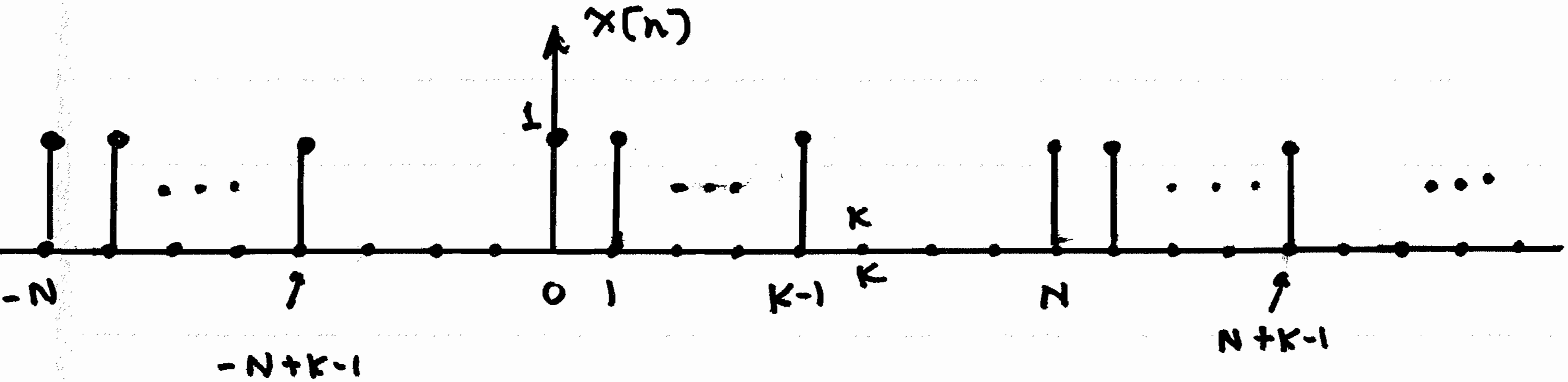
- THUS:

$$\boxed{x[n] = \sum_{\ell=-\infty}^{\infty} \delta[n-\ell N] = \sum_{k=0}^{N-1} \frac{1}{N} e^{j 2\pi \frac{k}{N} n}}$$

PROOF: $a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j 2\pi \frac{k}{N} n} = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{-j 2\pi \frac{k}{N} n}$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \delta[n] e^{j0} = \frac{1}{N} \quad \text{Q.E.D.}$$

• FS for Periodic Train of DT Rectangles:



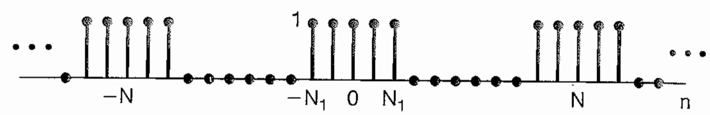
$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi\frac{k}{N}n} = \frac{1}{N}\sum_{n=0}^{K-1}\left(e^{-j2\pi\frac{k}{N}}\right)^n$$

$$= \frac{1}{N}\,\frac{1-e^{-j2\pi\frac{k}{N}(K)}}{1-e^{-j2\pi\frac{k}{N}}} = \frac{1}{N}\,\frac{e^{-j\pi\frac{K}{N}k}\left\{e^{+j\pi\frac{K}{N}k}-e^{-j\pi\frac{K}{N}k}\right\}\frac{1}{2j}}{e^{-j\pi\frac{1}{N}k}\left\{e^{j\pi\frac{1}{N}k}-e^{-j\pi\frac{1}{N}k}\right\}\frac{1}{2j}}$$

$$= \frac{\sin\left(k\pi\frac{K}{N}\right)}{N\sin\left(k\pi\frac{1}{N}\right)}\, e^{-j\pi\frac{(K-1)}{N}k}$$

• Consider K to be odd and shift to left by $n_0 = \frac{K-1}{2}$ so that one period is centered at $n=0$

See Fig. 3.16 on pg. 218 ⇒ Example 3.12



$N_1 = \frac{K-1}{2}$

Figure 3.16 Discrete-time periodic square wave.

$y[n] = x\left[n + \frac{K-1}{2}\right]$ ⇒ from time-shift property

new FS coeffs. are $a_k \, e^{+j 2\pi \frac{k}{N} \left(\frac{K-1}{2}\right)}$

$$= \frac{\sin\left(k\pi \frac{K}{N}\right)}{N \sin\left(k\pi \frac{1}{N}\right)}$$

• Compare to a_k for CT periodic train of rect. pulses

$$a_k = \frac{\sin(k\pi\, \tau/T)}{k\pi}$$