Example on CTFT-DTFT Relationship and DT LTI Systems Described by Difference Eqns and their frequency response of DT LTI Systems

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

i) $x[n] = x_a(nT_s)$ where:

$$x_a(t) = T_s \frac{\pi}{10}\left(\frac{\sin(10t)}{\pi t}\right)^2$$

and $T_s = \dfrac{2\pi}{30} \Rightarrow \omega_s = 30$

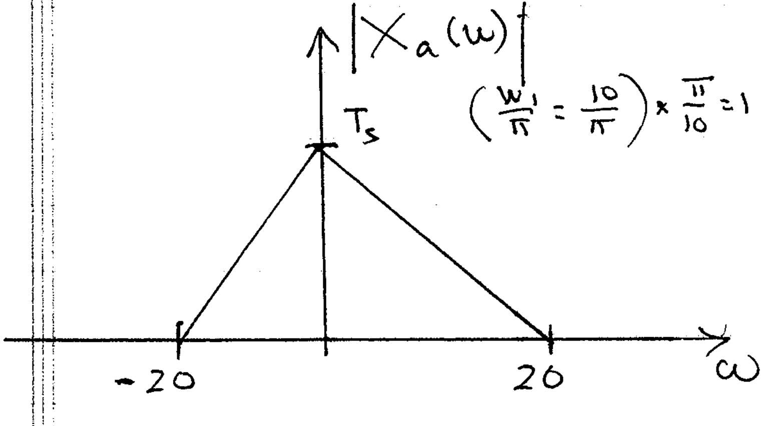
ii) $$h[n] = 2\,\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n}\cos\left(\frac{5\pi}{6}n\right)$$

a) Plot $|X(\omega)|$ where $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$

b) Plot $|Y(\omega)|$ where $y[n] \overset{DTFT}{\longleftrightarrow} Y(\omega)$

c) Determine $\displaystyle\sum_{n=-\infty}^{\infty} y^2[n] = ?$

a) First plot $|X_a(\omega)|$

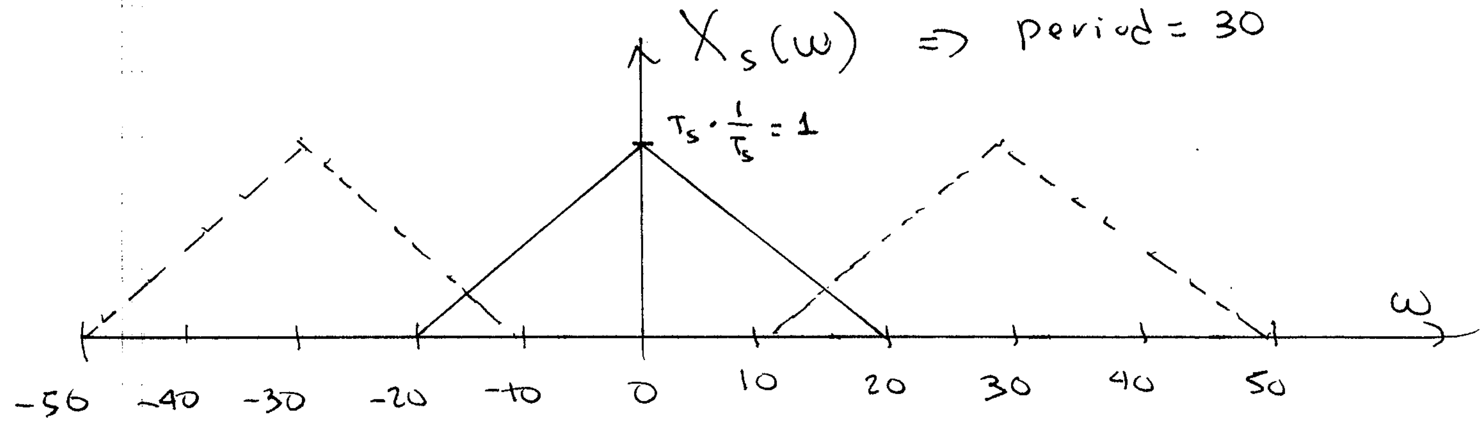


max frequency:
$\omega_M = 20$

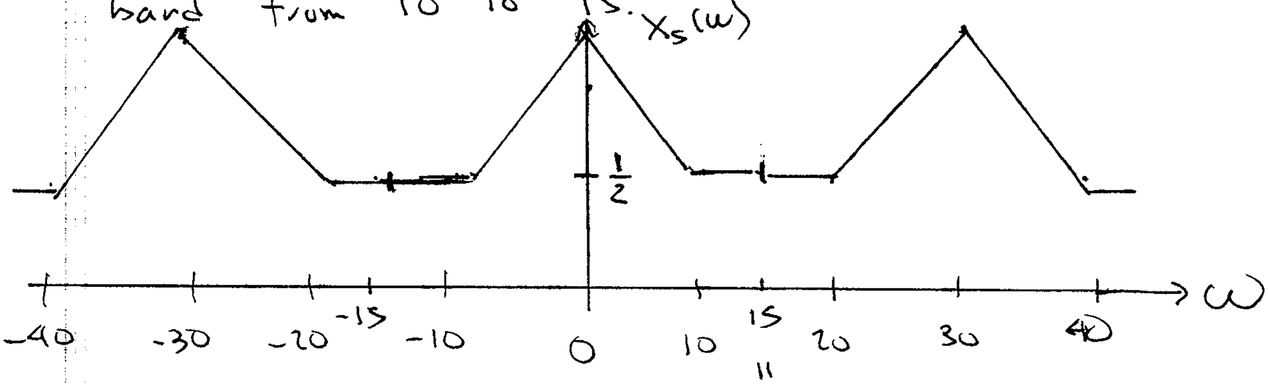
$\omega_s < 2\omega_M = 40$
$= 30$

$\Rightarrow$ aliasing

Plot $X_s(\omega) = \mathcal{F}\left\{ \sum_n x_a(nT_s)\,\delta(t-nT_s) \right\}$ (2)



half-sampling rate = 15 ⇒ frequency content from 15 thru $\omega_M = 20$ is aliased into the band from 10 to 15.



$\frac{1}{2}\omega_s = \frac{1}{2}(30$

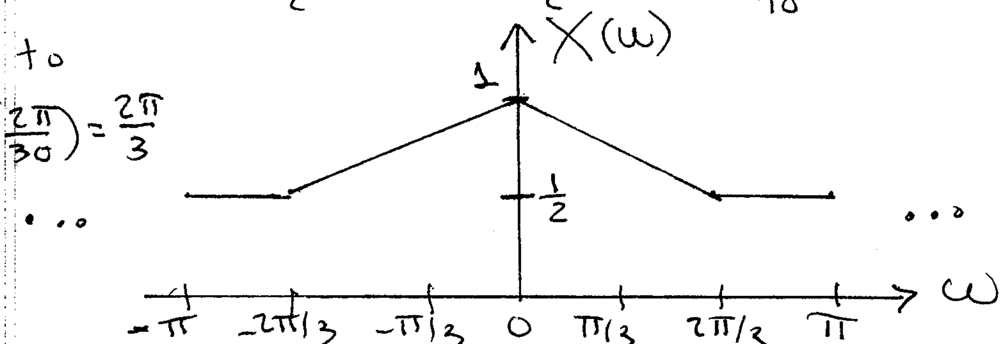
Finally, DTFT $X(\omega) = X_s(F_s\omega) = X_s\left(\frac{\omega}{T_s}\right)$

⇒ map to digital frequencies via $\omega_d = \omega_a T_s$

⇒ compress by sampling rate so that

$-\frac{\omega_s}{2} < \omega_a < \frac{\omega_s}{2}$ ⇒ mapped to ⇒ $-\pi < \omega < \pi$

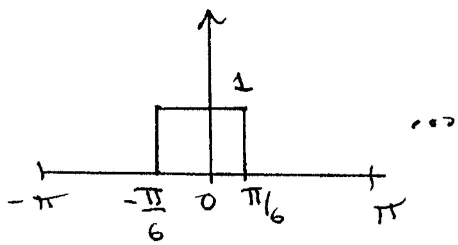
$\omega_a = 10$ mapped to $\omega_d = 10\left(\frac{2\pi}{30}\right) = \frac{2\pi}{3}$



period $= 2\pi$

(b) First need to determine $H(\omega)$

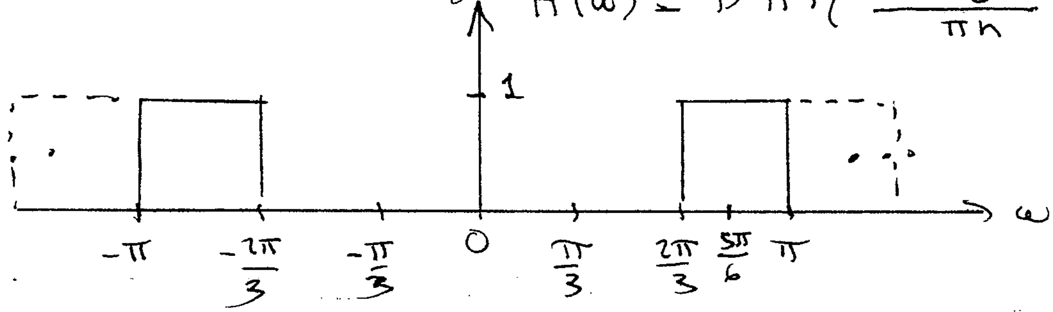
$$\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n} \xleftrightarrow{DTFT} \text{ from Table 5.2}$$



Since $\cos\left(\frac{5\pi}{6}n\right) = \frac{1}{2}e^{j\frac{5\pi}{6}n} + \frac{1}{2}e^{-j\frac{5\pi}{6}n}$

linearity plus $\underbrace{\text{shift}}_{\text{frequency}}$ prop. of DTFT in Table 5.1 yields

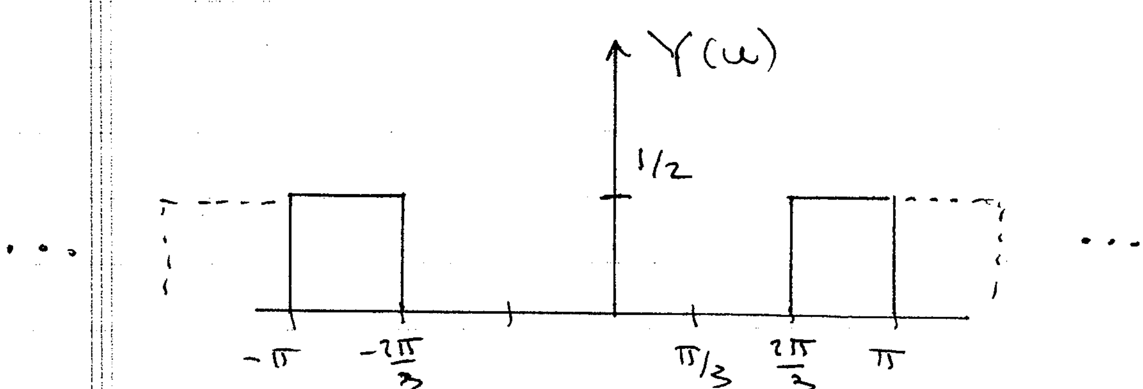
$$H(\omega) = DTFT\left\{2\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n}\cos\left(\frac{5\pi}{6}\right)\right\}$$



period $= 2\pi$

$$\frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3} \qquad \frac{5\pi}{6} + \frac{\pi}{6} = \pi$$

Since $Y(\omega) = H(\omega)\,X(\omega)$



(c) $$\sum_{n=-\infty}^{\infty} y^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}|Y(\omega)|^2\,d\omega = \frac{1}{2\pi}\,2\left[\left(\frac{1}{2}\right)^2\frac{\pi}{3}\right]$$

height X width

$$= \frac{1}{12}$$