DTFT: Properties and Examples

- **DTFT:**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- **Inverse DTFT:**

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$$

- **Note:** we will not use the book's notation $X(e^{j\omega})$
- Two primary differences between the CT Fourier Transform ((Chap. 4) and DTFT ((Chap 5)
  - in the DT domain, n is a discrete variable
  - the DTFT is periodic with period 2π

- as a result of these differences between the CTFT and DTFT, some of the properties of the DTFT are quite different than the properties of the CTFT, although some are similar

- <u>Similar properties:</u> $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$

- <u>Time-Shift</u> $x[n-n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\omega n_0} X(\omega)$

  n_0, integer

- <u>Convolution:</u>

$$y[n] = x[n] * h[n] \overset{DTFT}{\longleftrightarrow} Y(\omega) = H(\omega)X(\omega)$$

- The <u>Modulation</u> property is very similar

$$e^{j\omega_0 n} x[n] \overset{DTFT}{\longleftrightarrow} X(\omega - \omega_0)$$

but need to shift <u>all</u> periods of $X(\omega)$ to the right by ω_0 (example shortly)

- Since n is discrete while ω is continuous, there is <u>no</u> duality property for the DTFT
- Since n must be an integer, the time-scaling/frequency scaling property is <u>very</u> different
- recall for CTFT: $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$
  - true for <u>any</u> value of a

- For the DTFT, we have 2 corresponding props.

$$y[n] = x[Dn] \overset{\text{DTFT}}{\longleftrightarrow} Y(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega - k2\pi}{D}\right)$$

D = integer > 1

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = \ell L, \ \ell \text{ integer} \\ 0, & \text{otherwise} \end{cases} \overset{\text{DTFT}}{\longleftrightarrow} Y(\omega) = X(L\omega)$$

- These 2 properties are beyond the scope of this course => you will not be tested on them ☺

- Multiplication-in-time property:

$$z[n] = x[n]\,y[n] \overset{DTFT}{\longleftrightarrow} Z(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mu)\,Y(\omega-\mu)\,d\mu$$

$$= \frac{1}{2\pi}\, X(\omega) \circledast Y(\omega)$$

periodic convolution

- limits on integral are $-\pi$ to π (not $-\infty$ to ∞)
- $X(\omega)$ and $Y(\omega)$ are both periodic with period 2π } Must factor in when forming $Y(\omega-\mu)$ $= Y(-(\mu-\omega))$

- can't take derivative with respect to n in DT, but can take derivative wrt ω in the frequency domain:

$$n \cdot x[n] \overset{DTFT}{\longleftrightarrow} j \frac{d X(\omega)}{d\omega}$$

- <u>note</u>: if $x[n] = x_a(nT_s)$, and $T_s \ll 1$

$$\frac{1}{T_s}\Big(x_a((n+1)T_s) - x_a(nT_s)\Big) = \frac{1}{T_s}\big(x[n+1] - x[n]\big)$$

is a good approximation to the derivative of $x_a(t)$ at the time $t = nT_s$ a la basic calculus

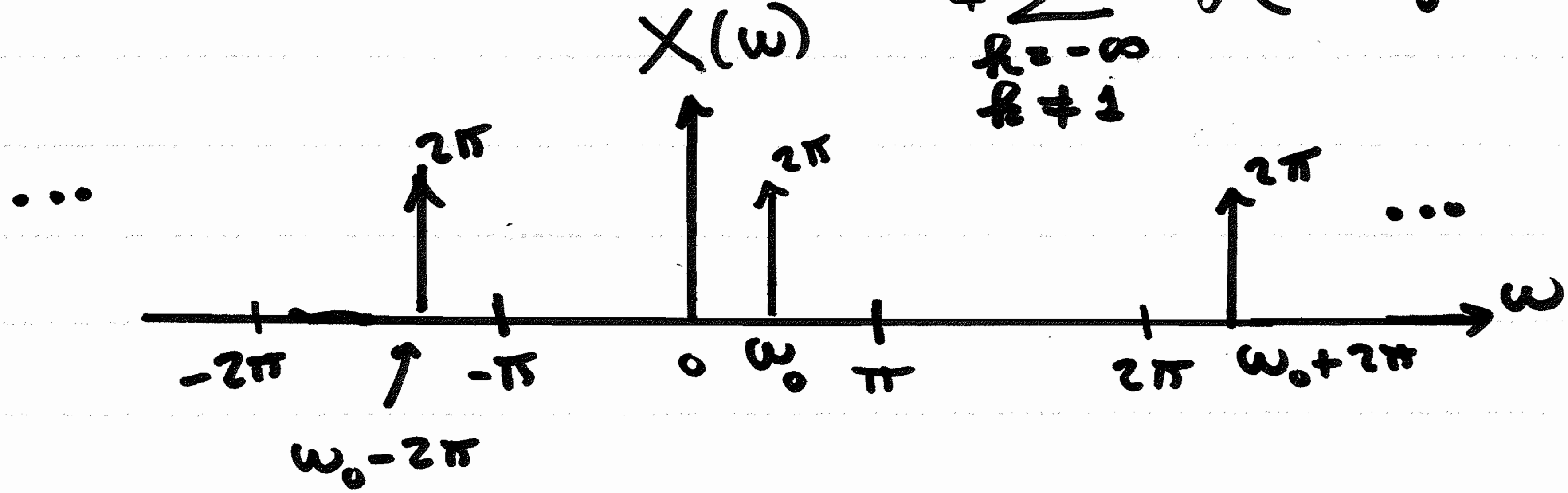
- Thus, might consider this property:

$$\frac{1}{T_s}\big(x[n+1] - x[n]\big) \overset{DTFT}{\longleftrightarrow} \frac{1}{T_s}\left(e^{j\omega} - 1\right)X(\omega)$$

- See Table 5.1 for list of DTFT properties

- DTFT pairs:

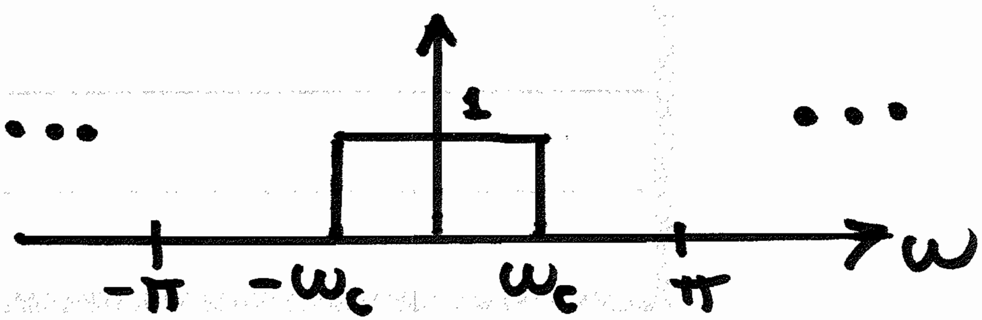
$$x[n] = e^{j\omega_0 n} \xleftrightarrow{\text{DTFT}} X(\omega) = 2\pi\,\delta(\omega - \omega_0) + \sum_{\substack{k=-\infty \\ k \neq 1}}^{\infty} 2\pi\,\delta(\omega - \omega_0 - k2\pi)$$



- will often plot only over $-\pi < \omega < \pi$ <u>but</u> must keep in mind $X(\omega)$ is periodic 2π

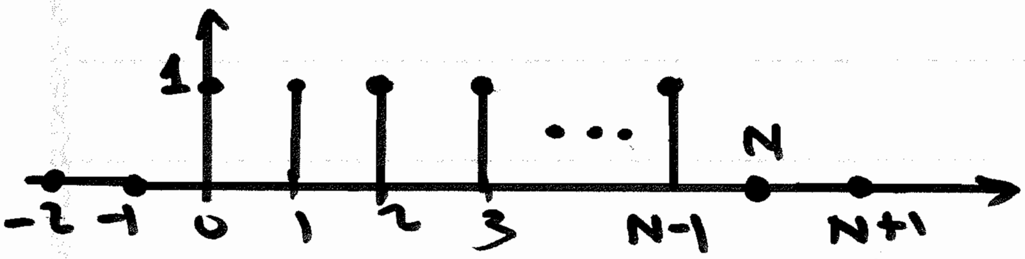
$$x[n] = \frac{\sin(\omega_c n)}{\pi n} \xleftrightarrow{\text{DTFT}} \ldots$$

$-\pi < \omega_c < \pi$



- DT rectangle:

$$x[n] = u[n] - u[n-N] \overset{\text{DTFT}}{\longleftrightarrow} X(\omega) = \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(N-1)}{2}\omega}$$



Proof: $$X(\omega) = \sum_{n=0}^{N-1} (1) e^{-j\omega n} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

since $$\left(e^{-j\omega}\right)^n = e^{-j\omega n}$$

- Use "half-angle trick" to simplify:

$$X(\omega) = \frac{\left(e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}\right)}{\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)} \frac{e^{-j\omega\frac{N}{2}}}{e^{-j\omega\frac{1}{2}}} \frac{\left(\frac{1}{2j}\right)}{\left(\frac{1}{2j}\right)}$$

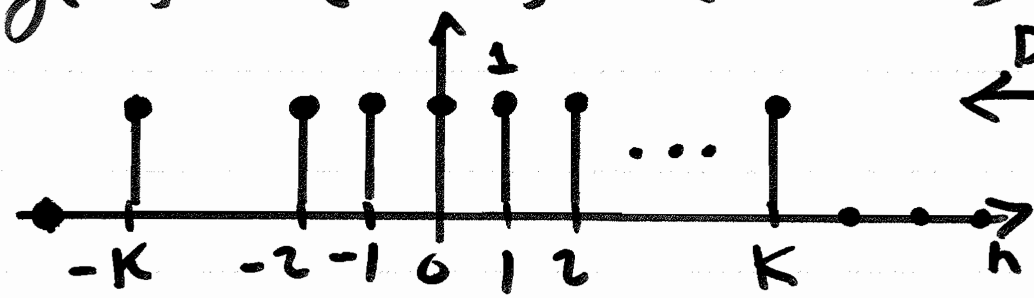
$$X(\omega) = \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} e^{-j\frac{(N-1)}{2}\omega}$$

- almost in polar form, but $\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$ can go negative for certain frequency bands
- Suppose N is odd such that $N = 2K+1$ and $K = \frac{N-1}{2}$ is an integer
- Form $y[n] = x[n+K] \Rightarrow$ shift to left by K so that DT rectangle is centered at $n=0$
- Time-Shift Property yields:

$$Y(\omega) = e^{jK\omega} X(\omega) = \underbrace{e^{j\frac{(N-1)}{2}\omega} e^{-j\frac{(N-1)}{2}\omega}}_{\text{cancel}} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

- This yields DTFT pair:

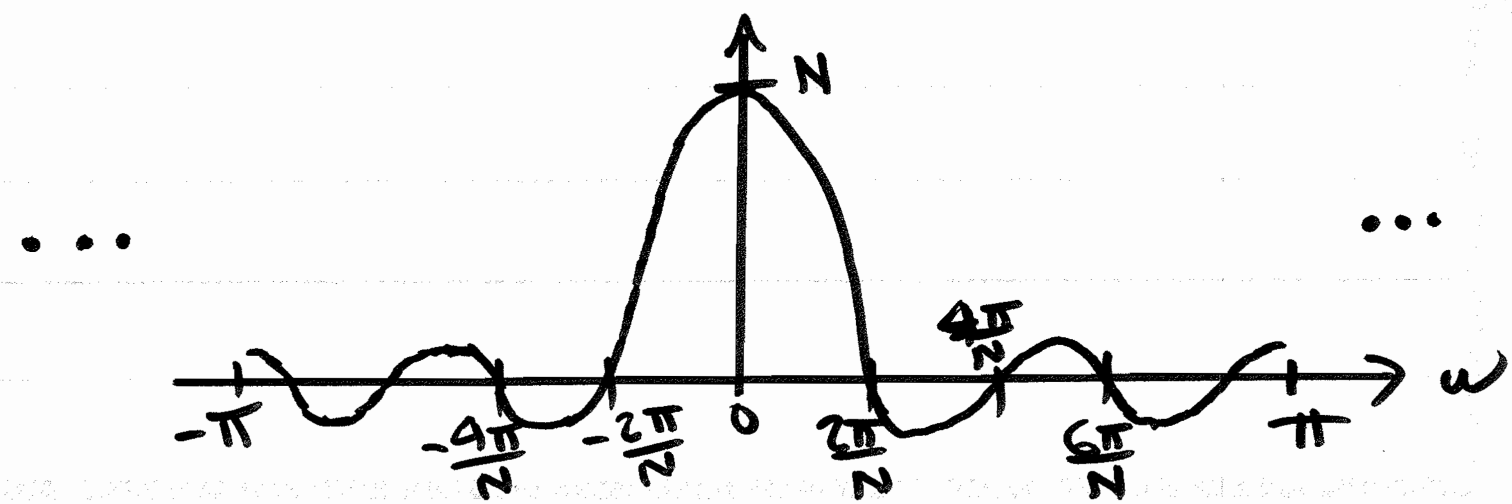
$$y[n] = u[n+K] - u[n-(K+1)] \xleftrightarrow{DTFT} Y(\omega) = \frac{\sin\left(\frac{(2K+1)}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$



let $N = 2K+1$.

since $\sin(\theta) = 0$ for $\theta = m\pi$, m integer

$\sin\left(\frac{N}{2}\omega\right) = 0$ for $\omega = m\frac{2\pi}{N}$, m integer



• Consider:

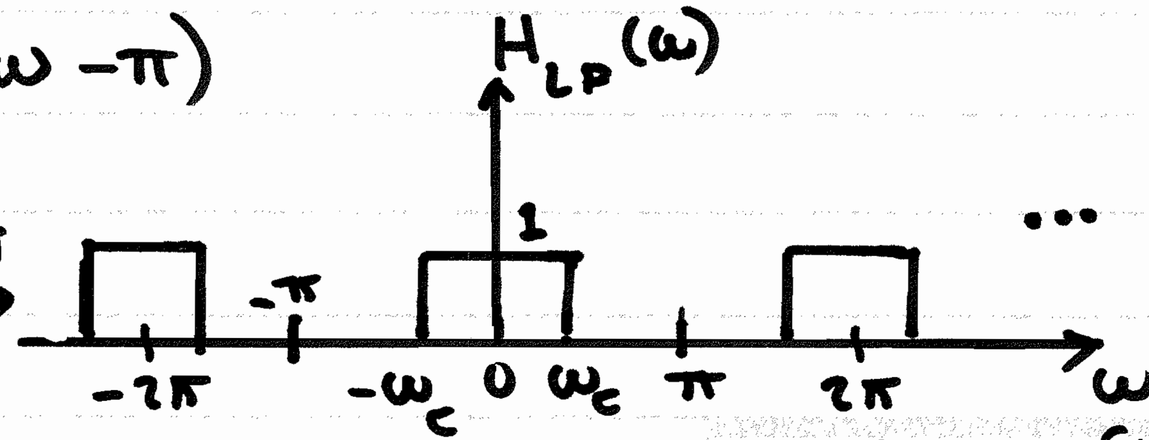
$$h_{HP}[n] = e^{j\pi n}\,\frac{\sin(\omega_c n)}{\pi n} = (-1)^n\,\frac{\sin(\omega_c n)}{\pi n} = (-1)^n\, h_{LP}[n]$$

• Modulation / Frequency-Shift property dictates

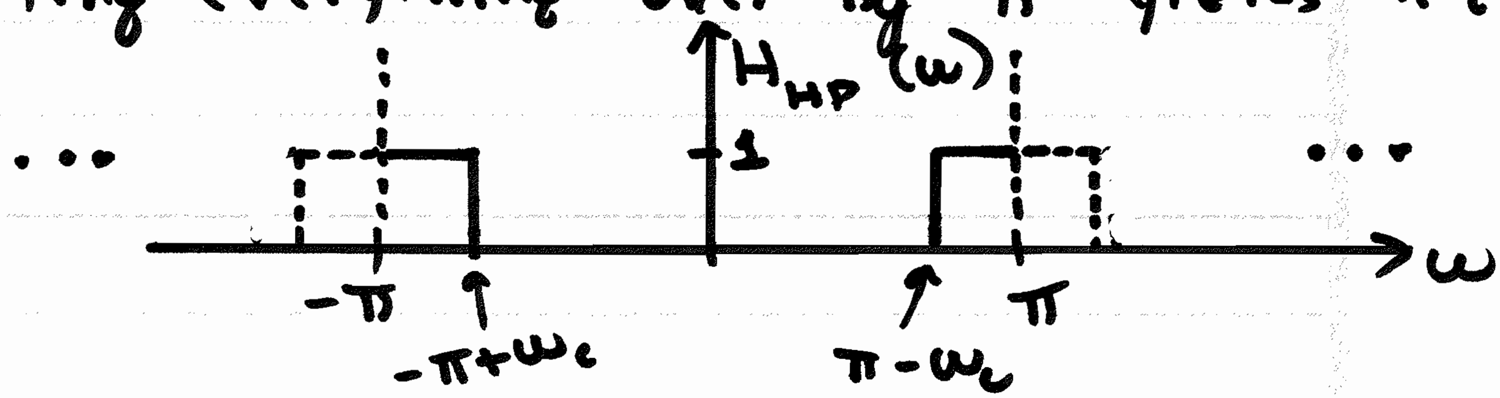
$$H_{HP}(\omega) = H_{LP}(\omega - \pi)$$

• where:

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n} \overset{DTFT}{\longleftrightarrow}$$

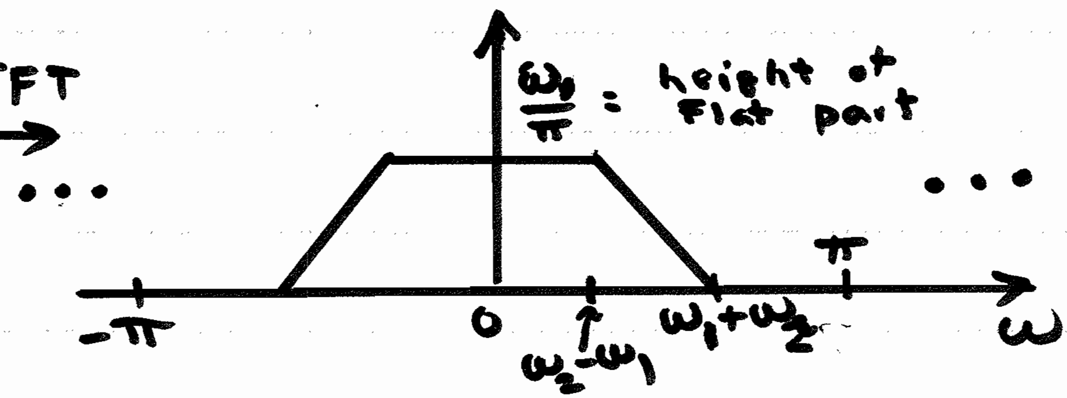


• shifting everything over by π yields highpass filter



• Important DTFT pair:

$$\frac{\sin(\omega_1 n)}{\pi n} \cdot \frac{\sin(\omega_2 n)}{\pi n} \overset{DTFT}{\longleftrightarrow}$$



restriction

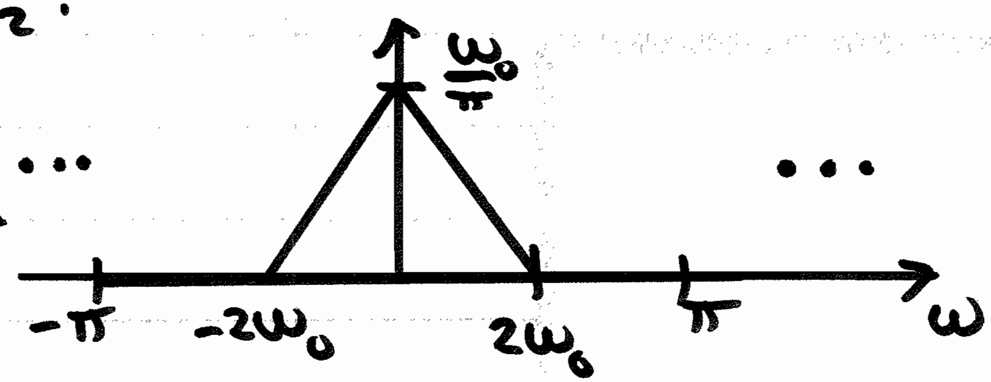
$$\boxed{\omega_1+\omega_2 < \pi}$$

$\omega_1 < \omega_2$
WLOG

remember: periodic with
period $= 2\pi$

• Special case: $\omega_1 = \omega_2$:

$$\left\{\frac{\sin(\omega_0 n)}{\pi n}\right\}^2 \overset{DTFT}{\longleftrightarrow}$$



$2\omega_0 < \pi$
or $\omega_0 < \frac{\pi}{2}$

• Frequency Response of DT LTI Systems Described by Difference Equations

• Digital Filter = Difference Equation implemented in either software or hardware

• used to perform frequency selective filtering and many other signal processing functions

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

• Interested in the frequency response of an LTI system = difference equation, which is the DTFT of the impulse response of the system

$$h[n] \xleftrightarrow{DTFT} H(\omega)$$

- Three properties of the DTFT - linearity, time-shift, convolution - allow us to find $H(\omega)$ without ever determining $h[n]$ (impulse response)
- Take DTFT of both sides of Diff. Eqn. using linearity and time-shift propterties:

$$Y(\omega) = -\sum_{k=1}^{N} a_k e^{-jk\omega} Y(\omega) + \sum_{k=0}^{M} b_k e^{-jk\omega} X(\omega)$$

- $$Y(\omega)\left\{1 + \sum_{k=1}^{N} a_k e^{-jk\omega}\right\} = \left\{\sum_{k=0}^{M} b_k e^{-jk\omega}\right\} X(\omega)$$

- $$Y(\omega) = \underbrace{\left\{\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{1 + \sum_{k=1}^{N} a_k e^{-jk\omega}}\right\}}_{\text{must be: } H(\omega) \text{ from convolution property}} X(\omega)$$

$$h[n] \xleftrightarrow{\text{DTFT}} H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Defining $a_0 = 1$

- Example: $y[n] = y[n-1] + x[n] - x[n-4]$

$N = M = 1$ $\quad a_1 = -1 \quad b_0 = 1 \quad b_4 = -1 \quad b_1 = 0,\ b_2 = 0,\ b_3 = 0$

$$H(\omega) = \frac{1 - e^{-j4\omega}}{1 - e^{-j\omega}}$$

- Use "half-angle trick" to evaluate how system responds at different frequencies

$$H(\omega) = \frac{\left(e^{j2\omega} - e^{-j2\omega}\right) e^{-j2\omega}}{\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) e^{-j\frac{\omega}{2}}} \cdot \frac{\frac{1}{2j}}{\frac{1}{2j}}$$

$$H(\omega) = \frac{\sin(4\omega)}{\sin\left(\frac{\omega}{2}\right)} e^{-\frac{3}{2}\omega}$$

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- compare with DTFT of $h[n] = u[n] - u[n-4]$

$$= u[n] - u[n-N]$$

with $N = 4$