Discrete-Time Fourier Transform (DTFT)

Chap 5

- Def'n of DTFT comes from passing DT sinewave thru a DT LTI system

$$e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h[n]\end{matrix}} \longrightarrow y[n] = e^{j\omega_0 n} * h[n]$$

$$= H(\omega_0)\, e^{j\omega_0 n}$$

- Since:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega_0 (n-k)} = \left\{ \sum_{k=-\infty}^{\infty} h[k]\, e^{-jk\omega_0} \right\} e^{j\omega_0 n}$$

DTFT: $$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

Since k is just a dummy variable for summation

- DTFT defined same way for DT signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- On the next 2 pages, we prove:

If $x[n] = x_a(nT_s)$, n integer, and

$$x_a(t) \overset{\mathcal{F}}{\longleftrightarrow} X_a(\omega) \qquad x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$$

then $X(\omega)$ is related to $X_a(\omega)$ as:

$$\boxed{\begin{aligned} &X(\omega) = X_s(F_s\,\omega) \\ &\text{where: } X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k2\pi F_s) \end{aligned}}$$

- $F_s = \frac{1}{T} =$ sampling rate $=$ no. of samples / second
per

- To derive this relationship, recall that for "sampled" signal, we have:

$$x_s(t) = x_a(t)\left(\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right) = \sum_{n=-\infty}^{\infty} x_a(t)\,\delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT_s)$$

- Taking the CT Fourier Transform:

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n T_s} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\left(\frac{\omega}{F_s}\right)n}$$

- Comparing with formula for DTFT obtained from passing DT sinewave thru DT LTI System

$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$, we have $X(\omega) = X_s(F_s\omega)$

- In our previous derivation for Sampling Theory for Chap. 7, rather than bring $x_a(t)$ inside the sum and taking the CTFT, we invoked the time-domain product property of the FT:

$$x_s(t) = x_a(t)\left(\sum_n \delta(t-nT_s)\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X_a(\omega) * \frac{1}{T_s}\sum_k \delta\left(\omega - \frac{k2\pi}{T_s}\right)$$

$$= X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k2\pi F_s)$$

- This completes proof:

$$X(\omega) = X_s(F_s\omega)$$

where: $X_s(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(\omega - k2\pi F_s)$

$$X(\omega) = F_s \sum_{k=-\infty}^{\infty} X_a(F_s\omega - k2\pi F_s) = F_s \sum_{k=-\infty}^{\infty} X_a\left(F_s(\omega - k2\pi)\right)$$

- Recall fundamental principle:

Sample in time domain $\longleftrightarrow$ replications in the frequency domain periodically spaced every integer multiple of $\omega_s = 2\pi F_s$

- The compression by F_s causes the replications in the DTFT to occur at every integer multiple of $\dfrac{\omega_s}{F_s} = 2\pi$

- Thus, a DTFT is periodic with period 2π
- This can be seen mathematically since

$$X(\omega + \underset{\text{integer}}{\ell}\, 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega + \ell 2\pi)n}$$

$$= \left\{ \sum_{n} x[n]\, e^{-j\omega n} \right\} \underbrace{\left(e^{-j2\pi}\right)^{\ell n}}_{=1}$$

- To further drive home the point, consider sampling a sinewave: $x_a(t) = e^{j\omega_a t}$

$\boxed{\text{subscript "a"} \equiv \text{analog}}$

$$x[n] = x_a(nT_s) = x_a\left(\frac{n}{F_s}\right) = e^{j\omega_a \frac{n}{F_s}} = e^{j\frac{\omega_a}{F_s} n}$$

- The frequency of the resulting DT sinewave is:

$$\boxed{\omega_d = \frac{\omega_a}{F_s}}$$

$\Rightarrow$ division by the sampling rate $\Rightarrow$ compression by the sampling rate

- The DT frequency variable may be viewed as a normalized frequency variable
  $\Rightarrow$ normalized by the sampling rate $F_s = \frac{1}{T_s}$

- That is, regardless of the sampling rate, the DTFT is periodic with period 2π
  $\Rightarrow$ the replications occur every integer multiple of 2π

- examine relationship between DT frequency variable and analog frequency variable ω_a

$$\boxed{\omega = \frac{\omega_a}{F_s}}$$

- If you sample at a rate $\omega_s = 2\pi F_s$, the highest frequency you can "see" is $\frac{\omega_s}{2} = 2\pi \frac{F_s}{2} = \pi F_s$

- This highest analog frequency is mapped to the DT frequeny: $\frac{\pi F_s}{F_s} = \pi$

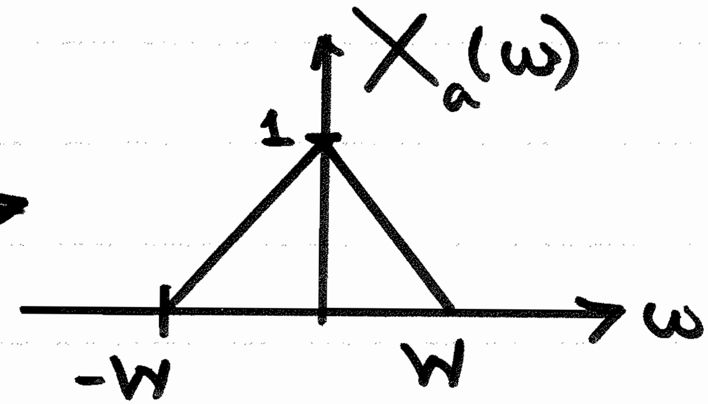
- That's why π is the highest DT frequency!!!
  - recall: any DT frequency outside the range $-\pi < \omega < \pi$ => you can subtract (or add) an integer multiple of 2π to put in $-\pi < \omega < \pi$

Example: $x_a(t) = \frac{2\pi}{W}\left\{\frac{\sin\left(\frac{W}{2}t\right)}{\pi t}\right\}^2$

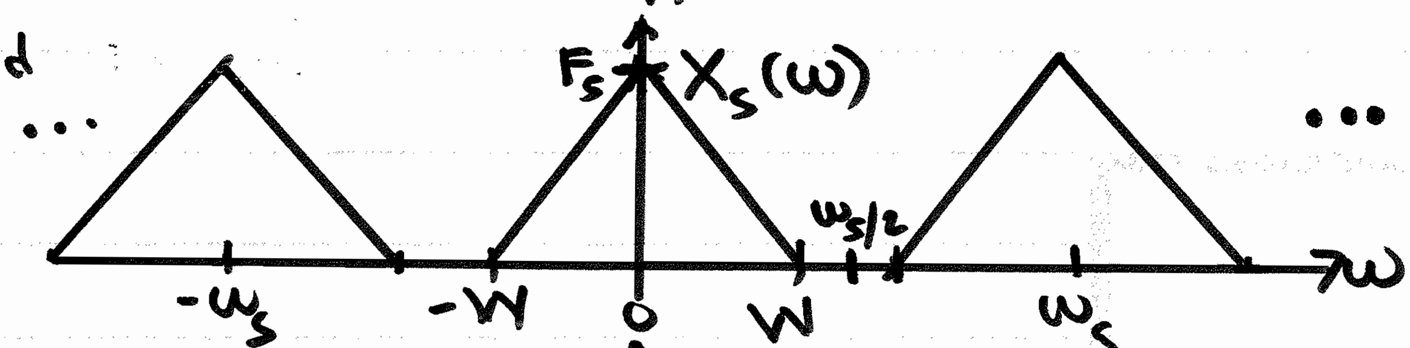
- sampled at a rate $\omega_s > 2\,\omega_{max}$

  where $\omega_{max} = W$

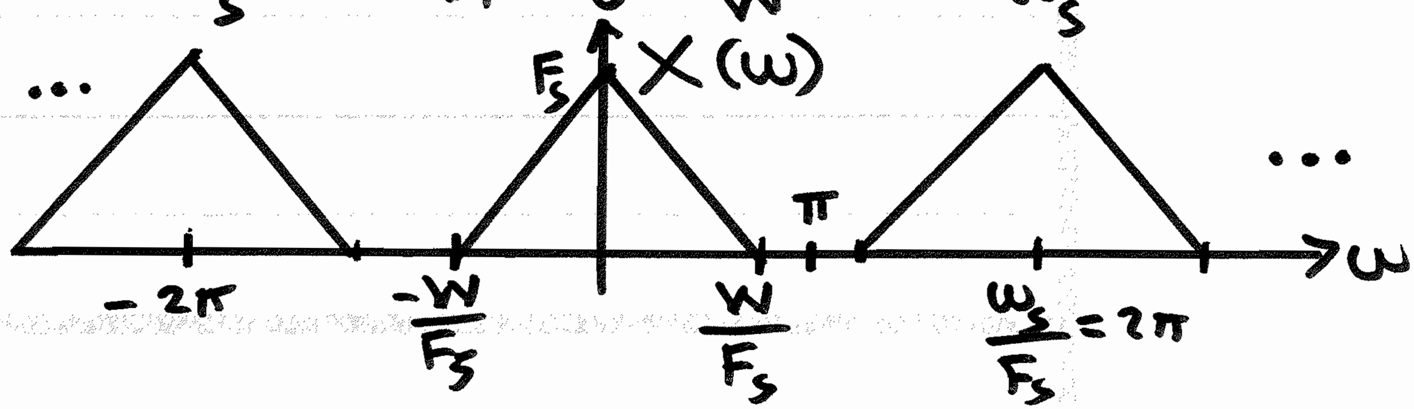
$$x_a(t) = \frac{2\pi}{W}\left\{\frac{\sin\left(\frac{W}{2}t\right)}{\pi t}\right\}^2 \xleftrightarrow{\mathcal{F}}$$



- For sampled signal: $x_s(t) \xleftrightarrow{\mathcal{F}}$



- Compress/Divide by sampling rate $F_s = \frac{1}{T}$



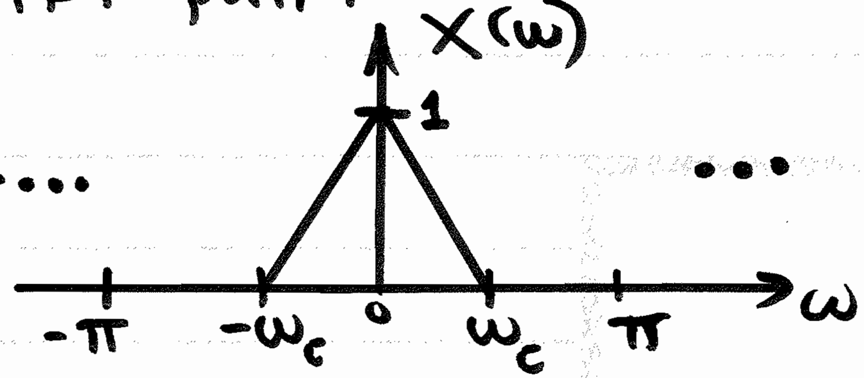
Note: $x[n] = x_a(nT_s) = \dfrac{2\pi}{W}\left\{\dfrac{\sin\left(\frac{W}{2}nT_s\right)}{\pi n T_s}\right\}^2$

$$= \frac{2\pi}{W}\left\{\frac{\sin\left(\frac{W}{2F_s}n\right)}{\pi n / F_s}\right\}^2 = \frac{2\pi F_s^2}{W}\left\{\frac{\sin\left(\frac{\omega_c}{2}n\right)}{\pi n}\right\}^2$$

- where: $\omega_c \triangleq \dfrac{W}{F_s}$
- now divide by F_s $\Rightarrow$ divide by F_s in freq. domain

and we got our first DTFT pair:

$$\frac{2\pi}{\omega_c}\left\{\frac{\sin\left(\frac{\omega_c}{2}n\right)}{\pi n}\right\}^2 \overset{DTFT}{\longleftrightarrow}$$



- See "CTFT-DTFT Relationship for a Sampled Sinewave" for DT sinewave:

$$x[n] = e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow}$$