Common Question asked about convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Why aren't limits on integral often not $-\infty$ to $+\infty$ when one actually computes the integral?

Consider course example: $T_2 > T_1$

$$x(t) = \left(1 - \frac{t}{T_2}\right)\left(u(t) - u(t-T_2)\right) * \left(u(t) - u(t-T_1)\right)$$

$$= \left(1 - \frac{t}{T_2}\right) \text{rect}\left(\frac{t - \frac{T_2}{2}}{T_2}\right) * \text{rect}\left(\frac{t - \frac{T_1}{2}}{T_1}\right)$$

$$y(t) = \int_{-\infty}^{\infty} \left(1 - \frac{\tau}{T_2}\right) \text{rect}\left(\frac{\tau - \frac{T_2}{2}}{T_2}\right) \text{rect}\left(\frac{t - \tau - \frac{T_1}{2}}{T_1}\right) d\tau$$

Since rect(x) is an even function

$$\mathrm{rect}\left(\frac{t-\tau-\frac{T_1}{2}}{T_1}\right)=\mathrm{rect}\left(\frac{-\left(\tau-t+\frac{T_1}{2}\right)}{T_1}\right)=\mathrm{rect}\left(\frac{\tau-\left(t-\frac{T_1}{2}\right)}{T_1}\right)$$

$$y(t)=\int_{-\infty}^{\infty}\left(1-\frac{\tau}{T_2}\right)\underbrace{\mathrm{rect}\left(\frac{\tau-\frac{T_2}{2}}{T_2}\right)\mathrm{rect}\left(\frac{\tau-\left(t-\frac{T_1}{2}\right)}{T_1}\right)}_{\substack{\text{over what range of } \tau \text{ is this} \\ \text{product } = 1\text{? denote product } f(\tau)}}\,d\tau$$

answer depends on t

- $t<0$: $f(\tau)=0$
- $0<t<T_1$: $f(\tau)=1$ for $0<\tau<t$
- $T_1<t<T_2$: $f(\tau)=1$ for $t-T_1<\tau<t$
- $T_2<t<T_1+T_2$: $f(\tau)=1$ for $t-T_1<\tau<T_2$
- $t>T_1+T_2$: $f(\tau)=0$

- If you had two functions that were "turned on" for all time AND they had the same functional form for all time, then limits would actually be $-\infty$ to $+\infty$
- Example: $x(t) = e^{-\frac{1}{2}\frac{t^2}{\sigma_1^2}}$ $\quad h(t) = e^{-\frac{t^2}{2\sigma_2^2}}$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2\sigma_1^2}} e^{-\frac{(t-\tau)^2}{2\sigma_2^2}} \, d\tau$$

<u>Answer:</u> $\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_2^2}}$

$$= \frac{1}{\sigma_3\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_3^2}} \qquad \sigma_3^2 = \sigma_1^2 + \sigma_2^2$$