Convolution Example:

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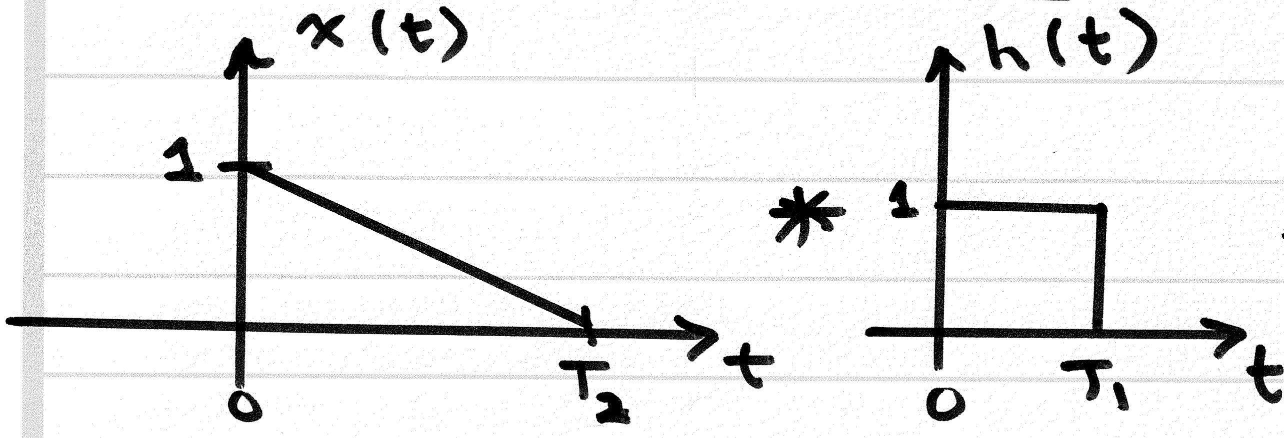


$= y(t) = ?$

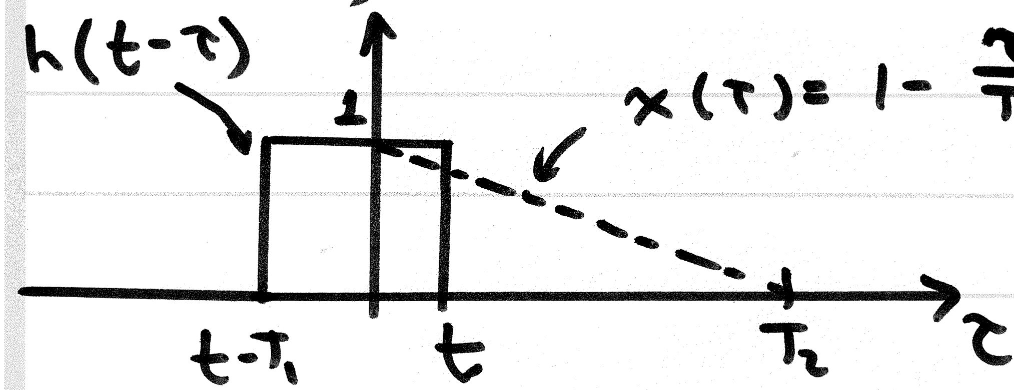
$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h(t) = \mathrm{rect}\left(\frac{t - T_1/2}{T_1}\right)\end{array}} \longrightarrow y(t)$$

$$y(t) = \int_{t-T_1}^{t} x(\tau)\, d\tau$$

Assume $T_2 > T_1$

For $t < 0$, $y(t) = 0$ (no overlap)

- for $t>0$ and $t-T_1<0$ $(t<T_1)$,
  that is, $0<t<T_1$ (partial overlap):



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_0^t \left(1 - \frac{\tau}{T_2}\right) d\tau = t\Big|_0^t - \frac{1}{T_2}\frac{\tau^2}{2}\Big]_0^t$$

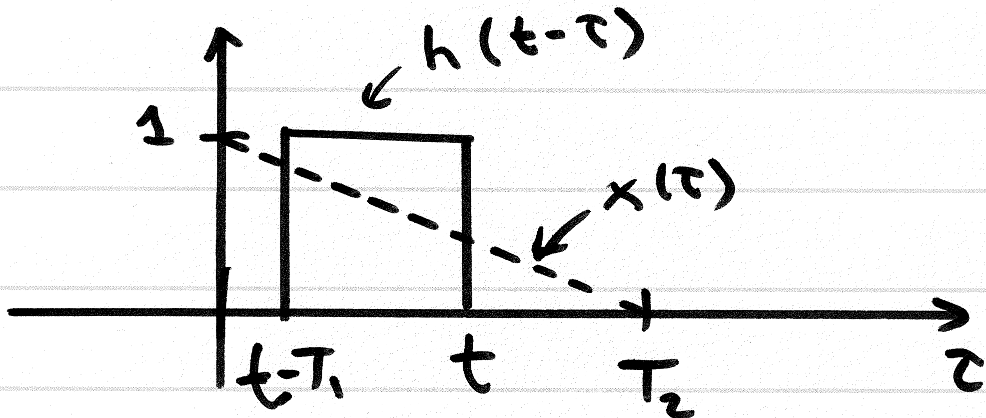
$$= (t-0) - \frac{1}{2T_2}(t^2 - 0^2)$$

$$= -\frac{1}{2T_2}t^2 + t \quad \text{for } 0<t<T_1$$

concave downwards

- for $t - T_1 > 0$ and $t < T_2$

$\Rightarrow \quad T_1 < t < T_2$ (full overlap):



$$y(t) = \int_{t-T_1}^{t} \left(1 - \frac{\tau}{T_2}\right) d\tau = \tau \Big]_{t-T_1}^{t} - \frac{1}{2T_2} \tau^2 \Big]_{t-T_1}^{t}$$

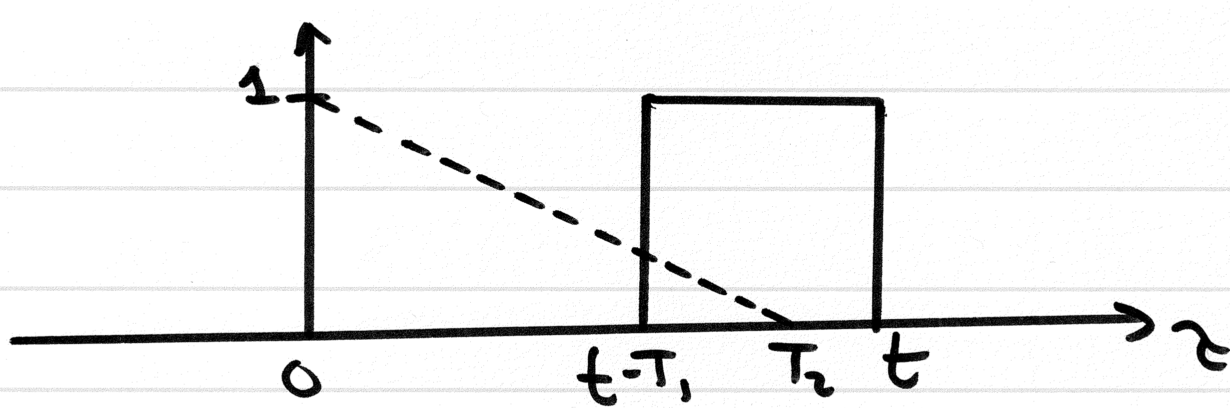
$$= \left[t - (t - T_1)\right] - \frac{1}{2T_2} \left\{ t^2 - (t - T_1)^2 \right\}$$

$$= T_1 - \frac{1}{2T_2} \left\{ t^2 - t^2 + 2T_1 t - T_1^2 \right\}$$

$$= -\frac{T_1}{T_2} t + \left(T_1 + \frac{T_1^2}{2T_2} \right)$$

• for $t > T_2$ and $t - T_1 < T_2$

$\Rightarrow$ $T_2 < t < T_1 + T_2$ (partial overlap):



$$y(t) = \int_{t-T_1}^{T_2} \left(1 - \frac{\tau}{T_2}\right) d\tau = \tau \Big]_{t-T_1}^{T_2} - \frac{1}{2T_2}\tau^2 \Big]_{t-T_1}^{T_2}$$

$$= \left[T_2 - (t - T_1)\right] - \frac{1}{2T_2}\left\{T_2^2 - (t-T_1)^2\right\}$$

$$= T_1 + T_2 - t - \frac{T_1}{T_2}t + \frac{t^2}{2T_2} - \left(\frac{T_2^2 - T_1^2}{2T_2}\right)$$

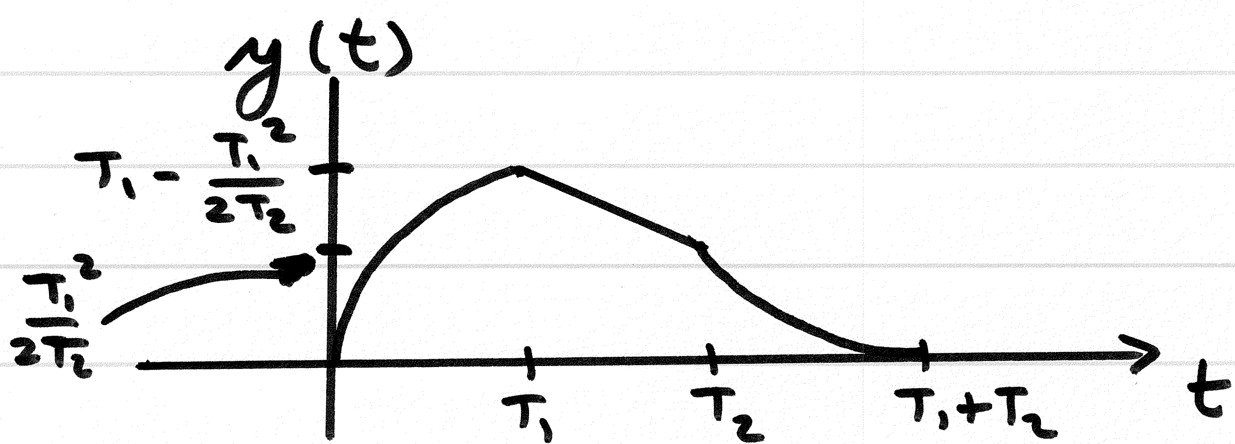
$$= +\frac{t^2}{2T_2} - \left(1 + \frac{T_1}{T_2}\right)t + \left\{T_1 + T_2 - \frac{T_2^2 - T_1^2}{2T_2}\right\}$$

concave upwards

over common denominator, constant equals: $\frac{(T_1+T_2)^2}{2T_1}$

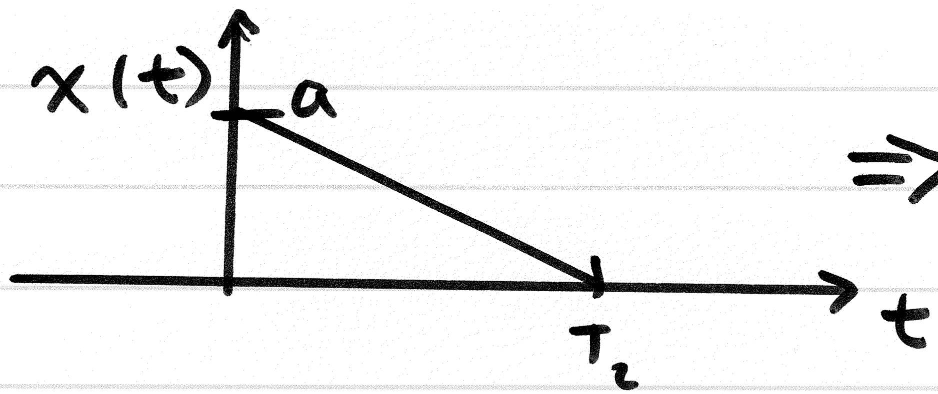
• Final answer:

$$y(t) = \begin{cases} 0, & t<0 \\ -\dfrac{t^2}{2T_2} + t, & 0<t<T_1 \quad \left(\begin{matrix}\text{quadratic,}\\ \text{partial overlap}\end{matrix}\right) \\ -\dfrac{T_1}{T_2}t + \left(T_1 + \dfrac{T_1^2}{2T_2}\right), & T_1<t<T_2 \quad \left(\begin{matrix}\text{linear,}\\ \text{full overlap}\end{matrix}\right) \\ +\dfrac{t^2}{2T_2} - \left(\dfrac{T_1+T_2}{T_2}\right)t + \left(\dfrac{(T_1+T_2)^2}{2T_2}\right), & \left(\begin{matrix}\text{quadratic,}\\ \text{partial}\\ \text{overlap}\end{matrix}\right) \\ & \text{for } T_2<t<T_1+T_2 \\ 0, & \text{for } t>T_1+T_2 \end{cases}$$



- Denote $y(t)$ on previous slide as $y_0(t)$

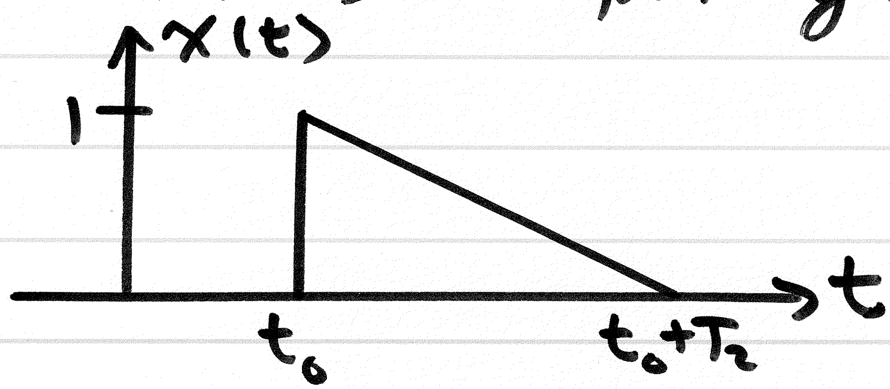
- What is output $y(t)$, when input is:



$\Rightarrow \quad y(t) = a\, y_0(t)$

$\Rightarrow$ linearity!

- What is output $y(t)$, when input is

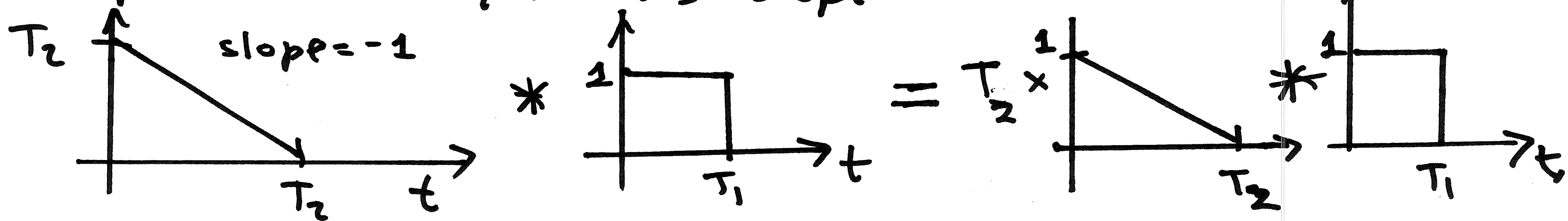


$\Rightarrow \quad y(t) = y_0(t - t_0)$

$\Rightarrow$ TI

Convolution Result Typically Given on Front Pg of Exam

- ramp down triangle has slope $= -1$

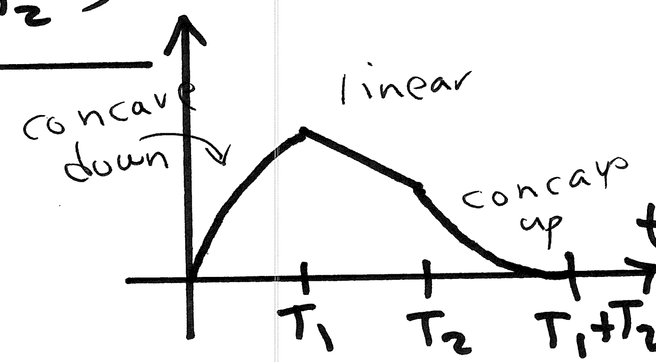


$$\underbrace{\{u(t)-u(t-T_1)\}}_{\mathrm{rect}\left(\frac{t-\frac{T_1}{2}}{T_1}\right)} * \underbrace{\{-(t-T_2)(u(t)-u(t-T_2))\}}_{\mathrm{rect}\left(\frac{t-\frac{T_2}{2}}{T_2}\right)} =$$

$$\left(T_2 t - t^2/2\right)\{u(t)-u(t-T_1)\}$$

$$+\left(-T_1 t + \frac{2T_1T_2+T_1^2}{2}\right)\{u(t)-u(t-T_2)\}$$

$$+\left(\frac{t^2}{2} - (T_1+T_2)t + \frac{(T_1+T_2)^2}{2}\right)\{u(t-T_2)-u(t-(T_1+T_2))\}$$



• What is output $y(t)$ when input is:

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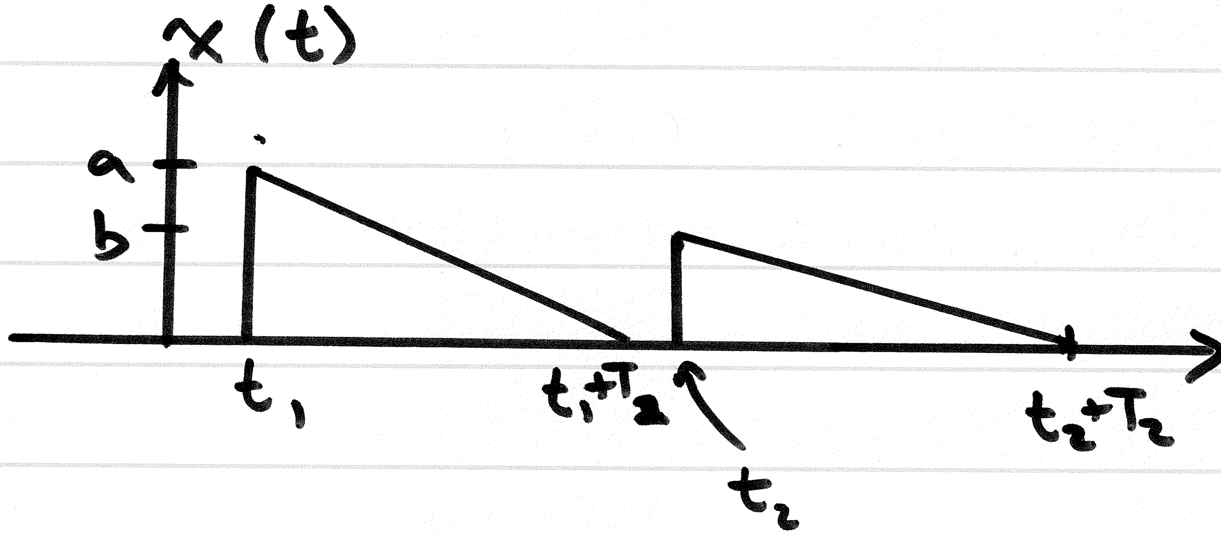

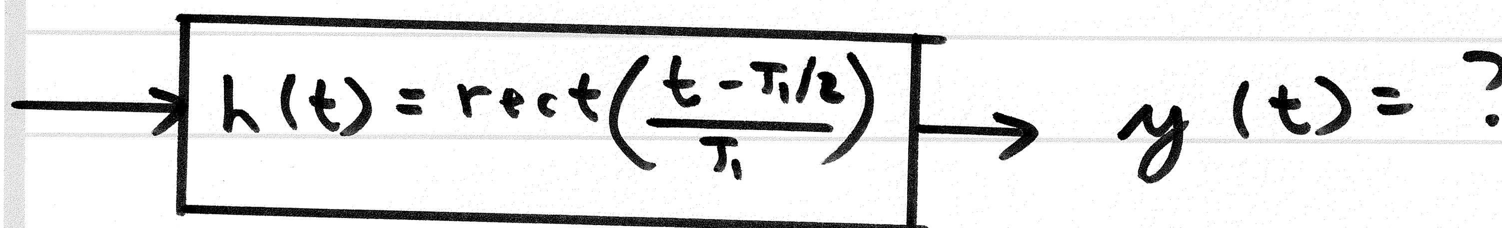

Answer:

$$y(t) = a\, y_0(t-t_1) + b\, y_0(t-t_2)$$

$\Rightarrow$ LTI