ECE301, Complex Numbers Overview

$x + jy$ (rectangular) Representation

Consider a complex number $z_1 = x_1 + jy_1$, where $x_1, y_1 \in \mathbb{R}$ and $j = \sqrt{-1}$. We define the "real part" of z as $\Re\{z_1\} = x_1$, and the "imaginary part" of z_1 as $\Im\{z_1\} = y_1$.
Now define another complex number $z_2 = x_2 + jy_2$ similarly. We have then that

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2), \\ \text{and } z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1), \end{aligned}$$

so that $\Re\{z_1 + z_2\} = x_1 + x_2$, $\Im\{z_1 + z_2\} = y_1 + y_2$; and $\Re\{z_1 z_2\} = x_1 x_2 - y_1 y_2$, $\Im\{z_1 z_2\} = x_1 y_2 + x_2 y_1$.

Magnitude of Complex Number

For a complex number $z = x + jy$, we define the magnitude, $|z|$, as follows:

$$|z| = \sqrt{x^2 + y^2}.$$

The magnitude can be thought of as the distance a complex number z lies from the origin of the complex plane.

Complex Conjugate

For a complex number $z = x + jy$, we define its conjugate, z^*, as follows:

$$z^* = x - jy.$$

It follows, then, that $zz^* = x^2 + y^2 = |z|^2$, and $(z^*)^* = z$ We may also reduce fractions of complex numbers by using the conjugate. Let

$z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\left(\frac{z_2^*}{z_2^*}\right) = \frac{z_1 z_2^*}{|z_2|} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} = \frac{(x_1 x_2 + y_1 y_2) + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

Euler's Identity

Euler's identity states the following

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

It follows from this and trigonometric identities $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, that:

$$\begin{aligned} \cos(\theta) &= \frac{e^{j\theta} + e^{-j\theta}}{2}, \\ \text{and } \sin(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{2j} \end{aligned}$$

Note that for all phase angles θ, $|e^{j\theta}| = \cos^2(\theta) + \sin^2(\theta) = 1$.

$|z|e^{j\angle z}$ (polar) Representation

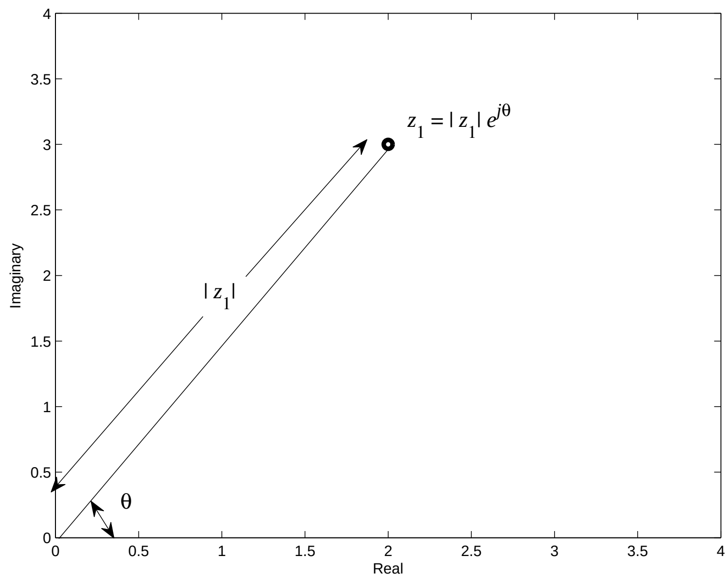
Let complex number $z_1 = x_1 + jy_1$. It follows from Euler's identity that $z_1 = |z_1|e^{j\theta_1}$, where $\theta_1 = \angle z_1 = \tan^{-1}(y_1/x_1)$.
We can thus represent a complex number z_1 in terms of a real and imaginary component (rectangular coordinates), or in terms of a magnitude, $|z_1|$, and a phase angle $\angle z_1$ (polar coordinates).

(Note that $\Re\{z_1\} = |z_1|\cos(\theta_1)$ and $\Im\{z_1\} = |z_1|\sin(\theta_1)$ define the inverse transformation back to rectangular coordinates.)
Let $z_2 = |z_2|e^{j\theta_2}$, and z_1 defined as above. We have then that

$$\begin{aligned} z_1 + z_2 &= (|z_1|\cos(\theta_1) + |z_2|\cos(\theta_2)) + j(|z_1|\sin(\theta_1) + |z_2|\sin(\theta_2)) \\ z_1 z_2 &= |z_1||z_2|e^{j\theta_1\theta_2}, \end{aligned}$$

so that

$$\begin{aligned} |z_1 + z_2| &= \sqrt{(|z_1|\cos(\theta_1) + |z_2|\cos(\theta_2))^2 + (|z_1|\sin(\theta_1) + |z_2|\sin(\theta_2))^2}, \\ \angle(z_1 + z_2) &= \tan^{-1}\left(\frac{|z_1|\cos(\theta_1) + |z_2|\cos(\theta_2)}{|z_1|\sin(\theta_1) + |z_2|\sin(\theta_2)}\right); \\ |z_1 z_2| &= |z_1||z_2|, \\ \angle(z_1 z_2) &= \angle z_1 + \angle z_2 \end{aligned}$$



Visualization of complex number z_1 on complex plane

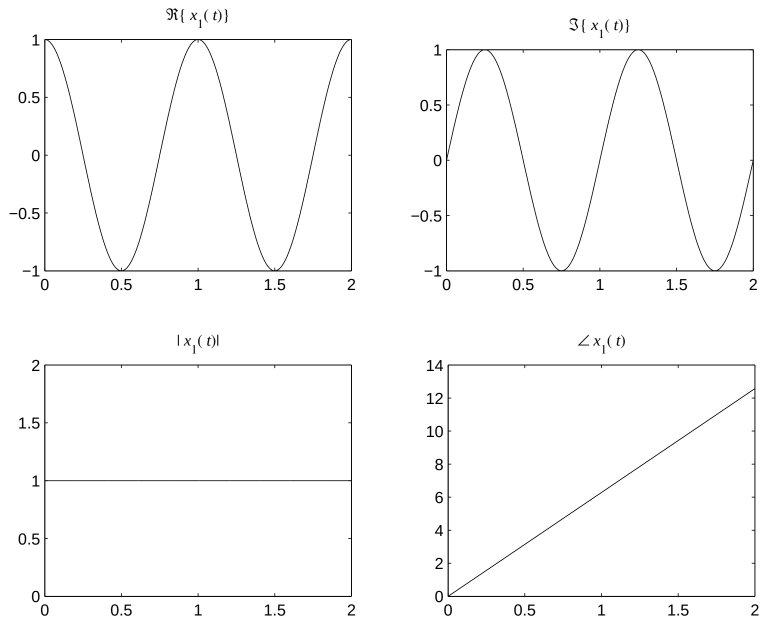
In general, it is easier to add complex numbers in rectangular coordinates, and multiply them in polar coordinates.
Note also that if $z_1 = |z_1|e^{j\theta_1} = |z_1|\cos\theta_1 + j|z_1|\sin\theta_1$, then

$$z_1^* = |z_1|\cos\theta_1 - j|z_1|\sin\theta_1 = |z_1|e^{-j\theta_1}.$$

Some Complex Signals

Consider the following signals: $x_1(t) = e^{j2\pi t}$, $x_2(t) = e^{(-2+j10\pi)t}$. For $x_1(t)$, we find its real and imaginary, magnitude and phase (all functions of t):

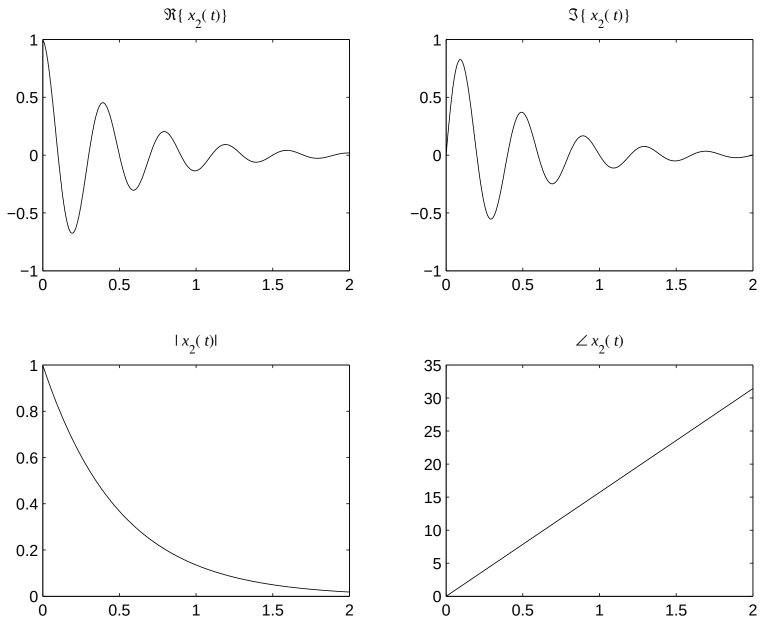
$$\begin{aligned} \Re\{x_1(t)\} &= \cos(2\pi t) \\ \Im\{x_1(t)\} &= \sin(2\pi t) \\ |x_1(t)| &= 1 \\ \angle x_1(t) &= 2\pi t. \end{aligned}$$



Plots of complex signal $x_1(t) = e^{j2\pi t}$

Similarly for $x_2(t)$, we have

$$\begin{aligned} \Re\{x_2(t)\} &= e^{-2t}\cos(10\pi t) \\ \Im\{x_2(t)\} &= e^{-2t}\sin(10\pi t) \\ |x_1(t)| &= e^{-2t} \\ \angle x_1(t) &= 10\pi t. \end{aligned}$$



Plots of complex signal $x_2(t) = e^{(-2+j10\pi)t}$