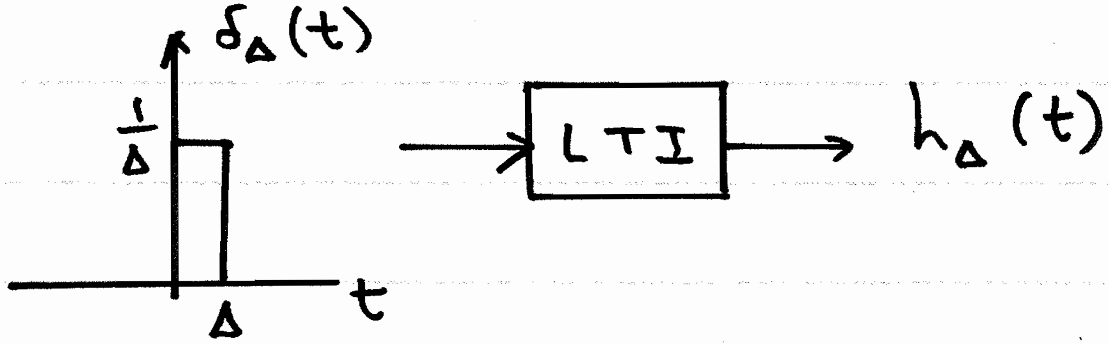
Derivation of CT Convolution



$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$$

impulse response

$$h(t) = \lim_{\Delta \to 0} h_\Delta(t)$$

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$$

- approximate $x(t)$ as:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \underbrace{x(k\Delta)\,\Delta}_{\text{view as amplitude of}} \; \underbrace{\delta_\Delta(t - k\Delta)}$$

- Due to both linearity (homogeneity <u>and</u> superposition) and time-invariance

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\,\delta_\Delta(t-k\Delta) \rightarrow \boxed{LTI} \rightarrow \hat{y}(t) =$$

$$\rightarrow \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\, h_\Delta(t-k\Delta)$$

} See Fig. 2.15 on pg 95 of text

as $\Delta \to 0$:

$\sum \to \int$ $\quad \Delta \to d\tau$

$k\Delta \to \tau$ $\Rightarrow$ continuous-valued variable since k goes all the way to ∞

$h_\Delta(t-k\Delta) \rightarrow h(t-\tau)$

- as $\Delta \to 0$: $y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Now, in ECE 202, you learned: Laplace Transform :

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = X(s)\,H(s)$$

- since multiplication is commutative, it follows:

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{L}} Y(s) = H(s)\,X(s)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$\boxed{\text{Convolution is commutative}}$

- Convolution also satisfies both distributive property and associative property:

- Distributive property has implications wrt two LTI systems in parallel:

$$x(t) * \{h_1(t) + h_2(t)\} \overset{\mathcal{L}}{\longleftrightarrow} X(s)\{H_1(s) + H_2(s)\}$$

$$= x(t) * h_1(t) + x(t) * h_2(t) \overset{\mathcal{L}}{\longleftrightarrow} = X(s)H_1(s) + X(s)H_2(s)$$

- <u>Note:</u> 2 LTI systems in parallel can be replaced by a single LTI system with impulse response $h(t) = h_1(t) + h_2(t)$ $\Rightarrow$ See Fig. 2.23 on pg. 105

- Associative property has implications wrt two LTI systems in series:

$$x(t) * h_1(t) * h_2(t) \xleftrightarrow{\mathcal{L}} X(s)\, H_1(s)\, H_2(s)$$

$$= \left(x(t) * h_1(t)\right) * h_2(t) \xleftrightarrow{\mathcal{L}} = \left(X(s) H_1(s)\right) H_2(s)$$

$$= x(t) * \left(h_1(t) * h_2(t)\right) \xleftrightarrow{\mathcal{L}} = X(s) \left(H_1(s) H_2(s)\right)$$

- Two LTI systems in series can be replaced by a single LTI system with impulse response

$$h(t) = h_1(t) * h_2(t)$$

- Further, commutativity dictates that $h(t) = h_2(t) * h_1(t)$ so order doesn't affect overall I/O relationship

See Fig 2.25 on pg. 109 of text

- Convolution is a general mathematical operation
  => doesn't have to be a signal with an impulse response
- can convolve two signals, for example
- also, later we will show that if you multiply two signals in the time domain, you get convolution in the frequency domain

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

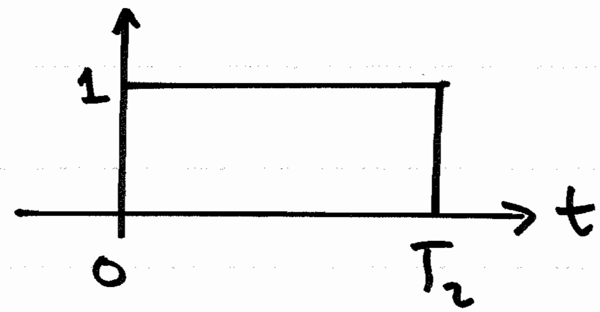
$$x_1(t) * \{x_2(t) + x_3(t)\} = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

$$(x_1(t) * x_2(t)) * x_3(t) = x_1(t) * (x_2(t) * x_3(t))$$

- **Example:** $$y(t) = \int_{t-T_1}^{t} x(\tau)\, d\tau$$

Input: $x(t) = u(t) - u(t-T_2) = \mathrm{rect}\left(\dfrac{t - \frac{T_2}{2}}{T_2}\right)$

$\boxed{T_1 < T_2}$



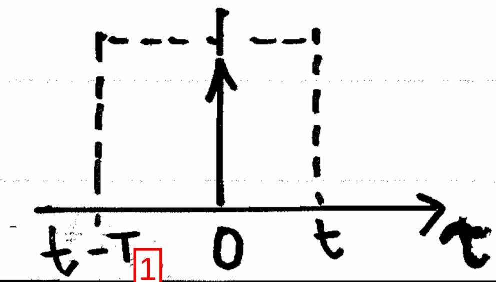
Impulse response $h(t) = ?$ input $x(t) = \delta(t)$

$$h(t) = \int_{t-T_1}^{t} \delta(\tau)\, d\tau$$

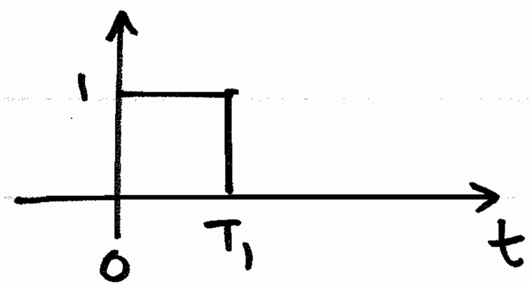
as long as $t - T_1 < 0$ and $t > 0$, capture area of delta function equal to one

- That is: $h(t) = 1 \qquad 0 < t < T_1$

$\qquad\qquad = 0 \qquad t < 0$ or $t > T_1$



$$h(t) = u(t) - u(t-T_1) = \text{rect}\left(\frac{t - \frac{T_1}{2}}{T_1}\right)$$



impulse response of: $y(t) = \int_{t-T_1}^{t} x(\tau)\, d\tau$

It follows that:

$h(t) = u(t)$ is the impulse response of

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t) \qquad (T_1 = \infty)$$

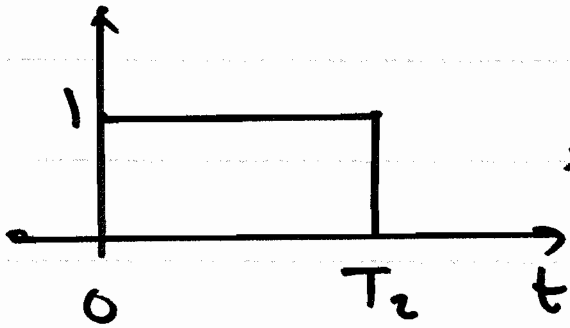
- Graphical method of convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

1. view both $x(\tau)$ and $h(t-\tau)$ as fns. of τ
2. $h(t-\tau) = h(-(\tau-t))$
   - flip $h(\tau)$ about $\tau=0$
   - shift to the right by t
3. point-wise multiply $x(\tau)\, h(-(\tau-t))$
4. find area under product (for all τ)

- Remember: convolution is commutative $\Rightarrow$ can flip either $x(t)$ or $h(t)$ and obtain same answer (i.e., reverse roles of $x(t)$ and $h(t)$ in procedure above)

$x(t) * h(t) = y(t) = ?$



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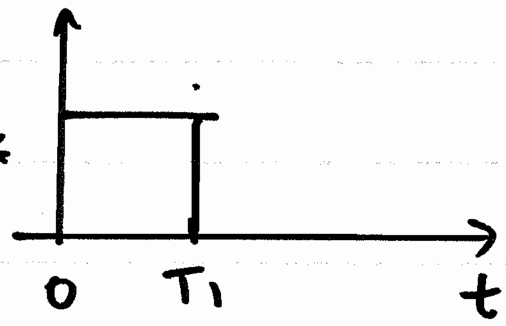
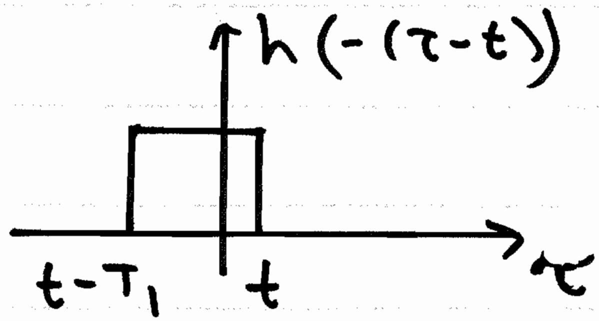
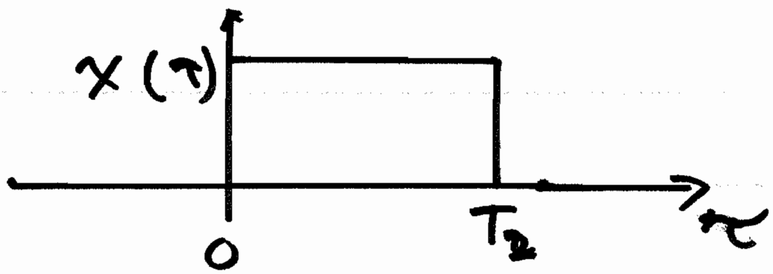
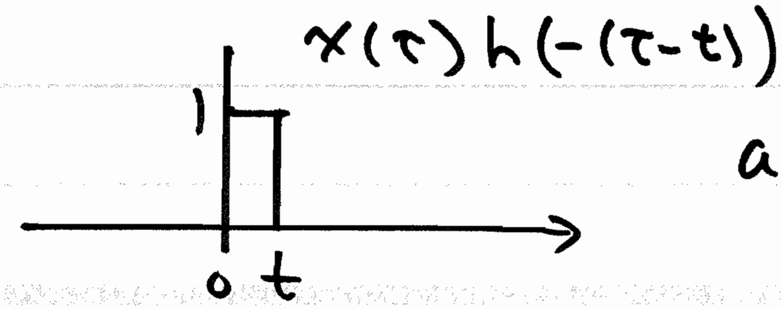


$= ??$







area $= t\,(1) = t$

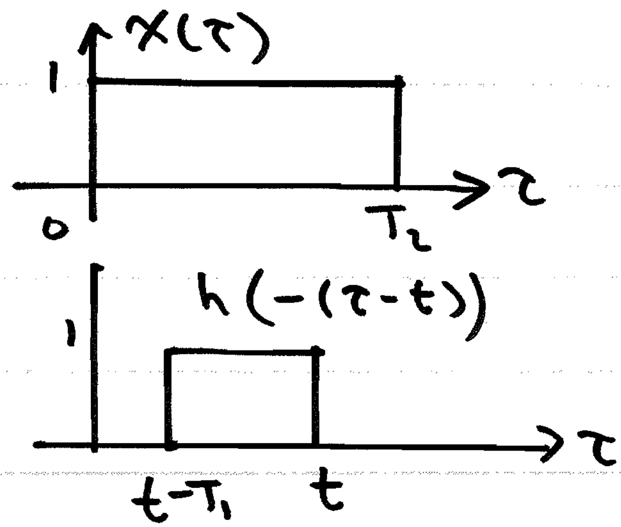
- so far: if $t<0$, no overlap $\Rightarrow y(t)=0$

- for $0<t<T_1$ $\boxed{t-T_1<0 \Rightarrow t<T_1}$

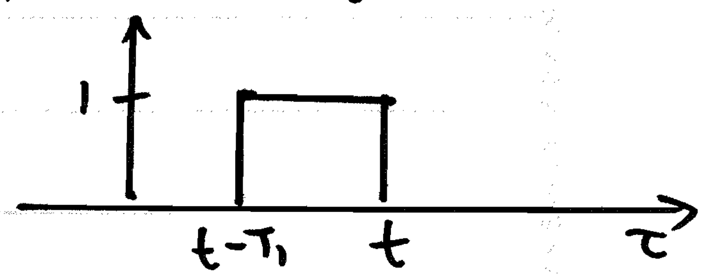
$\Rightarrow$ "partial overlap"

$$y(t)=t \quad \text{for } 0<t<T_1$$

- next: "full overlap" : $\left.\begin{array}{l} t-T_1>0 \\ t<T_2 \end{array}\right\} T_1<t<T_2$



product equals:



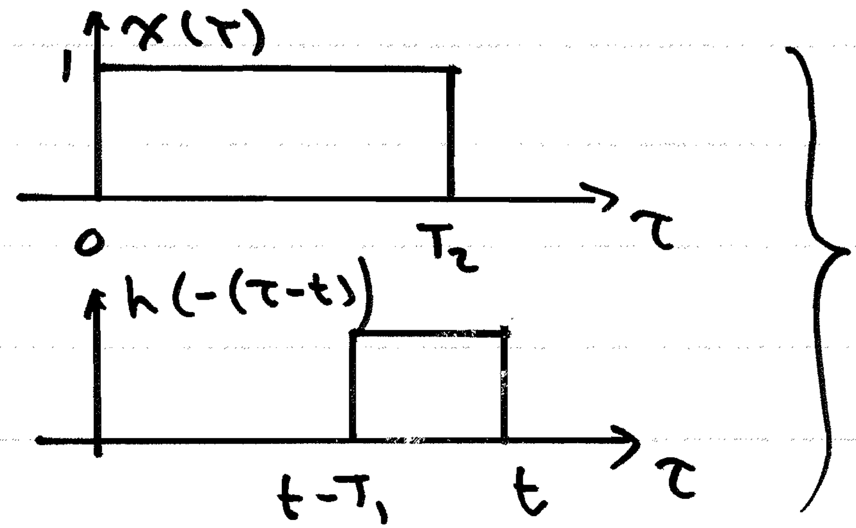
$$\text{area} = T_1(1) = T_1$$

- for $T_1 < t < T_2$: $\quad y(t) = T_1$

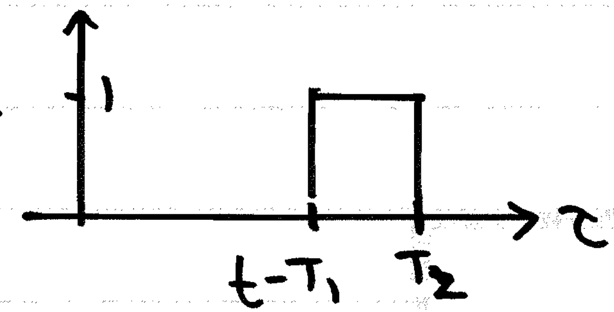
- next: "partial overlap" at end:

$$t > T_2 \quad \text{and} \quad t - T_1 < T_2$$

$$\Rightarrow \quad T_2 < t < T_1 + T_2$$



product:



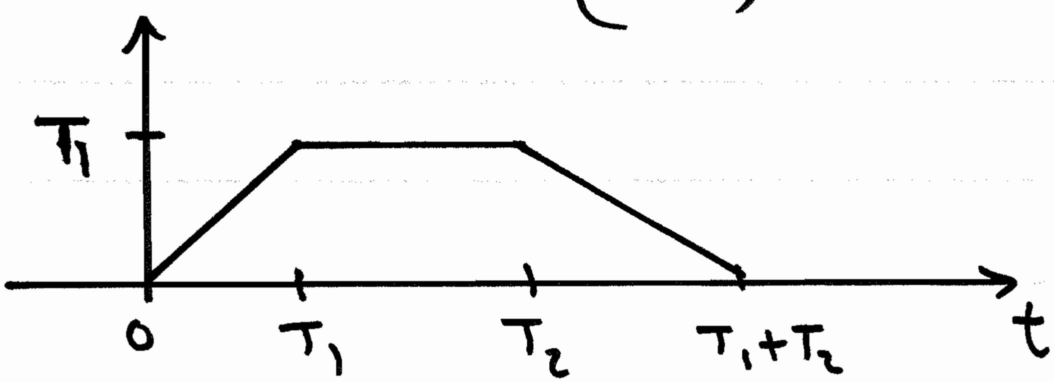
$$\text{area} = [T_2 - (t - T_1)]\,1$$

$$= T_1 + T_2 - t$$

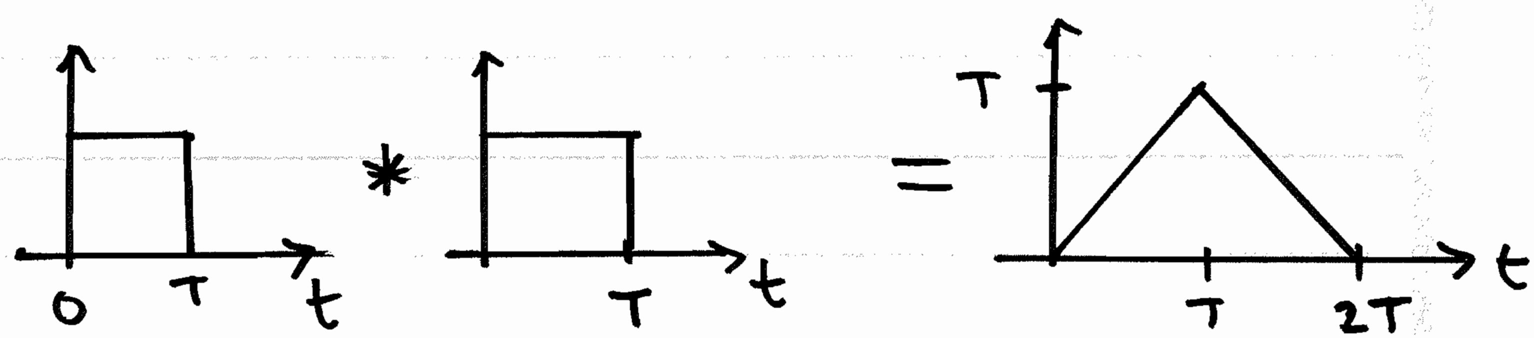
$$\boxed{\begin{array}{l} \bullet \text{ for } t - T_1 > T_2 \Rightarrow t > T_1 + T_2 \\ \quad y(t) = 0 \Rightarrow \text{no overlap} \end{array}}$$

• Summarizing:

$$y(t) = \begin{cases} 0, & t<0 \\ t, & 0<t<T_1 \\ T_1, & T_1<t<T_2 \\ T_1+T_2-t, & T_2<t<T_1+T_2 \\ 0, & t>T_1+T_2 \end{cases}$$

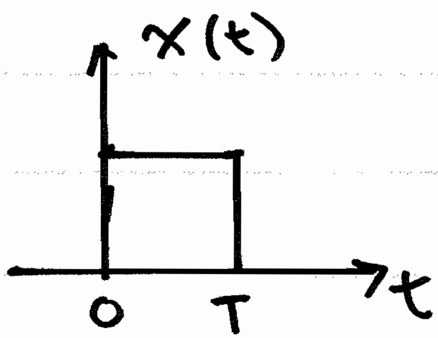


If $T_1 = T_2$:



- <u>General principle:</u> convolving $x_1(t)$ of duration T_1 with $x_2(t)$ of duration T_2 yields a result of duration $T_1 + T_2$

See Text Example 2.7 on pg. 99



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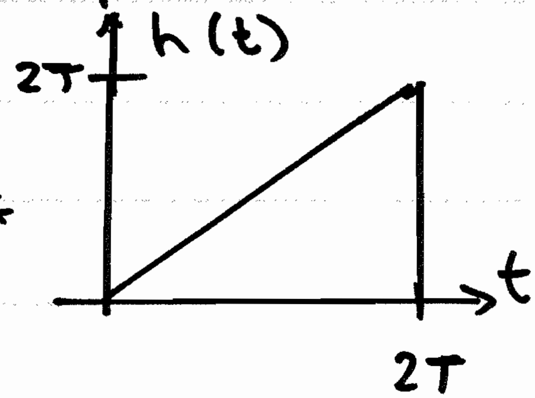

$= y(t) = ?$