Problem Statement

From The Weekly Challenge - 148 Task #2: Cardano Triplets retrieved on 2022-01-21 at 21:44:05:23-05:

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Write a script to find 5 Cardano Triplets.

A triplet of positive integers (a,b,c) is called a Cardano Triplet if it satisfies the below condition.

\[ 3 \sqrt[3]{a + b \sqrt{c}} + 3 \sqrt[3]{a - b \sqrt{c}} = 1 \]

Example (2,1,5) is the first Cardano Triplets.

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Cardano Triplets

A triplet of positive integers \((a, b, c)\) is called a Cardano triplet if it satisfies (1).

The math below is based on information in the Cardano Triplet transformation answer retrieved on 2022-01-21 at 22:00-05.

\[
\sqrt[3]{a + b \sqrt{c}} + \sqrt[3]{a - b \sqrt{c}} = 1
\]  
(1)

Subtract \(\sqrt[3]{a - b \sqrt{c}}\) from both sides—this is used in (8).

\[
\sqrt[3]{a + b \sqrt{c}} = 1 - \sqrt[3]{a - b \sqrt{c}}
\]  
(2)

Cube both sides of (1):

\[
(1 - (a - b \sqrt{c})^{1/3})^3 = 1 - \sqrt[3]{a - b \sqrt{c}} + \sqrt[3]{a - b \sqrt{c}}^2 + \sqrt[3]{a - b \sqrt{c}}
\]  
(3)

Add \(a\) to both sides:

\[
2a + \sqrt[3]{a + b \sqrt{c}} = 1 - 3\sqrt[3]{a - b \sqrt{c}} + 3\sqrt[3]{a - b \sqrt{c}}^2 + \sqrt[3]{a - b \sqrt{c}}
\]  
(4)

Subtract \(\sqrt[3]{a \sqrt{c}}\) from both sides:

\[
2a = 1 - 3\sqrt[3]{a - b \sqrt{c}} + 3\sqrt[3]{a - b \sqrt{c}}^2
\]  
(5)

Subtract 1 from both sides:

\[
2a - 1 = -3\sqrt[3]{a - b \sqrt{c}} + 3\sqrt[3]{a - b \sqrt{c}}^2
\]  
(6)

Factor out \(\sqrt[3]{a - b \sqrt{c}}\) from right hand side:

\[
2a - 1 = -3\sqrt[3]{a - b \sqrt{c}} \left(1 - 3\sqrt[3]{a - b \sqrt{c}}\right)
\]  
(7)

Use (2) to substitute for \(1 - \sqrt[3]{a - b \sqrt{c}}\) in right hand side:

\[
2a - 1 = -3\sqrt[3]{a - b \sqrt{c}} \sqrt[3]{a + b \sqrt{c}}
\]  
(8)

Cube both sides:

\[
8a^3 - 12a^2 + 6a - 1 = -27a^2 + 27b^2c
\]  
(9)

Add \(27a^2\) to both sides:

\[
8a^3 + 15a^2 + 6a - 1 = 27b^2c
\]  
(10)

Subtract \(27b^2c\) from both sides:

\[
8a^3 + 15a^2 + 6a - 1 - 27b^2c = 0
\]  
(11)

Add 1 to both sides:

\[
8a^3 + 15a^2 + 6a - 27b^2c = 1
\]  
(12)
**Parametrization of Cardano Triplet**

Parts of the below are copied from Parametrization of Cardano triplet answer retrieved on 2022-01-21 at 23:30-05.

Take (12) modulo 3:

\[ 8a^3 + 15a^2 + 6a - 27b^2c \equiv 1 \mod 3. \]  

(13)

Because 15, 6, and 27 are all multiples of 3, and \(8 \equiv 2 \mod 3\):

\[ 2a^3 \equiv 1 \mod 3. \]  

(14)

Multiply both sides by 2:

\[ 4a^3 \equiv 2 \mod 3 \implies a^3 \equiv 2 \mod 3 \]  

(15)

which has, by inspection of the three cases \((a = 0, 1, 2 \mod 3)\) the unique solution \(a \equiv 2 \mod 3\).

Thus \(a\) is of the form \(a = 2 + 3k\) for some integer \(k\).

Having this expression of \(a\), one needs only to expand the right hand side of:

\[ 27b^2c = 8a^3 + 15a^2 + 6a - 1 \]  

(16)

to get

\[ 27b^2c = 8(2 + 3k)^3 + 15(2 + 3k)^2 + 6(2 + 3k) - 1 \]  

(17)

Simplify the right hand side:

Use Wolfram Language to do the math.

Simplify \([8(2+3k)^3 + 15(2+3k)^2 + 6(2+3k) - 1]\)

\[ 27(1 + k)^2(5 + 8k). \]

\[ 27b^2c = 27(k + 1)^2(8k + 5) \]  

(18)

Divide both sides by 27:

\[ b^2c = (k + 1)^2(8k + 5) \]  

(19)

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Putting it all together

From the Problem Statement: the first Cardano triplet is (2, 1, 5).

From between equations (15) and (16): “a is of the form $a = 2 + 3k$. The first $a$ is 2, so the first $k$ is 0. The second $a$ is 5, the third 8, . . . .

Equation (19) is

$$b^2 c = (k + 1)^2(8k + 5)$$

Raku Solution

```raku
# Print first $n Cardano triplets.
my $n = 5;

# From https://engineering.purdue.edu/~mark/twc-148-2.pdf:
# From between equations (15) and (16):
# a = 2 + 3k
# From equation (19):
# b^2 c = (k+1)^2 (8k+5)

# The first element is not used.
my $square := 0, 1**2, 2**2, 3**3 .. Inf;

for (0 .. Inf) -> $k {
    # "rhs" is short for "right hand side".
    my $rhs = ($k+1)**2 * (8*$k+5);

    for (1 .. Inf) -> $b {
        # If b^2 > $rhs we've gone too far.
        ($square[$b] > $rhs) and last;

        # If $rhs is not evenly divisible by b^2 then c will not be an integer.
        ($rhs % $square[$b]) or next;

        # Found one.
        my $a = 2 + 3*$k;
        my $c = $rhs / $square[$b];
        say "($a,$b,$c)";

        (--$n) or exit 0;
    }
}
```