Cardano Triplets • Mark Senn • last updated on 2022-01-21 at 21:43-05

Problem Statement

From The Weekly Challenge - 148 Task #2: Cardano Triplets retrieved on 2022-01-21 at 21:44-05:23-05:

> **Submitted by:** Mohammad S Anwar
>
> Write a script to find 5 Cardano Triplets.
>
> > A triplet of positive integers (a,b,c) is called a Cardano Triplet if it satisfies the below condition.
> >
> > $$\sqrt[3]{a+b\sqrt{c}}+\sqrt[3]{a-b\sqrt{c}}=1$$
>
> **Example** (2,1,5) is the first Cardano Triplets.

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Cardano Triplets

A triplet of positive integers (a, b, c) is called a Cardano triplet if it satisfies (1).

The math below is based on information in the Cardano Triplet transformation answer retrieved on 2022-01-21 at 22:00-05.

$$\sqrt[3]{a+b\sqrt{c}}+\sqrt[3]{a-b\sqrt{c}}=1 \tag{1}$$

Subtract $\sqrt[3]{a-b\sqrt{c}}$ from both sides—this is used in (8).

$$\sqrt[3]{a+b\sqrt{c}}=1-\sqrt[3]{a-b\sqrt{c}} \tag{2}$$

Cube both sides of (1):

Use Wolfram Language to do the math.

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Expand[(1 - (a - b Sqrt[c])^(1/3))^3]
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$1-a-3\sqrt[3]{a-b\sqrt{c}}+3\sqrt[3]{a-b\sqrt{c}}^{\,2}+b\sqrt{c}$

$$a+b\sqrt{c}=1-a-3\sqrt[3]{a-b\sqrt{c}}+3\sqrt[3]{a-b\sqrt{c}}^{\,2}+b\sqrt{c} \tag{3}$$

Add a to both sides:

$$2a+b\sqrt{c}=1-3\sqrt[3]{a-b\sqrt{c}}+3\sqrt[3]{a-b\sqrt{c}}^{\,2}+b\sqrt{c} \tag{4}$$

Subtract $b\sqrt{c}$ from both sides:

$$2a=1-3\sqrt[3]{a-b\sqrt{c}}+3\sqrt[3]{a-b\sqrt{c}}^{\,2} \tag{5}$$

Subtract 1 from both sides:

$$2a-1=-3\sqrt[3]{a-b\sqrt{c}}+3\sqrt[3]{a-b\sqrt{c}}^{\,2} \tag{6}$$

Factor out $\sqrt[3]{a-b\sqrt{c}}$ from right hand side:

$$2a-1=-3\sqrt[3]{a-b\sqrt{c}}\left(1-\sqrt[3]{a-b\sqrt{c}}\right) \tag{7}$$

Use (2) to substitute for $1-\sqrt[3]{a-b\sqrt{c}}$ in right hand side:

$$2a-1=-3\sqrt[3]{a-b\sqrt{c}}\,\sqrt[3]{a+b\sqrt{c}} \tag{8}$$

Cube both sides:

$$8a^3-12a^2+6a-1=-27a^2+27b^2c \tag{9}$$

Add $27a^2$ to both sides:

$$8a^3+15a^2+6a-1=27b^2c \tag{10}$$

Subtract $27b^2c$ from both sides:

$$8a^3+15a^2+6a-1-27b^2c=0 \tag{11}$$

Add 1 to both sides:

$$8a^3+15a^2+6a-27b^2c=1 \tag{12}$$

Parametrization of Cardano Triplet

Parts of the below are copied from Parametrization of Cardano triplet answer retrieved on 2022-01-21 at 23:30-05.

Take (12) modulo 3:

$$8a^3 + 15a^2 + 6a - 27b^2c \equiv 1 \mod 3. \tag{13}$$

Because 15, 6, and 27 are all multiples of 3, and $8 \equiv 2 \mod 3$:

$$2a^3 \equiv 1 \mod 3. \tag{14}$$

Multiply both sides by 2:

$$4a^3 \equiv 2 \mod 3 \implies a^3 \equiv 2 \mod 3 \tag{15}$$

which has, by inspection of the three cases ($a = 0,\ 1,\ 2 \mod 3$) the unique solution $a \equiv 2 \mod 3$.

Thus a is of the form $a = 2 + 3k$ for some integer k.

Having this expression of a, one needs only to expand the right hand side of:

$$27b^2c = 8a^3 + 15a^2 + 6a - 1 \tag{16}$$

to get

$$27b^2c = 8(2+3k)^3 + 15(2+3k)^2 + 6(2+3k) - 1 \tag{17}$$

Simplify the right hand side:

Use Wolfram Language to do the math.

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Simplify[8(2+3k)^3 + 15(2+3k)^2 + 6(2+3k) - 1]
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$27(1+k)^2(5+8k)$.

$$27b^2c = 27(k+1)^2(8k+5) \tag{18}$$

Divide both sides by 27:

$$b^2c = (k+1)^2(8k+5) \tag{19}$$

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Putting it all together

From the Problem Statement: the first Cardano triplet is $(2,1,5)$.

From between equations (15) and (16): "a is of the form $a = 2+3k$". The first a is 2, so the first k is 0. The second a is 5, the third 8, … .

Equation (19) is

$$b^2c = (k+1)^2(8k+5)$$

Raku Solution

```
# Print first $n Candano triplets.
my $n = 5;

# From https://engineering.purdue.edu/~mark/twc-148-2.pdf:
#     From between equations (15) and (16):
#         a = 2 + 3k
#     From equation (19):
#         b^2 c = (k+1)^2 (8k+5)

# The first element is not used.
my $square := 0, 1**2, 2**2, 3**3 .. Inf;

for (0 .. Inf) -> $k {

    # "rhs" is short for "right hand side".
    my $rhs = ($k+1)**2 * (8*$k+5);

    for (1 .. Inf) -> $b {

        # If b^2 > $rhs we've gone too far.
        ($square[$b] > $rhs)  and  last;

        # If $rhs is not evenly divisible by b^2 then c will not be an integer.
        ($rhs %% $square[$b])  or   next;

        # Found one.
        my $a = 2 + 3*$k;
        my $c = $rhs / $square[$b];
        say "($a,$b,$c)";

        (--$n)  or  exit 0;
    }
}
```