CHAPTER 2
DIFFRACTED PROJECTIONS

2.1 Introduction

Tomography with diffracting energy can not be modeled with the same equations used to model projections in conventional, straight ray, tomography. Acoustic and electromagnetic waves do not travel along straight rays and the projections are not line integrals. Instead the flow of energy will be described with the wave equation and in the limit of very short wavelengths or objects where the effects of refraction are small it will be shown that the diffracted projections can be approximated by a non diffracting projection

First consider the propagation of waves in homogeneous media. The wave equation is a second order linear differential equation and under certain conditions it can be shown that an expression for the field at every other point in space can be written.

The problem is not to image a homogeneous media but one that is inhomogeneous. To solve the inhomogeneous wave equation, one of two approximations, the Born or the Rytov, must be used. With these two approximations expressions for the field scattered by the inhomogeneities of the media can be written.

The theory to be discussed will be applicable to both two and three dimensional structures. Even in a three dimensional world a two dimensional model can often be used if the object varies slowly in one direction. This assumption, for example, is often made in conventional computerized tomography where images are made of a single slice of the object. The theory of diffraction tomography will be presented almost entirely in two dimensions for two reasons. More importantly, the ideas behind the theory are often easier to visualize (and certainly to draw) in two dimensions. In addition technology has yet to make it practical to implement large three dimensional transforms and then to display the results. This limitation will certainly be eliminated in the near future and where the differences are significant both the two and three dimensional solutions will be indicated.

2.2 Homogeneous Wave Equation

In a constant or homogeneous media the propagation of acoustic or electromagnetic waves can be modeled with the scalar Helmholtz equation. For a temporal frequency of ω radians per second (rps) a field, $u(\vec{r})$, satisfies the equation

$$(\nabla^2+k_0^2)u(\vec{r}) = 0. \tag{2.1}$$

For homogeneous media the wavenumber, k_0, is a constant related to the wavelength, λ, of the wave by

$$k_0 = \frac{2\pi}{\lambda}. \tag{2.2}$$

The wavelength, λ, is related to the temporal frequency of the wave by the propagation speed in the media, c, or

$$\lambda = \frac{2\pi}{\omega}c \tag{2.3}$$

Since the theory of diffraction tomography is normally derived based on coherent fields the time dependence of most fields will be suppressed in this work. Thus all fields should be multiplied by $e^{-j\omega t}$ to find the measured field as a function of time. The extension of this theory to broadband fields is discussed in Section 3.4.3

For acoustic (or ultrasonic) tomography, $u(\vec{r})$ can be the pressure field at position $\vec{r}$. For the electromagnetic case, assuming the applicability of a scalar propagation equation, $u(\vec{r})$ may be set equal to the complex amplitude of the electric field along its polarization. In both cases the time dependence of the fields are suppressed and $u(\vec{r})$ represents the complex amplitude of the field. As a function of time and space the field is given by

$$u(\vec{r},t) = \text{Real Part}\left\{u(\vec{r})e^{-j\omega t}\right\}. \tag{2.4}$$

The vector gradient operator, ∇, can be expanded into its two dimensional representation and the wave equation becomes

$$\frac{\partial^2 u(\vec{r})}{\partial x^2} + \frac{\partial^2 u(\vec{r})}{\partial y^2} + k_0^2 u(\vec{r}) = 0. \tag{2.5}$$

As a trial solution let

$$u(\vec{r}) = e^{j\vec{k}\cdot\vec{r}} \tag{2.6}$$

where the vector $\vec{k} = (k_x,k_y)$ is the two dimensional propagation vector and

$u(\vec{r})$ represents a two dimensional plane wave of spatial frequency $|\vec{k}|$. This form of $u(\vec{r})$ represents the basis function for the two dimensional Fourier transform; using it, any two dimensional function can be represented as as a weighted sum of plane waves. Calculating the derivatives as indicated in equation (2.5) it can be seen that all plane waves that satisfy the condition

$$|\vec{k}|^2 = k_x^2 + k_y^2 = k_0^2 \tag{2.7}$$

are valid solutions to the wave equation. This condition is consistent with an intuitive picture of a wave and description of the wave equation above, since for any frequency wave only a single wavelength can exist no matter which direction it propagates.

The homogeneous wave equation is a linear differential equation so the general solution can be written as a weighted sum of each possible plane wave solution. In two dimensions, at a temporal frequency of ω, the field, $u(\vec{r})$ is given by

$$u(\vec{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(k_y) e^{j(k_x x + k_y y)} dk_y + \frac{1}{2\pi}\int_{-\infty}^{\infty} \beta(k_y) e^{j(-k_x x + k_y y)} dk_y \tag{2.8}$$

and by equation (2.7)

$$k_x = \sqrt{k_0^2 - k_y^2}. \tag{2.9}$$

The form of this equation might be surprising to the reader for two reasons. First, the integral has been split into two parts. The coefficients of waves traveling to the right are represented by $\alpha(k_y)$ and those traveling to the left by $\beta(k_y)$. In addition the limits of the integrals have been set to go from $-\infty$ to ∞. For k_y^2 greater than k_0^2 the radical in equation (2.9) becomes imaginary and the plane wave becomes an evanescent wave. These are valid solutions to the wave equation but because k_y is imaginary the exponential has a real or attenuating component. This real component causes the amplitude of the wave to either grow or decay exponentially. In practice, these evanescent waves only occur to satisfy boundary conditions, always decay rapidly far from the boundary, and can often be ignored at distance greater than 10λ from the inhomogeneity.

The limited range of valid solutions to the wave equation allows (under certain condition) an expression to be written for the field in all of two-space given the amplitude of the field along a line. The three dimensional version of this idea gives the field in three-space if the field is known at all points on a plane.

Consider a source of plane waves to the left of a vertical line as shown in Figure 2.1. By calculating the one-dimensional Fourier transform of the field along the line the field can be decomposed into a number of one-dimensional components. Each of these one dimensional components can then be attributed to one of the valid plane wave solutions to the homogeneous wave equation because for any one frequency component, k_y, there can exist only two plane waves that satisfy the wave equation. Since the incident field has already been constrained to propagate toward the right (all sources are to the left of the measurement line) then a one-dimensional Fourier component at a frequency of k_y can be attributed to a two dimensional wave with a propagation vector of $(\sqrt{k_0^2-k_y^2}, k_y)$.

This can be put on a more mathematical basis if the one-dimensional Fourier transform of the field is compared to the general form of the wave equation. If waves that are traveling to the left are ignored then the general solution to the wave equation becomes

$$u(\vec{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(k_y)e^{j(k_x x + k_y y)}dk_y \tag{2.10}$$

Now if the coordinate system is moved so that the measurement line is at $x = 0$ then the expression for the field becomes equal to the one-dimensional Fourier transform of the field or

$$u(\vec{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(k_y)e^{jk_y y}dk_y. \tag{2.11}$$

This equation establishes the link between the one-dimensional Fourier transform of the field along the line to the two-dimensional field. The coefficients $\alpha(k_y)$ are given from the one dimensional Fourier transform of the field by

$$\alpha(k_y) = \text{Fourier Transform}\left\{u(0,y)\right\}. \tag{2.12}$$

The simple form of a plane wave allows an expression to be written relating the field on two parallel lines. If *a priori* it is known that all the sources for the field are positioned, for example, left of the line at $x=l_0$ then the field $u(x=l_0,y)$ can be decomposed into its plane wave components. Given a plane wave $u_{plane\ wave}(x=l_0,y) = \alpha e^{j(k_x l_0+k_y y)}$ the field undergoes a phase shift as it propagates to the line $x=l_1$, and the field can be written

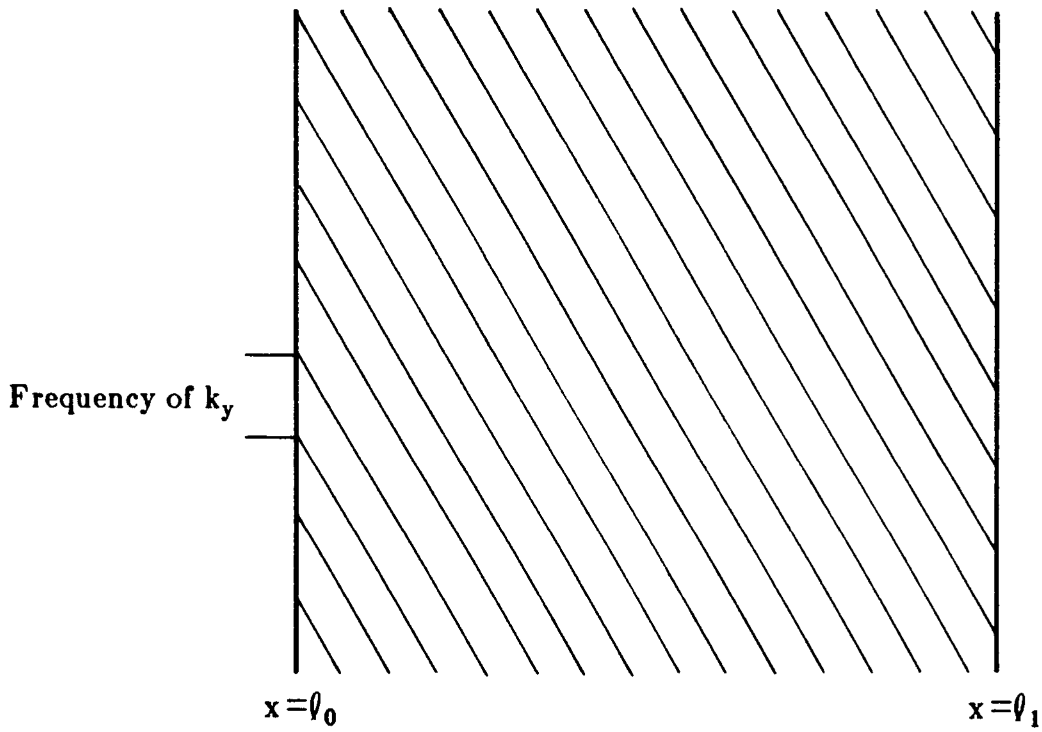

Figure 2.1 A plane wave with direction cosines $(\sqrt{k_0^2-k_y^2},k_y)$ is shown propagating between the lines $x=l_0$ and $x=l_1$.

$$u_{\text{plane wave}}(x=l_1,y) = \alpha e^{j(k_x l_0+k_y y)}e^{jk_x(l_1-l_0)} = u_{\text{plane wave}}(x=l_0,y)e^{jk_x(l_1-l_0)} \quad (2.13)$$

Thus the complex amplitude of the plane wave at $x=l_1$ is related to its complex amplitude at $x=l_0$ by a factor of $e^{jk_x(l_1-l_0)}$.

The complete process of finding the field at a line $x=l_1$ follows in three steps.

1) Take the Fourier transform of $u(x=l_0,y)$ to find the Fourier decomposition of u as a function of k_y.
2) Propagate each plane wave to the line $x=l_1$ by multiplying its complex amplitude by the phase factor $e^{jk_x(l_1-l_0)}$, where as before $k_x = \sqrt{k_0^2-k_y^2}$.
3) Find the inverse Fourier transform of the plane wave decomposition to find the field at $u(x=l_1,y)$.

2.3 Inhomogeneous Wave Equation

For imaging in an inhomogeneous media a more general form of the wave equation is written as

$$\left[\nabla^2+k(\vec{r})^2\right]u(\vec{r}) = 0. \quad (2.14)$$

For the electromagnetic case it is necessary to ignore the effects of polarization so that $k(\vec{r})$ is a scalar function representing the refractive index of the medium. Now write

$$k(\vec{r}) = k_0 n(\vec{r}) = k_0[1+n_\delta(\vec{r})] \quad (2.15)$$

where k_0 represents the average wavenumber of the media and $n_\delta(\vec{r})$ represents the refractive index deviations. In general it will be assumed that the object has a finite size and therefore $n_\delta(\vec{r})$ is zero outside the object. Rewriting the wave equation

$$(\nabla^2+k_0^2)u(\vec{r}) = -k_0^2[n(\vec{r})^2-1]u(\vec{r}) \quad (2.16)$$

where $n(\vec{r})$ is the electromagnetic refractive index of the media and is given by

$$n(\vec{r}) = \sqrt{\frac{\mu(\vec{r})\epsilon(\vec{r})}{\mu_0\epsilon_0}}. \quad (2.17)$$

Here μ and ϵ have been used to represent the magnetic permeability and dielectric constant and the subscript zero to indicate their average values. This new term, on the right hand side of equation (2.16), is known as a forcing function for the differential equation $(\nabla^2+k_0^2)u(\vec{r})$.

Note that equation (2.16) is a *scalar* wave propagation equation. Its use implies that there is no depolarization as the electromagnetic wave propagates through the medium. It is known [Ish78] that the depolarization effects can be ignored only if the wavelength is much smaller than the correlation size of the inhomogeneities in the object. If this condition is not satisfied, then strictly speaking the following vector wave propagation equation must be used

$$\nabla^2\vec{E}(\vec{r})+k_0^2n^2\vec{E}(\vec{r})-2\nabla\left[\frac{\nabla n}{n}\cdot\vec{E}\right]=0 \tag{2.18}$$

where $\vec{E}$ is the electric field vector. A vector theory for diffraction tomography based on this equation has yet to be developed.

For the acoustic case, first order approximations give the following wave equation [Kak84]

$$(\nabla^2+k_0^2)u(\vec{r}) = -k_0^2[n^2(\vec{r})-1]u(\vec{r}) \tag{2.19}$$

where n is the *complex refractive index* at position $\vec{r}$, and is equal to

$$n(\vec{r}) = \frac{c_0}{c(\vec{r})}, \tag{2.20}$$

where c_0 is the propagation velocity in the medium in which the object is immersed, and $c(\vec{r})$ is the propagation velocity at location $\vec{r}$ in the object. For the acoustic case where only the compressional waves in a viscous compressible fluid are involved, the speed of sound is given by

$$c(\vec{r}) = \frac{1}{\sqrt{\rho\kappa}} \tag{2.21}$$

where ρ and κ are the *local density* and the *complex compressibility* at location $\vec{r}$.

The forcing function in equation (2.19) is only valid provided the first and higher order derivatives of the medium parameters can be ignored. If the inhomogeneity can be modeled as a viscous compressible fluid, an exact form for the wave equation is given by

$$(\nabla^2+k_0^2)u(\vec{r}) = {k_0}^2\gamma_\kappa u - \nabla\cdot(\gamma_\rho\nabla u) \tag{2.22}$$

where

$$\gamma_\kappa = \frac{\kappa-\kappa_0}{\kappa_0} \tag{2.23}$$

$$\gamma_\rho = \frac{\rho - \rho_0}{\rho}. \tag{2.24}$$

κ_0 and ρ_0 are either the compressibility and density of the medium in which the object is immersed, or the average compressibility and the density of the object, depending upon how the process of imaging is modeled. On the other hand, if the object is a solid and can be modeled as a linear isotropic viscoelastic medium, the forcing function possesses another more complicated form. Since this form involves tensor notation, it will not be presented here and the interested reader is referred to [Iwa75].

Due to the similarities of the electromagnetic and acoustic wave equations a general form of the wave equation can be written as

$$(\nabla^2 + k_0^2)u(\vec{r}) = -o(\vec{r})u(\vec{r}) \tag{2.25}$$

where

$$o(\vec{r}) = k_0^2[n^2(\vec{r}) - 1] \tag{2.26}$$

To hide some of the mathematical details the term $o(\vec{r})$ will be used to represent all inhomogeneities of the object. Later the object will be reconstructed in terms of the object function, $o(\vec{r})$, and the reader is referred to equation (2.26) to put the reconstruction in terms of the refractive index.

Consider the field, $u(\vec{r})$, to be the sum of two components. The incident field, $u_0(\vec{r})$, is the field present without any inhomogeneities or a solution to the equation

$$(\nabla^2 + k_0^2)u_0(\vec{r}) = 0. \tag{2.27}$$

That leaves the scattered field, $u_s(\vec{r})$, as that part of the field due to the object inhomogeneities or

$$u_s(\vec{r}) = u(\vec{r}) - u_0(\vec{r}). \tag{2.28}$$

The wave equation becomes

$$(\nabla^2 + k_0^2)u_s(\vec{r}) = -u(\vec{r})o(\vec{r}). \tag{2.29}$$

The scalar Helmholtz equation (2.29) cannot be solved for $u_s(\vec{r})$ directly but a solution can be written in terms of the Green's function [Mor53]. The Green's function, which is a solution of the differential equation

$$(\nabla^2 + k_0^2)g(\vec{r}|\vec{r}\,') = -\delta(\vec{r} - \vec{r}\,'), \tag{2.30}$$

is written in three-space as

$$g(\vec{r}|\vec{r}') = \frac{e^{jk_0R}}{4\pi R} \tag{2.31}$$

with

$$R = |\vec{r}-\vec{r}'|. \tag{2.32}$$

In two dimensions the solution of (2.30) is written in terms of a zero-order Hankel function of the first kind, and can be expressed as

$$g(\vec{r}|\vec{r}') = \frac{j}{4}H_0^{(1)}(k_0R). \tag{2.33}$$

In both cases, the Green's function, $g(\vec{r}|\vec{r}')$, is only a function of the difference $\vec{r}-\vec{r}'$ so the function will often be represented as simply $g(\vec{r}-\vec{r}')$. Because the object function in equation (2.30) represents a point inhomogeneity, the Green's function can be considered to represent the field resulting from a single point scatterer.

It is possible to represent the forcing function of the wave equation as an array of impulses or

$$o(\vec{r})u(\vec{r}) = \int o(\vec{r}')u(\vec{r}')\delta(\vec{r}-\vec{r}')d\vec{r}'. \tag{2.34}$$

In this equation the forcing function of the inhomogeneous wave equation is represented as as a summation of impulses weighted by $o(\vec{r})u(\vec{r})$ and shifted by $\vec{r}$. The Green's function represents the solution of the wave equation for a single delta function; because the left hand side of the wave equation is linear, a solution can be written by summing the scattered field due to each individual point scatterer.

Using this idea, the total field due to the impulse $o(\vec{r}')u(\vec{r}')\delta(\vec{r}-\vec{r}')$ is written as a summation of scaled and shifted versions of the impulse response, $g(\vec{r})$. This is a simple convolution and the total radiation from all sources on the right hand side of (2.29) must be given by the following superposition:

$$u_s(\vec{r}) = \int g(\vec{r}-\vec{r}')o(\vec{r}')u(\vec{r}')d\vec{r}'. \tag{2.35}$$

At first glance it might appear that this is the solution needed for the scattered field, but it is not that simple. An integral equation for the scattered field, u_s, has been written in terms of the total field, $u = u_0 + u_s$. This equation needs to be solved for the scattered field and two approximations that allow this to be done will now be discussed.

2.4 Approximations to the Wave Equation

In the last section an inhomogeneous integral equation was derived to represent the scattered field, $u_s(\vec{r})$, as a function of the object, $o(\vec{r})$. This equation cannot be solved directly, but a solution can be written using either of the two approximations described here. These approximations, the Born and the Rytov, are valid under different conditions but the form of the resulting solutions are quite similar. These approximations are the basis of the Fourier Diffraction Theorem.

Mathematically speaking equation (2.35) is a Fredholm equation of the second kind. A number of mathematicians have presented works describing the solution of scattering integrals [Hoc73, Col83] and they should be consulted for the theory behind the approximations to be presented here.

2.4.1 The First Born Approximation

The first Born approximation is the simpler of the two approaches. Recall that the total field, $u(\vec{r})$, is expressed as the sum of the incident field, $u_0(\vec{r})$, and a small perturbation, $u_s(\vec{r})$, or

$$u(\vec{r}) = u_0(\vec{r}) + u_s(\vec{r}). \tag{2.36}$$

The integral of equation (2.35) is now written as

$$u_s(\vec{r}) = \int g(\vec{r}-\vec{r}\,')o(\vec{r}\,')u_0(\vec{r}\,')d\vec{r}\,' + \int g(\vec{r}-\vec{r}\,')o(\vec{r}\,')u_s(\vec{r}\,')d\vec{r}\,' \tag{2.37}$$

but if the scattered field, $u_s(\vec{r})$, is small compared to $u_0(\vec{r})$ the effects of the second integral can be ignored to arrive at the approximation

$$u_s(\vec{r}) \simeq u_B(\vec{r}) = \int g(\vec{r}-\vec{r}\,')o(\vec{r}\,')u_0(\vec{r}\,')d\vec{r}\,'. \tag{2.38}$$

The first Born approximation is valid only when the magnitude of the scattered field,

$$u_s(\vec{r}) = u(\vec{r}) - u_0(\vec{r}), \tag{2.39}$$

is smaller than the magnitude of the incident field, u_0. If the object is a homogeneous cylinder it is possible to express this condition as a function of the size of the object and the refractive index. Let the incident wave, $u_0(\vec{r})$, be a plane wave propagating in the direction of the vector, $\vec{k}_0$. For a large object, the field inside the object will not be well approximated by the incident field

$$u(\vec{r}) = u_{object}(\vec{r}) \neq Ae^{j\vec{k}_0\cdot\vec{r}} \tag{2.40}$$

but instead will be a function of the change in refractive index, n_δ. Along a ray through the center of the cylinder and parallel to the direction of propagation of the incident plane wave the field inside the object becomes a slow (or fast) version of the incident wave or

$$u_{object}(\vec{r}) = Ae^{j(1+n_\delta)\vec{k}_0 \cdot \vec{r}}. \tag{2.41}$$

Since the wave is propagating through the object the phase difference between the incident field and the field inside the object is approximately equal to the integral through the object of the change in refractive index. For a homogeneous cylinder of radius 'a' wavelengths the total phase shift through the object becomes

$$\text{Phase Change} = 4\pi n_\delta \frac{a}{\lambda} \tag{2.42}$$

where λ is the wavelength of the incident wave. For the Born approximation to be valid, a necessary condition is that the change in phase between the incident field and the wave propagating through the object be less than π. This condition can be expressed mathematically as [New66]

$$a n_\delta < \frac{\lambda}{4}. \tag{2.43}$$

2.4.2 The First Rytov Approximation

The Rytov approximation is another approximation to the scattered field and is valid under slightly different restrictions. It is derived by considering the total field to be represented as a complex phase or [Ish78]

$$u(\vec{r}) = e^{\phi(\vec{r})} \tag{2.44}$$

and rewriting the wave equation (2.14)

$$(\nabla^2 + k^2)u = 0 \tag{2.14}$$

as

$$\nabla^2 e^{\phi} + k^2 e^{\phi} = 0 \tag{2.45}$$

$$\nabla[\nabla\phi e^{\phi}] + k^2 e^{\phi} = 0 \tag{2.46}$$

$$\nabla^2\phi e^{\phi} + (\nabla\phi)^2 e^{\phi} + k^2 e^{\phi} = 0 \tag{2.47}$$

and finally

$$(\nabla\phi)^2 + \nabla^2\phi + k_0^2 = -o(\vec{r}). \tag{2.48}$$

(Although all the fields (ψ and ϕ) are a function of $\vec{r}$, to simplify the notation the argument of these functions will be dropped.) Expressing the total complex phase, ϕ, can be expressed as the sum of the incident phase function, ϕ_0, and the scattered complex phase, ϕ_s, or

$$\phi(\vec{r}) = \phi_0(\vec{r})+\phi_s(\vec{r}) \tag{2.49}$$

where

$$u_0(\vec{r}) = e^{\phi_0(\vec{r})} \tag{2.50}$$

to find that

$$(\nabla\phi_0)^2+2\nabla\phi_0\cdot\nabla\phi_s+(\nabla\phi_s)^2+\nabla^2\phi_0+\nabla^2\phi_s+k_0^2+o(\vec{r}) = 0. \tag{2.51}$$

As in the Born approximation it is possible to set the zero perturbation equation equal to zero. Doing this, the homogeneous wave equation can be written

$$k_0^2+(\nabla\phi_0)^2+\nabla^2\phi_0 = 0. \tag{2.52}$$

Substituting this into equation (2.51) the wave equation becomes

$$2\nabla\phi_0\cdot\nabla\phi_s+\nabla^2\phi_s = -(\nabla\phi_s)^2-o(\vec{r}). \tag{2.53}$$

This equation is still inhomogeneous but can be linearized by considering the following relation:

$$\nabla^2(u_0\phi_s) = \nabla(\nabla u_0\cdot\phi_s+u_0\nabla\phi_s) \tag{2.54}$$

or by expanding the first derivative on the right hand side of this equation

$$\nabla^2(u_0\phi_s) = \nabla^2u_0\cdot\phi_s+2\nabla u_0\cdot\nabla\phi_s+u_0\nabla^2\phi_s \tag{2.55}$$

Using a plane wave for the incident field,

$$u_0 = Ae^{j\vec{k}_0\cdot\vec{r}}, \tag{2.56}$$

the second gradient of the incident field is

$$\nabla^2u_0 = -k_0^2u_0 \tag{2.57}$$

so that equation (2.55) may be rewritten as

$$2u_0\nabla\phi_0\cdot\nabla\phi_s+u_0\nabla^2\phi_s = \nabla^2(u_0\phi_s)+k_0^2u_0\phi_s. \tag{2.58}$$

This result can be substituted into equation (2.53) to find

$$(\nabla^2+k_0^2)u_0\phi_s = -u_0\left[(\nabla\phi_s)^2+o(\vec{r})\right] \tag{2.59}$$

The solution to this differential equation can again be expressed as an integral equation. This becomes

$$u_0\phi_s = \int_{V'} g(\vec{r}-\vec{r}\,')u_0\Big[(\nabla\phi_s)^2 + o(\vec{r})\Big]dr'. \tag{2.60}$$

Using the Rytov Approximation it is necessary to assume that the term in brackets in the above equation can be approximated by

$$(\nabla\phi_s)^2 - o(\vec{r}) \simeq -o(\vec{r}). \tag{2.61}$$

When this is done, the first order Rytov approximation to the function $u_0\phi_s$ becomes

$$u_0\phi_s = \int_{V'} g(\vec{r}-\vec{r}\,')u_0(\vec{r}\,')o(\vec{r}\,')dr'. \tag{2.62}$$

Thus ϕ_s, the complex phase of the scattered field, is given by

$$\phi_s(\vec{r}) = \frac{1}{u_0(\vec{r})}\int_{V'} g(\vec{r}-\vec{r}\,')u_0(\vec{r}\,')o(\vec{r}\,')dr'. \tag{2.63}$$

Substituting the expression for u_B given in equation (2.38) the first Rytov approximation can be written

$$\phi_s(\vec{r}) = \frac{u_B(\vec{r})}{u_0(\vec{r})}. \tag{2.64}$$

The Rytov approximation is valid under a less restrictive set of conditions than the Born approximation [Che60, Kel69]. In deriving the Rytov approximation it was necessary to assume that

$$\int_{V'} g(\vec{r}-\vec{r}\,')u_0(\vec{r}\,')o(\vec{r}\,')dr' >> \int_{V'} g(\vec{r}-\vec{r}\,')u_0(\vec{r}\,')(\nabla\phi_s)^2 dr'. \tag{2.65}$$

If the object is smaller then a wavelength then both the field and the object can be assumed to be constant compared to the object function and the above relation can be written

$$g(\vec{r}-0)u_0(0)\int_{V'} o(\vec{r}\,')dr' >> g(\vec{r}-0)u_0(0)\int_{V'}(\nabla\phi_s)^2 dr'. \tag{2.66}$$

When the term $(\nabla\phi_s)^2$ is small outside the object this relation can be further simplified to find

$$o(\vec{r}) >> (\nabla\phi_s)^2. \tag{2.67}$$

If $o(\vec{r})$ is written in terms of the change in refractive index

$$o(\vec{r}) = k_0^2[n^2(\vec{r})-1] = k_0^2[(1+n_\delta(\vec{r}))^2-1] \tag{2.26}$$

and the square of the refractive index is expanded to find

$$o(\vec{r}) = k_0^2[(1+2n_\delta(\vec{r})+n_\delta^2(\vec{r}))-1] \tag{2.68}$$

$$o(\vec{r}) = k_0^2[2n_\delta(\vec{r})+n_\delta^2(\vec{r})]. \tag{2.69}$$

To a first approximation the object function is linearly related to the refractive index or

$$o(\vec{r}) \simeq 2k_0^2 n_\delta(\vec{r}). \tag{2.70}$$

The condition needed for the Rytov approximation (see equation (2.67) can be rewritten as

$$n_\delta >> \frac{(\nabla\phi_s)^2}{k_0^2}. \tag{2.71}$$

This can be justified by observing that to a first approximation the scattered phase, ϕ_s, is linearly dependent on the refractive index change, n_δ, and therefore the first term in equation (2.65) above can be safely ignored for small n_δ.

The term $\nabla\phi_s$ is the change in the complex scattered phase per unit distance and by dividing by the wavenumber

$$k_0 = \frac{2\pi}{\lambda} \tag{2.72}$$

a necessary condition for the validity of the Rytov approximation is

$$n_\delta >> \left[\frac{\nabla\phi_s\lambda}{2\pi}\right]^2. \tag{2.73}$$

Unlike the Born approximation, it is the change in scattered phase, ϕ_s, over one wavelength that is important and not the total phase. Thus, because of the ∇ operator, the Rytov approximation is valid when the phase change over a single wavelength is small.

Estimating $u_B(\vec{r})$ for the Rytov case is slightly more difficult. In an experiment the total field, $u(\vec{r})$, is measured. An expression for $u_B(\vec{r})$ is found by recalling the expression for the Rytov solution to the total wave

$$u(\vec{r}) = u_0+u_s(\vec{r}) = e^{\phi_0+\phi_s} \tag{2.74}$$

and then rearranging the exponentials to find

$$u_s = e^{\phi_0+\phi_s} - e^{\phi_0} \tag{2.75}$$

$$u_s = e^{\phi_0}\left[e^{\phi_s} - 1\right] \tag{2.76}$$

$$u_s = u_0\left[e^{\phi_s} - 1\right]. \tag{2.77}$$

Inverting this to find an estimate for the scattered phase, ϕ_s, the scattered phase is

$$\phi_s(\vec{r}) = \ln\left[\frac{u_s}{u_0} + 1\right]. \tag{2.78}$$

Then expand ϕ_s in terms of equation (2.64) to obtain the following estimate for the Rytov estimate of $u_B(\vec{r})$

$$u_B(\vec{r}) = u_0(\vec{r})\ln\left[\frac{u_s}{u_0} + 1\right]. \tag{2.79}$$

Since the natural logarithm is a multiple valued function, one must be careful at each position to choose the correct value. For continuous functions this is not difficult because only one value will satisfy the continuity requirement. On the other hand for discrete (or sampled) signals the choice is not nearly as simple and one must resort to a phase wrapping algorithm to choose the proper phase. Phase unwrapping has been described in a number of works [Tri77, OCo78, Kav84, McG82, Kav84]. Due to the "+1" factor inside the logarithmic term, this is only a problem if u_s is on the order of or larger than u_0. Thus both the Born and the Rytov techniques can be used to estimate $u_B(\vec{r})$.

While the Rytov approximation is valid over a larger class of objects, it is possible to show that the Born and the Rytov approximations produce the same result for objects that are small and deviate only slightly from the average refractive index of the medium. Consider first the Rytov expression to the total field. This is given by

$$u(\vec{r}) = e^{\phi_0+\phi_s}. \tag{2.80}$$

Substituting an expression for the scattered phase, (2.64) and the incident field, (2.56) into this expression

$$u(\vec{r}) = e^{-jk_0\vec{s}\cdot\vec{r} + \exp(-jk_0\vec{s}\cdot\vec{r})u_B(\vec{r})} \tag{2.81}$$

or

$$u(\vec{r}) = u_0(\vec{r}) e^{\exp(jk_0\vec{s}\cdot\vec{r}) u_B(\vec{r})}. \qquad (2.82)$$

For small u_B, the first exponential can be expanded in terms of its power series. Throwing out all but the first two terms the total field is approximately equal to

$$u(\vec{r}) = u_0(\vec{r}) \left[1 + e^{-jk_0\vec{s}\cdot\vec{r}} u_B(\vec{r}) \right] \qquad (2.83)$$

or

$$u(\vec{r}) = u_0(\vec{r}) + u_B(\vec{r}). \qquad (2.84)$$

Thus when the magnitude of the scattered field is small the Rytov solution is approximately equal to the Born solution given in equation (2.38).

The similarity between the expressions for the first order Born and Rytov solutions will form the basis of the reconstruction algorithms to be derived here. In the Born approximation the complex amplitude of the scattered field is measured and this is used as an estimate of the function u_B while in the Rytov case u_B is estimated from the complex phase of the scattered field. Since the Rytov approximation is considered more accurate than the Born approximation it should provide a better estimate of u_B. In Chapter 5 of this work, after deriving reconstruction algorithms based on the Fourier Diffraction Theorem, simulations comparing the Born and the Rytov approximations will be discussed.

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