The goal of this work is to investigate methods for simulating turbulent supersonic flows that include shock waves. Several different types of previously proposed characteristic filters, including total variation diminishing (TVD), monotone upstream-centered scheme for conservation laws (MUSCL), and weighted essentially non-oscillatory (WENO) filters, are investigated in this study. Similar to the traditional shock capturing schemes, these filters can not only eliminate the numerical instability caused by large gradients in flow fields, but also improve efficiency compared to classical shock-capturing schemes. Adding the nonlinear dissipation part of a classical shock-capturing scheme to a central scheme makes the method suitable for incorporation into any existing central-based high-order subsonic code. The amount of numerical dissipation to add is sensed by means of the artificial compression method (ACM) switch. In order to improve the performance of the characteristic filters, shock sensors are also employed and investigated. Through several numerical experiments (including a shock/density wave interaction, a shock/vortex interaction, and a shock/mixing layer interaction) we show that characteristic filters work well, and can be used for future turbulent flow simulations that include shocks.

Nomenclature

\(a_{i+1/2}^l\) Eigenvalue of the flux Jacobian
\(a_{i}\) Spatial filter coefficients
\(c_1, c_2\) Upper and lower stream speed of sounds
\(e\) Total energy
\(F, G\) Inviscid flux vectors in the Navier-Stokes equations
\(F_v, G_v\) Viscous flux vectors in the Navier-Stokes equations
\(M_c\) Convective Mach number
\(P\) Pressure
\(q_i\) Conductive heat flux vector
\(R\) Right eigenvector matrix of the flux Jacobian
\(r\) Radius from a vortex center
\(r_c\) Vortex core radius
\(S\) Entropy
\(T\) Temperature
\(t\) Nondimensional time
\(U\) Conservative vector in the Navier-Stokes equations
\(V_\theta\) Tangential velocity
\(u, v\) Velocities in \(x\) and \(y\) direction
\(u_1, u_2\) Upper and lower stream velocities
\(x, y\) Cartesian coordinates
\(\alpha_f\) Spatial filter parameter
\(\alpha_{i+1/2}^l\) Elements of characteristic vector in the Navier-Stokes equations

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I. Introduction

Numerical simulation of high speed turbulent flows can be applied in many fields, such as helicopter/propeller/fan blade tips operating at supersonic speeds, mixing enhancement in a supersonic combustion ramjet engine, and supersonic jet noise, which is the application we are interested in. The common features of these flow fields are vortices and shocks. A high-order central scheme is required to preserve the vortical flow structures; however, it does not have shock capturing ability. At the same time, traditional shock capturing schemes are usually too dissipative and not suitable for turbulence simulations. In order to capture the shocks while preserving the vortical flow structures, this study investigates numerical methods for extending subsonic turbulence calculations, such as large eddy simulations (LES), to supersonic flows that include shocks. This serves as our first step toward supersonic jet simulations.

Aviation plays a critical role in the development and expansion of the US economy. However, noise levels associated with operations of current commercial aircraft in communities surrounding airports increase the costs of operating these facilities and stifles expansion. Even when airplanes are cruising far from populated areas jet engine noise is still a serious issue for passenger comfort. Most of the commercial jets cruise at subsonic speed and one source of noise, called mixing layer noise, comes from the turbulent mixing layer caused by the instability of the shear layer near the jet lip.

Noise issues become more severe for supersonic (military) aircraft and some subsonic commercial jets (such as the Boeing 777) with high by-pass ratio engines. When a supersonic jet engine operates at an off-design condition, a series of compression and expansion waves, often called shock cells, form downstream of the engine exhaust. As the turbulence passes through these shock cells, an intense noise (called shock cell noise) is generated in addition to the mixing layer noise. The intense acoustic radiation from shock cell noise preferentially propagates in the upstream direction, and may damage the aircraft structure. Therefore, it has become necessary to develop new noise control strategies that will reduce noise emissions. In order to help us to gain a better understanding of supersonic jet noise and to develop a reliable prediction tool, computational fluid dynamics (CFD) provides an efficient and cheap way to achieve these goals.

High-order central numerical methods are typically used in direct numerical simulations (DNS) and large eddy simulation (LES) of turbulent flows as well as in computational aeroacoustics (CAA). This is because high-order schemes can resolve a larger range of length scales than traditional second-order accurate methods and they reduce dissipative numerical errors. However, they are often perceived as less robust and hard to code due to the large stencil size required. The compact method proposed by Lele\(^1\) can remedy this problem. For the same order of accuracy, by solving the flux derivative implicitly, a smaller stencil size is required than for explicit central-difference schemes and, therefore, the boundary condition treatment is simplified. Also the compact scheme contains a smaller truncation error compared to non-compact schemes of equal order. However, in order to eliminate high frequency errors that give rise to numerical instabilities while retaining high-order accuracy, filtering of the computed solutions is required. Either explicit or implicit
spatial filters can perform this task. The function of a spatial filter is similar to the more popular artificial dissipation method, but the derivation of the former approach is relatively less dependent on the governing equations being solved.\textsuperscript{2,3} The compact scheme combined with spatial filters has been used successfully in three-dimensional LES for turbulent jet noise prediction by several investigators.\textsuperscript{4–6}

When the flow field includes shocks or large gradients, the application of a compact scheme and a spatial filter in these regions results in spurious numerical oscillations and causes numerical instabilities. Therefore, unlike the artificial dissipation method, the spatial filter is inadequate for predicting high speed flows that involve shocks. Yee et al.\textsuperscript{7} proposed characteristic-type filters, which add the dissipative part of traditional shock capture schemes to nondissipative central based schemes in order to damp out numerical instabilities. Due to this feature, characteristic filters are very suitable to incorporate into existing LES codes based on high-order methods, and they allow the codes to have shock capturing capability. Also, since these filters are typically applied to solutions once after each full time step, their computational cost is considerably less than that of traditional shock capturing schemes such as TVD, MUSCL, and ENO/WENO schemes.

It has been shown that in turbulent flows or in aeroacoustics the application of these traditional shock capturing schemes in the entire domain is not very suitable, because they can lead to significant damping of the turbulent or acoustic fluctuations.\textsuperscript{8,9} Our numerical experiments show that the same phenomenon happens for the application of characteristic filters. In order to remedy this problem, we must rely on a shock detector, which is used as a sensor in order to apply the filter locally. In other words, the characteristic filter is only applied to large gradient regions (i.e. shocks) and the implicit spatial filter is applied to other smooth regions instead. This approach is similar to the hybrid compact-Roe scheme.\textsuperscript{10}

In this work, we examine several different types of characteristic filters, including the TVD and MUSCL types proposed by Yee et al.\textsuperscript{7}, and the ENO/WENO type proposed by Garnier et al.\textsuperscript{11} The formulations of the various types of filters are presented in the next section. In order to evaluate their capabilities of capturing shocks and turbulence, a one-dimensional shock/turbulence interaction test case proposed by Shu and Osher\textsuperscript{12} is used. For two-dimensional studies, both a shock vortex interaction\textsuperscript{13,14} and a shock/mixing layer interaction\textsuperscript{7} are studied to demonstrate the performance of the high-order scheme with characteristic filters. This is because vortical flow fields are very challenging for low-order numerical methods, and require fine grid resolution to preserve the vortex strength.

\section*{II. Numerical Methods}

\subsection*{A. Governing equations}

The governing equations are the nondimensional Navier-Stokes equations written in conservation form. In two-dimensional space the governing equations written in Cartesian coordinates have the following form

\begin{equation}
\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = \frac{\partial E_v}{\partial x} + \frac{\partial F_v}{\partial y}
\end{equation}

The conservative vector $U$, convective fluxes $E$ and $F$, and viscous fluxes $E_v$ and $F_v$ are given by

\begin{equation}
U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ (e + P)u \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + P \\ (e + P)v \end{bmatrix}
\end{equation}

\begin{equation}
E_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + q_x \end{bmatrix}, \quad F_v = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} + q_y \end{bmatrix}
\end{equation}

where $t$ is nondimensional time, $\rho$ is the density, $u$ and $v$ are the $x$ and $y$ velocity components, respectively, $e$ is the total energy, and $P$ is the pressure. $\tau_{ij}$ and $q_i$ are the viscous stress tensor and conductive heat flux, respectively.

\subsection*{B. Time advancement scheme}

Two available time advancement schemes are used in the current study. The first one is the third-order accurate total variation diminishing (TVD) Runge-Kutta method proposed by Shu and Osher\textsuperscript{12}

\begin{equation}
U^{(1)} = U^n + \Delta t L(U^n)
\end{equation}
\[ U^{(2)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{1}{4} \Delta t L(U^{(1)}) \]  
\[ \dot{U}^{n+1} = \frac{1}{3} U^n + \frac{2}{3} U^{(2)} + \frac{2}{3} \Delta t L(U^{(2)}) \]

The second one is the classical fourth-order Runge-Kutta method which has the form

\[
k_1 = L(U^n) \]
\[
k_2 = L(U^n + \frac{\Delta t}{2} k_1) \]
\[
k_3 = L(U^n + \frac{\Delta t}{2} k_2) \]
\[
k_4 = L(U^n + \Delta t k_3) \]
\[
\dot{U}^{n+1} = U^n + \frac{\Delta t}{6} [k_1 + 2k_2 + 2k_3 + k_4]
\]

where \( L \) represents the right hand side of the Navier-Stokes equations evaluated using a base scheme which can be any high-order non-dissipative method, such as that shown in the following section, and \( \Delta t \) is the time step.

C. Base spatial discretization scheme

To compute spatial derivatives, the sixth-order non-dissipative compact finite difference scheme developed by Lele\(^1\) is used for the current study. The method is formulated as follows

\[
\frac{1}{3} f'_{i-1} + f'_i + \frac{1}{3} f'_{i+1} = \frac{7}{9 \Delta x} (f_{i+1} + f_{i-1}) + \frac{1}{36 \Delta x} (f_{i+2} + f_{i-2})
\]

where \( f' \) is the derivative of the function \( f \). Computation of the derivative \( f' \) requires solution of a tridiagonal linear system of equations. However, the implicit nature of the scheme allows for a smaller stencil size (hence the name compact scheme), which makes implementation near boundaries easier. In addition, this method has better spatial wavenumber resolution than explicit methods of the same order of accuracy.

D. Low-Pass Spatial filter

The compact scheme is a high-order accurate, non-dissipative, centered scheme. However, like other centered schemes, it produces high-frequency spurious modes that originate from nonperiodic boundary conditions, stretched grids, or nonlinear interactions, and cause numerical instabilities. In order to eliminate these spurious modes and keep the scheme stable, filtering of the computed solution is required. The tenth-order tri-diagonal spatial filter proposed by Gaitonde and Visbal\(^{15}\) is used for the current study. This compact type filter is applied on the conservative variables once per time step. Denoting the pre-filtered value by \( \dot{U} \) (which is the solution of a full Runge-Kutta time step described in the above section), the filtered value, \( \bar{U} \), is obtained by solving the system

\[
\alpha_f \dot{U}_{i-1} + \dot{U}_i + \alpha_f \dot{U}_{i+1} = \sum_{n=0}^{N} a_n (\dot{U}_{i+n} + \dot{U}_{i-n})
\]

where \( a_n \) are filter coefficients and \( N = 5 \) for tenth-order accuracy. \( \alpha_f \) is a filter parameter that controls how much of the high wavenumber range is filtered. In this study, \( \alpha_f \) is set to 0.48. Since the filter is applied only once per time step, its cost is not as significant as that of the compact spatial derivative when time advancing the governing equations for multi-dimensional LES.
E. Characteristic (ACM) Filter

Even though the compact spatial filters can suppress the numerical instability due to the high-frequency modes, they fail around large gradients and discontinuities. Yee et al.\textsuperscript{7} proposed characteristic-based filters which can maintain the non-dissipative nature of high-order spatial differencing schemes away from shocks, while being capable of capturing shocks. The basic idea is to add the nonlinear dissipation from a TVD, MUSCL, or ENO/WENO scheme to the non-dissipative high-order central scheme. The filter operator, $L_f$ is

$$L_f(F^*, G^*)_{i,j} = \frac{1}{\Delta x} (\hat{F}^*_{i+1/2,j} - \hat{F}^*_{i-1/2,j}) + \frac{1}{\Delta y} (\hat{G}^*_{i,j+1/2} - \hat{G}^*_{i,j-1/2})$$

(14)

where $\hat{F}^*$ and $\hat{G}^*$ are the $x$ and $y$ direction filter numerical fluxes evaluated using $\hat{U}^{n+1}$ (which is the solution of a full Runge-Kutta step described in the above section). Then, the filtered value, $U_{i,j}$, at the new time level $n + 1$ becomes

$$U_{i,j}^{n+1} = U_{i,j}^n - \Delta t L_f(F^*, G^*)_{i,j}$$

(15)

This filtering process can be applied either at the end of a full time step or after each sub-stage of a Runge-Kutta integration. For computational efficiency, in our current study we use the former approach.

The Harten switch\textsuperscript{16} (originally designed for the self-adjusting hybrid schemes) can switch from a higher-order scheme to Harten’s first-order artificial compression method (ACM) scheme for shock capturing. Instead of switching to a first-order scheme, following the idea of the ACM method, a low-dissipative high-order shock-capturing scheme can be achieved, while maintaining the accuracy of the high-order non-dissipative property, by using a nonlinear characteristic filter\textsuperscript{7}. The nonlinear dissipation is multiplied by the Harten entropy correction function, $\psi$. The limiters used with them are fixed at 1 for all the cases in this report.

The MUSCL-type scheme\textsuperscript{18} — the Harten-Yee upwind (HYTVD), the Yee-Roe-Davis symmetric (YRDVTVD), and the Roe-Sweby upwind (RSTVD) — are compared in the following section. For brevity, the detailed TVD scheme formulas can be found in the original references. The limiters used with them are min mod($a_{i-1/2}^t, a_{i+1/2}^t$) for the HYTVD, min mod(2$a_{i-1/2}^t$, 2$a_{i+1/2}^t$, 2$a_{i+1/2}^t$, 2$a_{i-1/2}^t$, 2$a_{i+3/2}^t$, 2$a_{i-3/2}^t$, 2$a_{i+1/2}^t$, 2$a_{i-1/2}^t$)/2 for the YRDVTVD, and min mod(1, r) for the RSTVD, where the formula for $r$ can be found in the same reference.

The MUSCL-type scheme\textsuperscript{18} is summarized as follows

$$\phi_{i+1/2}^t = \psi(a_{i+1/2}^t) a_{i+1/2}^t$$

(20)

$$a_{i+1/2}^t = R_{i+1/2}^{-1} (U_{i+1/2}^R - U_{i+1/2}^L)$$

(21)

where $\psi$ is the Harten entropy correction function, and $a_{i+1/2}^t$ and $R_{i+1/2}^{-1}$ are the eigenvalues and eigenvectors of the flux Jacobian $\partial F/\partial U$ evaluated using a symmetric average of $U_{i+1/2}^R$ and $U_{i+1/2}^L$ (i.e. Roe average).
In here, \( U_{i+1/2} \) and \( U_{i+1/2}^L \) are evaluated by upwind-biased interpolation from neighboring grid points with the minmod limiter imposed and the compression factor equal to 4.

Garnier et al.\cite{garnier2015} proposed a characteristic filter using the ENO/WENO scheme. The dissipative numerical flux is written as

\[
F_{i+1/2,j}^{\text{ENO/WENO}} = R_{i+1/2} \phi_{i+1/2}^j
\]  

(22)

\[
\phi_{i+1/2}^j = R_{i+1/2} \phi_{i+1/2}^j
\]  

(23)

Here, \( \phi_{i+1/2}^j \) is obtained by subtracting an \( m \)-th order centered scheme \( F_C \) from an \( r \)-th order ENO/WENO scheme \( F \), as follows

\[
\phi_{i+1/2}^j = F - F_C
\]  

(24)

where \( F \) and \( F_C \) are numerical fluxes, which are interpolated from nearby grid points by the ENO/WENO approach. Either the Roe-type or the flux splitting type ENO/WENO scheme can be chosen. Our current study implements the later one with the local Lax-Friedrichs splitting technique. There are other similar WENO filters in Ref.\cite{deng2013, liu2016} and the detailed formulation of the ENO/WENO scheme can be found in Ref.\cite{shu2016}.

F. Shock detectors

As reported by Lee et al.\cite{lee2017} the use of a 6-th order accurate ENO scheme for the entire domain adds too much dissipation such that turbulent fluctuations are excessively damped. Our numerical experiments show that a similar situation happens when using the characteristic filter in the entire domain. Another problem is that when the high-order compact spatial filters are used with shocks or large gradients, spurious oscillations occur resulting in a numerical instability; therefore, they cannot be used in the entire domain either. In order to remedy these problems, a shock detector must be used as a sensor to switch between the spatial filters and the ACM filters. For our one-dimensional study, a WENO-type smoothness criterion\cite{shu2016} is used, which is given by

\[
\Omega_i = [a(P_{i+1} - P_{i-1})^2 + b(P_{i+1} - 2P_i + P_{i-1})^2]^{1/n}
\]  

(25)

where \( \Omega_i \) is the shock detector function. \( a \) and \( b \) are defined in the fifth-order WENO scheme (i.e., \( a = 1/4 \) and \( b = 13/12 \)), and \( n \) is set equal to 2 in order to sharpen the shock detection region. \( P \) is the pressure. Here the WENO smoothness criterion is used only for the shock detector and it is not related to the accuracy of the spatial discretization.

The original reference specifies a threshold parameter for \( \Omega_i \) to define a shock region directly. In the following one-dimensional case, the threshold parameter is computed by \( \text{max}(\Omega_i)/\text{const} \), where \( \text{const} \) is a constant. This slight modification can prevent an unsuitable threshold parameter specification which may not detect the shocks. Therefore, with the shock detector and the adaptive filter approach,\cite{shu2016} the characteristic and spatial filters can be applied in shock regions and smooth regions, respectively.

For multi-dimensional turbulent simulations, Ducros et al.\cite{ducros2001} proposed another shock detector which is capable of distinguishing turbulent fluctuations from large gradients and shocks. This is done by multiplying the Jameson sensor\cite{ducray2003} by another sensor proposed by them, that is

\[
\Omega_i = \left| \frac{P_{i+1} - 2P_i + P_{i-1}}{P_{i+1} + 2P_i + P_{i-1}} \right| \left( \frac{(\text{div}(\bar u))^2}{(\text{div}(\bar u))^2 + (\text{rot}(\bar u))^2} \right)
\]  

(26)

where \( P \) and \( \bar u \) are the pressure and velocity vector, respectively. Obviously, the second part of the sensor cannot be used in one-dimensional problems; therefore, it is only used in our two-dimensional test cases.

Once the shock detector function is calculated, the shock region is then defined by \( \Omega_i > \sigma \), and the ACM filters can be applied locally.

III. Numerical Experiments and Discussion of Results

A. Shock/density oscillation interaction

The first test case we consider is a 1-D inviscid moving shock/density wave interaction.\cite{lee2017} This test case consists of an interaction of a moving Mach 3 shock with a density fluctuation. The nondimensional initial condition is specified as

\[
[\rho, u, P] = [3.857143, 2.269369, 10.33333] \quad \text{for} \quad x < -4
\]  

(27)

\[
[\rho, u, P] = [1 + 0.2 \sin(5x), 0, 1] \quad \text{for} \quad x \geq -4
\]  

(28)

\[
\rho, u, P\text{.}
\]
and the computation covers a domain \( x \in [-5, 5] \). The solution is integrated with a CFL number equal to 0.3 until \( t = 1.8 \). A uniform mesh with a total of 401 grid points is used in this case. The reference solution, computed with a fifth-order accurate WENO scheme with 1601 grid points, and the initial conditions are shown in figure 1.

The results include three different TVD, MUSCL, and WENO filters and can be divided into two categories. In the first category the ACM filter is applied in the entire computational domain; therefore, no spatial filter is required or applied. These results are shown in figures 2 and 3. In the second category, shown in figures 4 and 5, the ACM filter is applied in the high gradient regions and the 10th-order spatial filter in the other regions. The shock detection function (Eq. 25) is used in this category and the \( \text{const} \) equals to \( 10^4 \). As the results show, the high-order filters (MUSCL and WENO filters, figures 3 and 5) are less dissipative than the low-order filters (TVD filters, figures 2 and 4) using the same constant \( \kappa \). Generally speaking, the WENO-ACM method performs better than the other ones. Also, when the ACM filter is applied in the entire computational domain, it may damp out not only the oscillations caused by a numerical instability, but also the turbulence fluctuations. The same situation is more pronounced for the TVD filter as shown in figures 2 and 4. Adding dissipation to the turbulence fluctuations should be avoided in turbulent flow simulations such as LES or DNS, because this will degrade the accuracy of the solutions. For high-order filters such as the WENO filter, the difference between the local and global ACM application is less significant as seen by comparing figures 3 and 5. The instantaneous distributions of the shock detector function for the case using the WENO-ACM filter at \( t = 1.8 \) is shown in figure 6. The instantaneous threshold value is also shown by a dashed line. Therefore, the main moving shock and three shocklets (seen in figure 1) arising due to the interaction are captured.

The effect of local versus global ACM filter application and the advantage of the high-order ACM filters are more pronounced in figure 7, which is computed by using 201 grid points. All the filters cannot resolve properly the oscillations behind the main shock without the shock sensor. The result using the local WENO filter application shows a significant improvement over the local MUSCL and TVD filter approaches. As the figure shows, even with the local application, all the TVD filters are still too dissipative to resolve the density oscillations behind the main shock.

B. Shock/vortex interactions

1. 2-D inviscid shock/vortex interactions

This two-dimensional inviscid test case is adapted from Shu.\(^1\)\(^3\) It contains an interaction between a stationary Mach 1.1 shock and a moving isentropic vortex. The initial condition is set following the exact Rankine-Hugoniot condition and a vortex is added to the main flow with its center at \((x_0, y_0) = (0.25, 0.5)\) The left and right state of the shock are specified as

\[
[p, u, v, P] = [1, 1.111, 0, 1] \quad \text{for} \quad x < 0.5
\]

\[
[p, u, v, P] = [1.169, 1.134, 0, 1.245] \quad \text{for} \quad x \geq 0.5
\]

The vortex is described by the perturbations to the velocities \((u, v)\), temperature \((T = P/\rho)\), and entropy \((S = \ln(P/\rho^\gamma))\) of the mean flow and has the values

\[
u' = \varepsilon \tau e^{\alpha(1-\tau^2)} \sin \theta
\]

\[
\nu' = -\varepsilon \tau e^{\alpha(1-\tau^2)} \cos \theta
\]

\[
T' = -\frac{(\gamma - 1)e^{2\alpha(1-\tau^2)}}{4\alpha \gamma}
\]

\[
S' = 0
\]

where \( \tau = r/r_c, r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \varepsilon = 0.3, r_c = 0.05, \) and \( \alpha = 0.204 \). A non-dimensional computational domain \([0, 2] \times [0, 1]\) is used, which contains 251\(\times\)100 grid points uniform in the \( y \) direction, but clustered around the shock in the \( x \) direction. The Robert transformation function\(^2\)\(^3\) with the cluster parameter equal to 5 is used in the \( x \) direction. Periodic boundary conditions are used at the top and bottom faces. The calculation is performed by using a constant time step which equals \( 10^{-3} \). The Ducros\(^2\)\(^1\) sensor with a threshold parameter 0.01 is used for shock detection. The initial vortex core has a minimum pressure 0.844, and the results by HYTVD, MUSCL, and WENO filters at \( t = 0.05, 0.20, \) and 0.35 are shown in
figures 8 and 9. All the cases show a clear shock/vortex resolution except the MUSCL filter, which contains some oscillations near the shock. This situation may be due to the limiter types, or control parameters used in current study and further investigation may be needed. All the cases resolve the vortex properly, and the effect of local versus global ACM filter application is not significant in this case.

Due to the deformation of the shock wave during a shock/vortex interaction, a series of compression and rarefaction regions form behind the main shock. Several sound waves (e.g. precursor, second, and third sounds) with a quadrupolar nature are then generated and propagate in the radial direction with respect to the vortex center. In order to capture these sound waves (not considered in the current study), a large computational domain and a very fine grid are required. The more detailed mechanisms of the sound generated by a shock/vortex interaction are described by Inoue and Hattori\textsuperscript{24} and the references cited therein.

2. 2-D viscous shock/vortex interactions

This case is used to investigate the ability of shock-capturing schemes to predict the generation and transport of acoustic waves during a shock vortex interaction.\textsuperscript{14} The nondimensional computational domain is $[0, 2] \times [0, 2]$. Two uniform grids same as the original reference with $101 \times 101$ and $201 \times 201$ points are used. The initial condition satisfies the exact Rankine-Hugoniot condition and a stationary shock is located at $x = 1$. An isolated Taylor vortex is added to the uniform flow and described by the tangential velocity only.

$$V_\theta = C_1 r e^{-C_2 r^2}$$

(35)

with

$$C_1 = \frac{U_c}{r_c} e^{1/2}, \quad C_2 = \frac{1}{2r_c^2}, \quad r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

where $r_c = 0.075$, and $U_c = 0.25$. The initial position of the vortex center is $(x_0, y_0) = (0.5, 1)$, and the inflow Mach number is 1.1588. The Reynolds number based on the uniform velocity and the length of the computational domain is 2000. A nondimensional time step $\Delta t = 4 \times 10^{-3}$ is used and the final output time is $t = 0.7$. For the local filter cases, the Ducros\textsuperscript{21} sensor with a threshold parameter 0.01 is used. The difference between this case and previous one is that in this case as the vortex starts to move it generates a circular acoustic wave which propagates in the radial direction with respect to the vortex center. Therefore, some portions of this acoustic wave propagate in the downstream direction with the vortex and then pass through the stationary shock. Different ACM filter schemes are then used to compare the accuracy at both the vortex core and this circular acoustic wave behind the shock. Again, the sound waves generated by the shock/vortex interaction are not considered here.

The reference solution is obtained by the local WENO-ACM approach on a fine uniform grid ($401 \times 401$). The pressure contour of the reference solution at $t = 0.7$ is shown in figure 10. An acoustic waves is seen spreading outward as discussed above. Figure 11 shows the density distributions of the reference solution at $t = 0.7$ and the initial condition along $y = 1$. Along the line of $y = 1$, the vortex core at $t = 0.7$ is located at $x = 1.16$ and a downstream propagating acoustic wave is around $x = 1.75$. The upstream propagating acoustic wave is located at $x = 0.55$. The density distributions using each type of filter (including local and global ACM filter application) on a coarse uniform grid ($101 \times 101$) along $y = 1$ at $t = 0.7$ are shown in figure 12. It can be observed that the TVD filter cannot resolve the acoustic wave and vortex core properly without the shock sensor. The WENO filter demonstrates a superior capability of capturing the acoustic wave and resolving the vortex core correctly compared to the other filters. The results computed using a finer grid ($201 \times 201$) are shown in figure 13. With this grid resolution, all the cases (except the global HYTVD-ACM approach) resolve the vortex core and the acoustic wave accurately and become grid independent.

C. Shock/mixing layer interaction

This case is used to test the performance of the shock capturing schemes for interactions of shock waves and shear layers.\textsuperscript{7} A spatially developing mixing layer has an initial convective Mach number of 0.6, and a 12\textdegree oblique shock originating from the upper-left corner interacts with the vortices developed from the instability of the shear layer. This oblique shock is deflected by the shear layer and then reflects from the bottom slip wall. At the same time, an expansion fan forms above the shear layer. Downstream, a series of shock waves form around the vortices. The outflow boundary has been arranged to be supersonic everywhere; therefore, a simple first-order extrapolation is used in the outflow boundary condition.

The inflow boundary condition has a hyperbolic tangent velocity profile.

$$u = 2.5 + 0.5 \tanh(2y)$$

(36)
and the velocities in the upper and lower streams are \( u_1 = 3 \) and \( u_2 = 2 \), respectively. The convective Mach number is

\[
M_c = \frac{u_1 - u_2}{c_1 + c_2} = 0.6
\]  

(37)

where \( c_1 \) and \( c_2 \) are free stream sound speeds which equal 0.5333 and 1.1333, respectively. These nondimensional velocities and speeds of sound are taken directly from the original reference. The Prandtl number is 0.72 and the Reynolds number is 500. More detailed boundary information can be found in the same reference. Fluctuations are added to the inflow as

\[
v' = \sum_{k=1}^{2} a_k \cos(2\pi kt/\tau + \varphi_k) \exp(-y^2/b)
\]  

(38)

with period \( \tau = \lambda/u_c \), wavelength \( \lambda = 30 \), and convective velocity \( u_c = 2.68 \). Other constants are \( a_1 = 0.05, \varphi_1 = 0, a_2 = 0.05, \varphi_2 = \pi/2, \) and \( b = 10 \). This case was run on a 200×40 domain with a 321×81 grid, which is uniform in the \( x \) direction but stretched in the \( y \) direction. A constant time step is used with \( \Delta t = 0.12 \) and the final output time is \( t = 120 \). The Ducros\cite{Ducros} sensor with a threshold parameter 0.005 is used for the local filter case. Figures 14 to 16 show the density and pressure contours for each case. As expected, the solutions by the TVD filter are more diffused and the shape of vortices cannot be resolved properly. Both the MUSCL (figure 15) and WENO (figure 16) filters provide high quality vortices and downstream shocklet resolution. However, the effect of the shock detector (i.e. local versus global application of the ACM filter) in this case is difficult to identify. Further numerical experiments, such as a three-dimensional shock homogeneous turbulence interaction,\cite{Bodony} can verify this.

IV. Conclusions

A high-order Navier-Stokes solver based on the compact finite difference scheme and characteristic filters has been developed and tested for several numerical experiments. These test cases include the interactions of a 1-D moving shock with a sinusoidal density wave, a 2-D stationary shock with a moving vortex, and a 2-D shock/mixing layer. The numerical experiments show that the high-order characteristic filters, such as the WENO filter, perform better than the low-order filters (such as the TVD and MUSCL filters). This is due to the less dissipation of the high-order characteristic filters, and to the insensitivity to the parameters that appear in both the filter schemes and the original shock capturing schemes. One disadvantage of the ACM filters for turbulent flows is the difficulty of distinguishing turbulent fluctuations from shocks. Therefore, smaller eddies away from the shocks may be dissipated. This drawback can be remedied by using a shock sensor and then applying the ACM filter locally in the shock regions. Two shock sensors are presented, and both of them work quite well. Through several numerical experiments, the results using the compact scheme and the WENO filter are satisfactory. The same approach will be used in the future for three-dimensional LES of supersonic jets.

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References


Figure 1. Initial density distribution and final reference solution.

Figure 2. Density distribution at $t = 1.8$ by TVD filters applied globally.

Figure 3. Density distribution at $t = 1.8$ by MUSCL and WENO filters applied globally.

Figure 4. Density distribution at $t = 1.8$ by TVD filters applied locally.

Figure 5. Density distribution at $t = 1.8$ by MUSCL and WENO filters applied locally.

Figure 6. Shock detection function at $t = 1.8$ computed for the case using WENO filter. Dashed line is the instantaneous threshold value.
Figure 7. Density distributions at $t = 1.8$ using (from top to bottom) TVD, MUSCL, and WENO filters applied globally (left) and locally (right) on a coarse grid with 201 points.
Figure 8. Pressure contours (from top to bottom) by TVD, MUSCL, and WENO filters applied globally at (from left to right) $t=0.05$, 0.20, and 0.35.
Figure 9. Pressure contours (from top to bottom) by TVD, MUSCL, and WENO filters applied locally at (from left to right) $t=0.05, 0.20, \text{ and } 0.35$. 

(a) $(\text{max, min})=(1.249, 0.844)$.

(b) $(\text{max, min})=(1.276, 0.837)$.

(c) $(\text{max, min})=(1.475, 0.995)$.

(d) $(\text{max, min})=(1.267, 0.844)$.

(e) $(\text{max, min})=(1.311, 0.837)$.

(f) $(\text{max, min})=(1.580, 0.907)$.

(g) $(\text{max, min})=(1.249, 0.844)$.

(h) $(\text{max, min})=(1.270, 0.839)$.

(i) $(\text{max, min})=(1.504, 0.994)$.
Figure 10. Isocontours of the static pressure at \( t = 0.7 \) using WENO filter with Ducros sensor.

Figure 11. Initial density distribution and final reference solution along \( y = 1 \) at \( t = 0.7 \).

Figure 12. Instantaneous density distribution at \( t = 0.7 \) along \( y = 1 \) by a grid \( 101 \times 101 \).

(a) filter applied globally.

(b) filter applied locally.

Figure 13. Instantaneous density distribution at \( t = 0.7 \) along \( y = 1 \) by a grid \( 201 \times 201 \).

(a) filter applied globally.

(b) filter applied locally.
(a) global filtering, (max,min)=(2.577,0.335).

(b) local filtering, (max,min)=(2.544,0.333).

(c) global filtering, (max,min)=(0.717,0.258).

(d) local filtering, (max,min)=(0.696,0.256).

Figure 14. Density (top two) and pressure (bottom two) contours using the HYTVD filter.
Figure 15. Density (top two) and pressure (bottom two) contours using the MUSCL filter.

(a) global filtering, $(\text{max,min})=(2.703,0.282)$

(b) local filtering, $(\text{max,min})=(2.892,0.267)$

(c) global filtering, $(\text{max,min})=(0.708,0.146)$

(d) local filtering, $(\text{max,min})=(0.713,0.151)$
Figure 16. Density (top two) and pressure (bottom two) contours using the WENO filter.