# Understanding the Capacity Region of the Greedy Maximal Scheduling Algorithm in Multi-hop Wireless Networks

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***Abstract*—In this paper, we characterize the performance of an important class of scheduling schemes, called *Greedy Maximal Scheduling* (*GMS*), for multi-hop wireless networks. While a lower bound on the throughput performance of *GMS* has been well known, empirical observations suggest that it is quite loose, and that the performance of *GMS* is often close to optimal. In this paper, we provide a number of new analytic results characterizing the performance limits of *GMS*. We first provide an equivalent characterization of the efficiency ratio of *GMS* through a topological property called the local-pooling factor of the network graph. We then develop an iterative procedure to estimate the local-pooling factor under a large class of network topologies and interference models. We use these results to study the worst-case efficiency ratio of *GMS* on two classes of network topologies. We show how these results can be applied to tree networks to prove that *GMS* achieves the full capacity region in tree networks *under the $K$-hop interference model*. Then, we show that the worst-case efficiency ratio of *GMS* in geometric unit-diskgraphs is between $\frac{1}{6}$ and $\frac{1}{3}$.**

***Index Terms*—Communication systems, multi-hop wireless networks, greedy maximal scheduling, longest queue first, capacity region.**

## I. Introduction

Over the last few years there has been significant interest in studying the *scheduling* problem for multi-hop wireless networks [1]–[8]. In general, this problem involves determining which links should transmit (i.e., which node-pairs should communicate) at what times, what modulation and coding schemes should be used, and at what power levels should communication take place. While the optimal solution of this scheduling problem has been known for a long time [1], the resultant solution has high computational complexity and is difficult to implement in multi-hop networks. For example, consider the simplest 1-hop interference model (also known as the *node-exclusive* or *primary* interference model), where two links interfere with each other only if they are within a 1-hop distance. Under this model, the throughput-optimal policy of [1] corresponds to a *Maximum Weighted Matching* (*MWM*) policy and its complexity is roughly $O(N^3)$ [9], where $N$ is the total number of nodes in the network. While the 1-hop interference model has been used as a reasonable approximation to Bluetooth or FH-CDMA networks [2], [10], [11], a large class of systems can be modeled using the more general $K$-hop interference models, in which any two links within a $K$-hop distance cannot be activated simultaneously. For example, the ubiquitous IEEE 802.11 DCF (Distributed Coordination Function) wireless networks is often modeled using the 2-hop interference model [12], [13], when the carrier-sensing range is equal to the transmission range. On the other hand, when the carrier-sensing range is $(K-1)$ times the transmission range, we can model these networks with $K$-hop interference models [14]. The complexity of the throughput-optimal policy of [1] for the $K$-hop interference model is NP-Hard [14], and hence, it is difficult to implement in practice.

In this paper, we are interested in a well-known suboptimal scheduling policy called the *Greedy Maximal Scheduling* (*GMS*) [2], [15] (also known as *Longest Queue First* (*LQF*) in [16], [17]), which determines a schedule by choosing links in a decreasing order of the backlog, while conforming to interference constraints. *GMS* has low complexity [2], [15], [16] and may be implemented in a distributed manner [18]. However, to date its performance is not well-understood. We characterize the performance of *GMS* through its efficiency ratio $\gamma^*$, which is defined as the achievable fraction of the optimal capacity region (see Definition 2 for a precise definition). Under the 1-hop interference model, it is relatively straightforward to show that the efficiency ratio of *GMS* is at least $\frac{1}{2}$, i.e., *GMS* can sustain at least a half of the throughput of the optimal policy. However, simulation results suggest that the performance of *GMS* is often much better than this lower bound in most network settings [6]. For the $K$-hop interference model, the known performance guarantees of *GMS* are also quite pessimistic [12], [14], [19].

Recently, Dimakis and Walrand [17] have shown that if the network topology satisfies the so-called *local-pooling* condition, then *GMS* can in fact achieve the full capacity region. The idea is extended in [20], [21] to find network topologies that maximize the throughput under *GMS*. Unfortunately, realistic network topologies may not satisfy the local-pooling condition. Hence, the true efficiency ratio of *GMS* in many network scenarios remains unknown.

The main objective of this paper is to understand the achievable efficiency ratio of *GMS* for a large class of network topologies and interference models. Understanding the performance limits of *GMS* is important for the following reasons. First, it has been empirically observed in [6] that the centralized *GMS* outperforms many distributed scheduling schemes and achieves virtually the same throughput as the

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throughput-optimal scheduler for a variety of networking scenarios. Second, although there have been some recently developed distributed scheduling schemes [5], [8] that can achieve the maximum achievable throughput, the study of *GMS* continues to remain attractive because, empirically, *GMS* performs better than these schemes in terms of the resultant queueing delay [3], [6]. Third, it has been known[1] in [18] that *GMS* can be also implemented in a distributed manner, which is critical from the point of view of many multi-hop wireless systems and applications. Finally, recent studies have proposed even simpler constant-time-complexity random algorithms [4], [6], [7] that appear to approximate the performance of *GMS* by giving a larger weight to a link with a larger queue length.

In this paper, we provide a number of new analytical results along this direction. We first generalize the notion of local-pooling in [17] to the notion of the local-pooling factor, which is a topological property of a graph. We show that, under arbitrary interference models, the efficiency ratio of *GMS* for a given network graph is equal to the local-pooling factor of the graph. We then develop an iterative procedure to determine a lower bound on the local-pooling factor of a network graph, and a sufficient condition for a lower bound on the worst-case local-pooling factor over a class of network topologies. We next apply these results to two classes of network topologies. First, we show how these results can be applied to tree networks to prove that *GMS* achieves the full capacity for any tree network under the $K$-hop interference model. (This result is also shown in [21] by using a different approach.) Second, we develop much sharper bounds on the worst-case efficiency ratio for geometric unit-diskgraphs other than those known in the literature.

The rest of the paper is organized as follows. We first describe our system model in Section II. In Section III, we provide an equivalent characterization of the efficiency ratio of *GMS* through the local-pooling factor of the underlying network graph. We develop an iterative analysis method estimating the local-pooling factor of a network graph in Section IV. Using the new methodology, we show that *GMS* achieves the full capacity region in tree topologies under the $K$-hop interference model in Section V. In Section VI, we also provide new results bounding the efficiency ratio of *GMS* in geometric unit-diskgraphs. We conclude in Section VII.

## II. Network Model

We model a wireless network by a graph $G(V, E, I)$, where $V$ is the set of nodes, $E$ is the set of undirected links, and $I$ represents interference constraints (e.g., an $|E|$x$|E|$ interference matrix). For each link $l$, let $I(l)$ denote the set of links that interfere with $l$. For convenience, we adopt the convention that $l \in I(l)$. We define the interference degree $d(l)$ as the maximum number of links in $I(l)$ that do not interfere with each other. We assume a time-slotted system, where the length of each time slot is of unit length. We assume that in each time slot, link $l$ can transmit one packet provided that no other links in $I(l)$ are transmitting at the same time. If two interfering links transmit at the same time, neither of these can transmit any data. This assumption of either collision or perfect reception ignores the possibility of errors due to background noise and also ignores the capture effect [22]. A set of active (i.e., transmitting) links forms a *feasible schedule* in $E$ if none of them interfere with each other. The model is very general representing a large class of wireless networks. For example, in the so-called $K$-hop interference model, two links within a $K$-hop "distance" interfere with each other. We can correspondingly define $I(l)$ as, for all links $l \in E$,

$$I(l) = \{k \in E \mid \text{the distance between links } l \text{ and } k \text{ is less than or equal to } K \text{ hops}\}.$$

A *maximal schedule* $\vec{M}$ on $E$ is defined as a feasible schedule such that when all links in $\vec{M}$ are activated, no more links can be activated without violating the interference constraints. We use a vector in $\{0,1\}^{|E|}$ to denote a maximal schedule $\vec{M}$ such that the $k$-th element $M_k$ is set to 1 if link $k \in E$ is included in $\vec{M}$, and to 0 otherwise. Let $\mathcal{M}_E$ be the set of all possible $\vec{M}$'s and let $Co(\mathcal{M}_E)$ denote its convex hull, where the convex hull $Co(A)$ of a set $A$ is defined as

$$Co(A) := \{\textstyle\sum_i w_i \vec{\alpha}_i \mid w_i \geq 0, \sum_i w_i = 1, \vec{\alpha}_i \in A\}.$$

We define a *maximal scheduling vector* $\vec{\phi}$ in $E$ as a vector $\vec{\phi} \in Co(\mathcal{M}_E)$.

We assume that packets arrive at each link $l$ according to a stationary and ergodic process, and that the average arrival rate is $\lambda_l$. Further, we assume that the arrival process satisfies the conditions for the fluid limit to hold (e.g., as in [23]). The *capacity region* (or *stability region*) under a given scheduling policy is defined as the set of arrival rate vectors $\vec{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_{|E|}\}$ for which the system is stable (i.e., all queues are kept finite). We define the *optimal* capacity region $\Lambda$ as the union of the capacity regions of all scheduling policies. The optimal capacity region is known to be,

$$\Lambda = \left\{ \vec{\lambda} \mid \vec{\lambda} \preceq \vec{\phi}, \text{ for some } \vec{\phi} \in Co(\mathcal{M}_E) \right\}, \tag{1}$$

where $\vec{x} \preceq \vec{y}$ denotes that $\vec{x}$ is component-wise dominated by $\vec{y}$. Let $\mathring{\Lambda}$ denote the interior of $\Lambda$. This expression can be explained as follows. Assume that $\vec{\lambda} \prec \vec{\phi}$ and $\vec{\phi}$ can be written as a convex combination of vectors in $\mathcal{M}_E$, i.e., $\vec{\phi} = \sum_i w_i \vec{M}_i$, where $w_i \geq 0$ and $\sum_i w_i = 1$. Then by choosing the maximal schedule $\vec{M}_i$ with probability $w_i$, the service rate at each link will be larger than the arrival rate. Hence the system will be stable. On the other hand, if no vector $\vec{\phi} \in Co(\mathcal{M}_E)$ exists such that $\vec{\lambda} \preceq \vec{\phi}$, then one can show that the system is unstable under any scheduling algorithms [1], [24], [25].

It is well-known that the scheduling policy of [1], which we refer to as the Maximum Weighted Scheduling (*MWS*) policy, achieves the capacity region $\mathring{\Lambda}$. *MWS* chooses a schedule at

[1]Although the distributed algorithm in [18] has been devised to compute matching, it is not difficult to generalize the idea to $K$-hop interference models. Specifically, we can let each link $l$ decide either to schedule itself or to give up, as follows: Link $l$ will schedule itself if its weight is larger than other interfering links (i.e., links within the $K$-hop distance from $l$). Ties can be broken by pre-assigned link IDs. Otherwise, link $l$ will wait until all interfering links with larger weights have decided. If any one of the interfering links with larger weight has been scheduled, link $l$ will give up. If all the interfering links with larger weights have given up, link $l$ schedules itself.

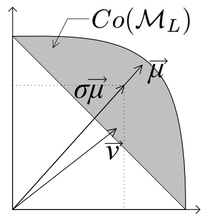



Fig. 1. Given a network graph $G(V,E,I)$, if there exist two vectors $\vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)$ for some subset of links $L \subset E$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$, then the graph is said to be $\sigma$-dominant.

each time slot $t$ that maximizes the total queue weighted rate sum as

$$\vec{M}^*(t) = \operatorname*{argmax}_{\vec{M}\in\mathcal{M}_E} \sum_{l\in E} q_l(t) M_l,$$

where $q_l(t)$ is the backlog of link $l$ at time $t$. However, this policy has high computational complexity. The complexity is $O(N^3)$ under the 1-hop interference model and is in general NP-Hard under $K$-hop interference models ($K \geq 2$). In this paper, we are interested in a suboptimal (but much simpler) policy called *Greedy Maximal Scheduling* (*GMS*) or *Longest Queue First* (*LQF*) policy. *GMS* can be viewed as an approximation to *MWS*. It operates as follows: start with an empty schedule; first pick the link $l$ with the largest backlog; add $l$ into the schedule, and disable other links in $I(l)$; next pick the link $l'$ with the largest backlog from the remaining links, add $l'$ into the schedule, and disable other links in $I(l')$; and this process continues until all links are either chosen or disabled. All chosen links $\{l, l', \cdots\}$ will be scheduled during time slot $t$. Our goal of the paper is to characterize the efficiency ratio of *GMS* under arbitrary network topologies. The efficiency ratio is defined as follows.

*Definition 1:* We say that a scheduling policy achieves a *fraction $\gamma$ of the capacity region* under a given network topology if it can keep the system stable for any offered load $\vec{\lambda} \in \gamma\Lambda$, where $0 \leq \gamma \leq 1$.

*Definition 2:* The *efficiency ratio* $\gamma^*(G)$ of a scheduling policy under a given network graph $G(V,E,I)$ is the supremum of all $\gamma$ such that the policy can achieve a fraction $\gamma$ ($0 \leq \gamma \leq 1$) of the capacity region, i.e.,

$$\begin{aligned}\gamma^*(G) := \sup\{\gamma \mid\ & \text{the system is stable under all offered} \\ & \text{load vectors } \vec{\lambda} \text{ such that } \vec{\lambda} \preceq \gamma\vec{\phi} \\ & \text{for some } \vec{\phi} \in Co(\mathcal{M}_E)\}.\end{aligned} \tag{2}$$

## III. An Equivalent Characterization of the Efficiency Ratio of *GMS*

In this section, we provide an equivalent characterization of the efficiency ratio of *GMS* through its topological properties. We start with the following definition.

*Definition 3:* A graph $G(V,E,I)$ is said to be *$\sigma$-dominant*, if there exist two vectors $\vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)$ for a subset of links $L \subset E$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$, i.e., $\sigma\mu_i \geq \nu_i$ for all $i$. The vectors $\vec{\mu}$ and $\vec{\nu}$ are called $\sigma$-dominant vectors.

Fig. 1 depicts the convex hull of maximal schedules $Co(\mathcal{M}_L)$ for some subset $L$ and two vectors $\vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)$ satisfying that $\sigma\vec{\mu} \succeq \vec{\nu}$. Then the graph $G$ is said to be $\sigma$-dominant.

The reason that we are interested in $\sigma$-dominance is as follows. Suppose that links in a subset $L$ have larger queues than the rest of the links. Since *GMS* will pick these links first, its service vector will belong to $Co(\mathcal{M}_L)$. However, there is still some uncertainty as to which vector in $Co(\mathcal{M}_L)$ is the actual service vector. It turns out that if there exist two $\sigma$-dominant vectors $\vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$, then we can construct a traffic pattern that i) has an arrival rate equal to $\sigma\vec{\mu}$ and ii) induces the service vector of *GMS* to be $\vec{\nu}$. (This point is made rigorously in Proposition 1.) Thus the system is unstable at an arrival rate $\sigma\vec{\mu}$, while the arrival rate $\vec{\mu}$ could have been stabilized under a throughput-optimal policy. Hence, the efficiency ratio of *GMS* will be no greater than $\sigma$.

Clearly, if $\sigma$ is too small, we will no longer be able to find such a subset $L$ and two $\sigma$-dominant vectors $\vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)$. Intuitively, if we can find the smallest value of $\sigma$, for which the graph is $\sigma$-dominant, then the smallest value will have some relationship to the efficiency ratio of *GMS*. This notion is reflected in the following definition.

*Definition 4:* The *local-pooling factor* $\sigma^*(G)$ of a graph $G(V,E,I)$ is the infimum of all $\sigma$ such that the graph $G$ is $\sigma$-dominant. In other words,

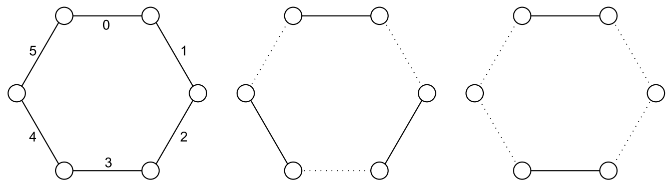
$$\begin{aligned}\sigma^*(G) &:= \inf\{\sigma \mid G \text{ is } \sigma\text{-dominant}\} \\ &= \inf\{\sigma \mid \sigma\vec{\mu} \succeq \vec{\nu} \text{ for some } L \text{ and some } \vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)\} \\ &= \sup\{\sigma \mid \sigma\vec{\mu} \not\succeq \vec{\nu} \text{ for all } L \text{ and all } \vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)\}.\end{aligned} \tag{3}$$

The notion of local-pooling and local-pooling factor was first introduced in [17] and [26], respectively. The definition of *local-pooling* in [17] is equivalent to the definition of a *local-pooling factor of* 1. (We refer to [26] for the details.) It was shown in [17] that, if the local-pooling factor of an arbitrary graph is 1, *GMS* can achieve the efficiency ratio of 1. However, realistic network topologies often do not have a local-pooling factor of 1. In our earlier work [26], we show that under the 1-hop interference model, the efficiency ratio of *GMS* under a given network graph is equivalent to the local-pooling factor of the graph. We next generalize this result to arbitrary interference models.

*Proposition 1:* The efficiency ratio $\gamma^*(G)$ of *GMS* under a given network graph $G(V,E,I)$ is equal to its local-pooling factor $\sigma^*(G)$.

*Remark:* Since both $\gamma^*(G)$ and $\sigma^*(G)$ are determined by the network $G$, in the sequel we will simply use $\gamma^*$ and $\sigma^*$ when there is no source of confusion regarding the network $G$.

The proof of Proposition 1 is a straightforward extension of that of Proposition 8 in [26] and its supporting lemmas. We next sketch the main idea of the proof and refer the readers to [26] for the details. First, as we discuss at the beginning of this section, we can show that $\gamma^* \leq \sigma^*$ by constructing a particular traffic pattern with rate outside $\sigma^*\Lambda$ such that the system is unstable under *GMS*. Specifically, for any $\sigma < \sigma^*$, we can find two $\sigma$-dominant vectors $\vec{\mu},\vec{\nu} \in Co(\mathcal{M}_L)$ satisfying $\sigma\vec{\mu} \succeq \vec{\nu}$. Then for all $\epsilon > 0$, we can construct a traffic pattern with offered load $\vec{\lambda} = \vec{\nu} + \epsilon\vec{e}_L$, under which *GMS* selects the service vector $\vec{\nu}$ on average, where $\vec{e}_L$ is a vector with $e_k = 1$ for $k \in L$ and $e_k = 0$ for $k \notin L$. Thus the




(a) Topology (b) Maximal schedule $\vec{M}_0$ (c) Maximal schedule $\vec{M}_2$

Fig. 2. The 6-link cyclic network and the instances of maximal schedule under the 1-hop interference model. The solid lines in (b) and (c) are the active links.

system becomes unstable. Since $\vec{\nu} \preceq \sigma\vec{\mu}$, we have $\gamma^* \leq \sigma$ for all $\sigma \leq \sigma^*$. In the other direction, we can obtain $\gamma^* \geq \sigma^*$ by showing that the network is stable under *GMS* for any offered load strictly in $\sigma^*\Lambda$. To elaborate, we can show that in the fluid limit, the longest queue always decreases under *GMS*. To see this, suppose that the set $L$ of links have the longest queue in the fluid limit and they all grow at the same rate $\epsilon > 0$. *GMS* will pick a service rate $\vec{\pi}$ such that $\vec{\pi}|_L \in Co(\mathcal{M}_L)$, where $\cdot|_L$ denotes the projection of a vector onto $L$. Hence, we have $\vec{\pi}|_L = \vec{\lambda}|_L - \epsilon\vec{e}_L \preceq \vec{\lambda}|_L$. However, since $\vec{\lambda} \in \sigma^*\mathring{\Lambda}$, there must exist a vector $\vec{\phi} \in Co(\mathcal{M}_L)$ such that $\vec{\lambda}|_L \prec \sigma\vec{\phi}$ for some $\sigma < \sigma^*$. Then we obtain that $\vec{\pi}|_L \prec \sigma\vec{\phi}$, which implies that $\vec{\pi}|_L$ and $\vec{\phi}$ are $\sigma$-dominant vectors. This contradicts to the definition of the local-pooling factor $\sigma^*$. Hence, the longest queue cannot grow. The result of the proposition then follows.

In the following, we further explain the first part of the proof (i.e. $\gamma^* \leq \sigma^*$) using an example. Specifically, we illustrate how, from two $\sigma$-dominant maximal scheduling vectors, we can construct a traffic pattern with which the system is unstable under *GMS*. This example will also illustrate how the performance limits of *GMS* are related to maximal scheduling vectors.

*Example:* We consider the 6-link cyclic network graph under the 1-hop interference model. We illustrate its topology in Fig. 2(a) and number all links clockwise from 0 to 5. All possible maximal schedules under this network graph are listed below.

- $\vec{M}_0 = \{1,0,1,0,1,0\}$, $\vec{M}_1 = \{0,1,0,1,0,1\}$,
- $\vec{M}_2 = \{1,0,0,1,0,0\}$, $\vec{M}_3 = \{0,0,1,0,0,1\}$, $\vec{M}_4 = \{0,1,0,0,1,0\}$.

Note that the number of links included in a maximal schedule is three for $\vec{M}_0$ and $\vec{M}_1$, and is two for $\vec{M}_2$, $\vec{M}_3$, and $\vec{M}_4$. Figs. 2(b) and 2(c) show the two instances of the maximal schedules, i.e., $\vec{M}_0$ and $\vec{M}_2$. Note that we can take two convex combinations $\vec{\mu}, \vec{\nu}$ from maximal schedules (i.e., $\vec{\mu}, \vec{\nu} \in Co(\{\vec{M}_i\})$ as

$$\vec{\mu} = \tfrac{1}{2}\vec{M}_0 + \tfrac{1}{2}\vec{M}_1 = \left\{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\},$$
$$\vec{\nu} = \tfrac{1}{3}\vec{M}_2 + \tfrac{1}{3}\vec{M}_3 + \tfrac{1}{3}\vec{M}_4 = \left\{\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right\},$$

and hence, $\frac{2}{3}\vec{\mu} \succeq \vec{\nu}$. This implies that the 6-link cyclic network is $\sigma$-dominant with $\sigma = \frac{2}{3}$, i.e., its local-pooling factor $\sigma^*$ must be no larger than $\frac{2}{3}$.

We now show that the efficiency ratio of *GMS* is no larger than $\frac{2}{3}$ by constructing a particular traffic pattern with offered load $\vec{\lambda} = \vec{\nu} + \frac{\epsilon}{3}\vec{e}$ such that the system is unstable under *GMS*, where $\vec{e} = \{1,1,1,1,1,1\}$ and $\epsilon$ is a small positive number. Assume that all queues in the system are of the same length at time 0.

1) *1st time slot:* One packet is applied to links 0 and 3. Since *GMS* gives priority to links with a longer queue, it will serve links 0 and 3. Therefore, at the end of time slot 1, all queues will still have the same length.
2) *2nd time slot:* One packet is applied to links 1 and 4. For the same reason as above, *GMS* will serve links 1 and 4, and all queues will still have the same length at the end of time slot 2.
3) *3rd time slot:* With probability $1-\epsilon$, one packet is applied to links 2 and 5. With probability $\epsilon$, two packets are applied to links 2 and 5, and one packet is applied to all other links. In both cases, links 2 and 5 have the longest queue and will be served by *GMS*. At the end of time slot 3, all queues still have the same length. However, with probability $\epsilon$, the queue length increases by 1.

The pattern then repeats itself.

Over all links, the arrival rate is $\frac{1}{3} + \frac{\epsilon}{3}$ and the queue length increases by 1 with probability $\epsilon$ every three time slots. Hence, the system with offered load $\vec{\nu} + \frac{\epsilon}{3}\vec{e}$ is unstable under *GMS*. However, the optimal policy (*MWS*) can support an offered load $\vec{\mu} = \frac{3}{2}\vec{\nu}$ in this example. Hence, the efficiency ratio of *GMS* is no greater than $\frac{2}{3}$, i.e., $\gamma^* \leq \frac{2}{3}$ in this 6-link cyclic network under the 1-hop interference model.

*Remark:* Note that the key in constructing the above traffic pattern is that (i) we keep all queues in $L$ of the same length at all time, (ii) we inject packets according to the maximal schedules that form the vector $\vec{\nu}$ so that these maximal schedules will be picked by *GMS* at all time, and (iii) the offered load is slightly larger than $\vec{\nu}$, i.e., $\vec{\lambda} = \vec{\nu} + \epsilon\vec{e}_L$ so that the queues of $L$ grow to infinity together. In [26], we show that such a traffic pattern can be constructed for all $\vec{\mu}, \vec{\nu}$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$.

Proposition 1 provides an equivalent characterization of the efficiency ratio of *GMS* through the topological properties (i.e., the local-pooling factor) of the given graph. However, it can still be quite difficult to compute the local-pooling factor for an arbitrary network graph. In the next section, we will extend the methodology of Proposition 1 to develop new approaches to estimate the efficiency ratio and the local-pooling factor of arbitrary network graphs.

## IV. Estimates of the Local-Pooling Factor for Arbitrary Network Graphs

In this section, we would like to answer the following questions: (i) how do we estimate the local-pooling factor of a given graph? and (ii) what types of graphs will have low local-pooling factors? We now argue that both questions are intimately related to the characterization of a set of unstable links. We first state the following lemma. The proof is provided in Appendix A.

*Lemma 1:* Given a network graph $G(V,E,I)$ with local-pooling factor $\sigma^*$, there exist a subset of links $L \subset E$, and two maximal scheduling vectors $\vec{\mu}^*, \vec{\nu}^* \in Co(\mathcal{M}_L)$ such that $\sigma^*\vec{\mu}^* \succeq \vec{\nu}^*$.

*Remark:* According to Definition 4, for any $\sigma > \sigma^*$, there exist two $\sigma$-dominant vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$. However, since $\sigma^*$ is the infimum of such $\sigma$, it could be possible that no $\sigma^*$-dominant vectors exist in $Co(\mathcal{M}_L)$. Lemma 1 shows that this is not the case.

The idea in the rest of the section is as follows. Suppose that we want to show that $\sigma^* \geq \sigma$ for some $\sigma > 0$. We want to prove by contradiction. Assume in contrary that $\sigma^* < \sigma$. Given a network graph $G(V, E, I)$ with local-pooling factor $\sigma^*$, there exist a set $Y \subset E$ and two $\vec{\mu}^*, \vec{\nu}^* \in Co(\mathcal{M}_Y)$ such that $\sigma^*\vec{\mu}^* \succeq \vec{\nu}^*$ from Lemma 1. According to the proof of Proposition 1 (see the example in Section III), we can then construct a traffic pattern with offered load $\vec{\nu}^* + \epsilon\vec{e}_Y$ such that the queues of all links in $Y$ increase to infinity together under *GMS*. Let $\vec{\lambda}^*(\epsilon) = \vec{\nu}^* + \epsilon\vec{e}_Y \in \sigma\mathring{\Lambda}$ denote this offered load[2]. We refer to this set $Y$ as the unstable links. Clearly, if we can show that $Y = \emptyset$, then this leads to a contradiction, which then implies that $\sigma^* \geq \sigma$.

Towards this end, we first study the properties of this set $Y$ of unstable links.

### A. Properties of unstable links

For a subset $L \subset E$, we let $I_L(l) = I(l) \cap L$ denote the set of links in $L$ that interfere with link $l$, and define the interference degree $d_L(l)$ as the maximum number of links in $I_L(l)$ that can be scheduled at the same time without interfering with each other. We begin with the following two lemmas.

*Lemma 2:* If $\vec{\lambda} \in \mathring{\Lambda}$, then $\sum_{j \in I_L(l)} \lambda_j \leq d_L(l)$ for all $l \in L$ and all $L \subset E$.

We note that when $L = E$, Lemma 2 reduces to Lemma 1 in [12]. Lemma 2 is a generalization since the set $L$ can be any subset of $E$.

*Proof:* The lemma can be proven by contradiction. We assume that there exist a subset $L \subset E$ and a link $l \in L$ such that $\sum_{j \in I_L(l)} \lambda_j > d_L(l)$. Since $\vec{\lambda}$ is within $\mathring{\Lambda}$, it can be stabilized by some scheduling policy. However, at any time, any schedule must satisfy the interference constraints and thus, cannot serve more than $d_L(l)$ links out of $I_L(l)$. Hence, the summation of any feasible service rate over $I_L(l)$ cannot exceed $d_L(l)$, which is smaller than the sum of the rates with which packets arrive at $I_L(l)$. Therefore, the network is unstable, which contradicts our assumption. The result of Lemma 2 then follows. ∎

*Lemma 3:* Assume that $Y$ is the set of unstable links under *GMS* with offered load $\vec{\lambda}^*(\epsilon) = \vec{\nu}^* + \epsilon\vec{e}_Y$, then for all $l \in Y$, $\sum_{j \in I_Y(l)} \lambda_j^*(\epsilon) > 1$.

*Proof:* Note that $\vec{\nu}^* \in Co(\mathcal{M}_Y)$ is a convex combination of elements of $\mathcal{M}_Y$. For each element $\vec{M} \in \mathcal{M}_Y$, if none of the links in $I_Y(l) \backslash \{l\}$ is picked, then $l$ must be picked. Hence, $\sum_{j \in I_Y(l)} M_l \geq 1$. We then have $\sum_{j \in I_Y(l)} \nu_l \geq 1$ and $\sum_{j \in I_Y(l)} \lambda_l^*(\epsilon) > 1$ for all $\epsilon > 0$. ∎

*Lemma 4:* Assume that $\vec{\nu}^* \in \frac{1}{d}\mathring{\Lambda}$ for some $d \geq 1$ and that $Y$ is the corresponding set of unstable links under *GMS*, then for all links $l \in Y$, its interference degree in $Y$ must be larger than $d$, i.e., $d_Y(l) > d$.

[2]Note that there exists some small $\epsilon > 0$ such that $\vec{\lambda}^*(\epsilon) \in \sigma\mathring{\Lambda}$ because $\vec{\nu}^* \in \sigma^*\Lambda \subset \sigma\mathring{\Lambda}$.

*Proof:* Again, we prove by contradiction. Suppose that there is a link $l \in Y$ with $d_Y(l) \leq d$. Pick $\vec{\lambda}^*(\epsilon) = \vec{\nu}^* + \epsilon\vec{e}_Y$ such that $\vec{\lambda}^*(\epsilon)$ is strictly within $\frac{1}{d}\Lambda \subset \frac{1}{d_Y(l)}\Lambda$, we have $\sum_{j \in I_Y(l)} \lambda_j^*(\epsilon) \leq 1$ from Lemma 2. However, by Lemma 3, we should have $\sum_{j \in I_Y(l)} \lambda_j^*(\epsilon) > 1$. This is a contradiction. Hence, $d_Y(l)$ must be larger than $d$. ∎

From Lemma 4, we can derive the following main result.

*Proposition 2:* Given a network graph $G(V, E, I)$, assume that a sequence of links $\{l_1, l_2, \ldots, l_{|E|}\}$ and a sequence of sets $\{L_1, L_2, \ldots, L_{|E|}, L_{|E|+1}\}$ with $L_1 = E$ and $L_{|E|+1} = \emptyset$ satisfy that $L_{i+1} = L_i \backslash \{l_i\}$ and $d_{L_i}(l_i) \leq d$ for all $1 \leq i \leq |E|$ with some $d \geq 1$. Then the local-pooling factor is bounded by $\frac{1}{d}$, i.e., $\sigma^* \geq \frac{1}{d}$.

*Proof:* We prove the proposition by a contradiction. Suppose that $\sigma^* < \frac{1}{d}$, then there exists $\vec{\mu}^*, \vec{\nu}^* \in Co(\mathcal{M}_Y)$ such that $\sigma^*\vec{\mu}^* \succeq \vec{\nu}^*$. Further, $\vec{\nu}^* \in \frac{1}{d}\mathring{\Lambda}$ since $\sigma^* < \frac{1}{d}$. Let $Y$ denote the corresponding set of unstable links. We now show that $Y$ must be $\emptyset$, which is a contradiction.

If $Y \neq \emptyset$, we can pick the link $l \in Y$ with the smallest index in the set $\{l_1, l_2, \ldots, l_{|E|}\}$, say $l = l_j$. Then we have that all links $l_i \notin Y$ for $1 \leq i < j$ and hence, $Y \subset L_j$. Since $\vec{\nu}^* \in \frac{1}{d}\mathring{\Lambda}$ and $l_j \in Y$, we have $d_Y(l_j) > d$ from Lemma 4. Since we also have $d_Y(l_j) \leq d_{L_j}(l_j)$ from $Y \subset L_j$, and $d_{L_j}(l_j) \leq d$ from our assumption, we arrive at a contradiction and, thus $Y = \emptyset$. ∎

Clearly, if there exists a number $d$ such that $d_E(l) \leq d$ for all $l \in E$, then the assumption of Proposition 2 holds for any sequence of links $\{l_1, l_2, \ldots, l_{|E|}\}$. Hence, $\sigma^* \geq \frac{1}{d}$ and the efficiency ratio of *GMS* is no smaller than $\frac{1}{d}$. Note that a similar conclusion has been drawn for Maximal Scheduling. In [12], it has been shown that if $d_E(l) \leq d$ for all $l \in E$, then given $\vec{\lambda} \in \frac{1}{d}\mathring{\Lambda}$, a Maximal Scheduling policy can stabilize the network. However, Proposition 2 is in fact much stronger than the results in [12] and the efficiency ratio of *GMS* can often be shown to be larger than that of Maximal Scheduling. We highlight this important difference with the following example.

*Example (Edge Effect)*: Consider $N + 1$ nodes $n_1, n_2, \ldots, n_{N+1}$ lying in a straight line from left to right. Each node is connected only to its immediate neighbors. We denote link $(n_i, n_{i+1})$ by $l_i$. Assume the 1-hop interference model. For this network graph, since $d_E(l) \leq 2$ for all links, the efficiency ratio of *GMS* is no smaller than $\frac{1}{2}$. However, *GMS* in fact achieves the full capacity for this graph The reason is that there always exists a link at the end of the line with interference degree of 1. The existence of this link in fact determines the efficiency ratio of *GMS*. To see this, we pick the sequence of links in Proposition 2 as $\{l_1, l_2, \ldots, l_N\}$. We first look at link $l_1$ on the end of the line. Let $L_1 = E$. Since $d_{L_1}(l_1) = 1$, the assumption of Proposition 2 holds for $i = 2$. Now, we let $L_2 = L_1 \backslash \{l_1\}$ and move our attention to the next link $l_2$. Since $d_{L_2}(l_2) = 1$, the assumption of Proposition 2 holds for $i = 2$. We can apply this procedure iteratively to links $l_3, l_4, \ldots, l_N$. Therefore, after the $N$-th iteration, we will have sequences of $\{l_1, l_2, \ldots, l_N\}$ and $\{E = L_1, L_2, \ldots, L_{N+1} = \emptyset\}$ satisfying $L_{i+1} = L_i \backslash \{l_i\}$ and $d_{L_i}(l_i) \leq 1$ for all $1 \leq i \leq N$. Then from Proposition 2, $\sigma^* \geq 1$.

Although the techniques of [17] can also be used to draw the same conclusion that *GMS* achieves full capacity in the simple linear network discussed above, our emphasis here is to illustrate an interesting "edge effect" of *GMS*, which has not been studied in prior works [17], [20], [21]. The example illustrates that the worst-case efficiency ratio of *GMS* depends more on those links with the smallest interference degree. For uniform networks, such links often fall on the edge of the network. Hence, we refer to this property as the "edge effect." However, note that in general such links could also lie in the interior of the network. In the next two sections, we make this intuition rigorous by providing a procedure to derive a lower bound of the efficiency ratio of arbitrary network graphs, and a condition for the worst-case efficiency ratio for a class of graphs.

### B. An iterative approach

We present the procedure in Algorithm 1, which bounds the local-pooling factor of the underlying graph, i.e., the efficiency ratio of *GMS*. At each iteration, the algorithm picks up a link and check the interference degree of the chosen link in the remaining network graph.

**Algorithm 1** Iterative analysis procedure

Initialization: $L_1 \leftarrow E$, $d \leftarrow 1$

1: **for** $1 \leq i \leq |E|$ **do**
2: $\quad \boxed{\text{Choose a link } l_i \text{ from } L_i}$
3: $\quad$ **if** $d_{L_i}(l_i) \geq d$ **then**
4: $\quad\quad d \leftarrow d_{L_i}(l_i)$
5: $\quad$ **end if**
6: $\quad L_{i+1} \leftarrow L_i \backslash \{l_i\}$
7: **end for**
8: **return** $d$

Let $d_e$ denote the returned value at the end of the algorithm. We show that the local-pooling factor $\sigma^*$ of the graph $G(V, E, I)$ is at least $\frac{1}{d_e}$.

*Lemma 5:* Given $G(V, E, I)$, if we obtain $d_e$ from Algorithm 1 with a sequence of $\{l_1, l_2, \ldots, l_{|E|}\}$, then $\sigma^* \geq \frac{1}{d_e}$.

*Proof:* The lemma directly follows from Proposition 2. Note that the resulting sequence of links $\{l_1, l_2, \ldots, l_{|E|}\}$ and the corresponding sequence of sets $\{E = L_1, L_2, \ldots, L_{|E|+1} = \emptyset\}$ satisfy two conditions of $L_{i+1} = L_i \backslash \{l_i\}$ and $d_{L_i}(l_i) \leq d_e$ for all $1 \leq i \leq |E|$. Hence, by Proposition 2, $\sigma^* \geq \frac{1}{d_e}$. ∎

The outcome of the algorithm depends on the sequence of links chosen. One possibility is to choose at each iteration $i$ the link with the smallest interference degree in $L_i$, i.e., in line 2 of Algorithm 1, we choose $l_i$ such that

$$l_i \leftarrow \operatorname*{argmin}_{k \in L_i} d_{L_i}(k). \tag{4}$$

This choice of $l_i$ tends to produce a smaller value of $d_e$. This procedure can be used to estimate the local-pooling factors of arbitrary network graphs.

It is worth noting that there is some similarity between our iterative procedure and the scheduling algorithms proposed in [19]. Given a network graph, they both order links based on some topological structure of the graph, and tackle the links in the corresponding order. However, they were meant to serve completely different purposes: Algorithm 1 is merely an analytical procedure used to compute the performance bounds of *GMS*. In contrast, the algorithms of [19] are actually used to generate link schedules, and they are not related to *GMS*.

### C. The worst-case local-pooling factor over a class of graphs

We are often interested in the worst-case efficiency ratio of a scheduling policy for a class of network graphs. This information is useful when the exact network topology is unknown. Let $\mathbb{P}$ be a set of network graph with certain topological properties and let $\sigma^+(\mathbb{P})$ and $\gamma^+(\mathbb{P})$ denote the worst-case local-pooling factor and the worst-case efficiency ratio, respectively, over all graphs in $\mathbb{P}$, i.e.,

$$\sigma^+(\mathbb{P}) := \inf\{\sigma^*(G) \mid G \in \mathbb{P}\}, \gamma^+(\mathbb{P}) := \inf\{\gamma^*(G) \mid G \in \mathbb{P}\}.$$

We have $\sigma^+(\mathbb{P}) = \gamma^+(\mathbb{P})$ from Proposition 1.

We next use the methodology of Section IV-B to derive a condition for a lower bound of $\sigma^+(\mathbb{P})$. Given $\mathbb{P}$, define $d^+(\mathbb{P})$ to be a positive integer with the following property: *For any $G \in \mathbb{P}$, there must exist a link $l^*$ such that $d(l^*) \leq d^+(\mathbb{P})$ and further, $G \backslash \{l^*\} \in \mathbb{P}$.* We call $d^+(\mathbb{P})$ as the *recurrent interference degree* of $\mathbb{P}$. The following proposition shows that if we can find such a recurrent interference degree $d^+(\mathbb{P})$, the worst-case efficiency ratio of *GMS* is bounded by $\frac{1}{d^+(\mathbb{P})}$, i.e., $\sigma^+(\mathbb{P}) \geq \frac{1}{d^+(\mathbb{P})}$.

*Proposition 3:* Given a network graph $G(V, E, I) \in \mathbb{P}$ with a recurrent interference degree $d^+(\mathbb{P})$, the local-pooling factor is bounded by $\sigma^*(G) \geq \frac{1}{d^+(\mathbb{P})}$.

*Proof:* Proposition 3 can be proven as Lemma 5. Since $G(V, E, I) \in \mathbb{P}$, there exists link $l_1^* \in E$ with $d(l_1^*) \leq d^+(\mathbb{P})$ and $G \backslash \{l_1^*\} \in \mathbb{P}$. Let $L_1 = E$ and $L_2 = E \backslash \{l_1^*\}$. Since $G \backslash \{l_1^*\} \in \mathbb{P}$, there exists link $l_2^* \in L_2$ with $d_{L_2}(l_2^*) \leq d^+(\mathbb{P})$ and $G \backslash \{l_1^*, l_2^*\} \in \mathbb{P}$. Repeating this procedure until no link remains, we obtain a sequence of links $\{l_1^*, l_2^*, \ldots, l_{|E|}^*\}$ and a sequence of sets $\{E = L_1, L_2, \ldots, L_{|E|+1} = \emptyset\}$ satisfying $L_{i+1} = L_i \backslash \{l_i^*\}$ and $d_{L_i}(l_i^*) \leq d^+(\mathbb{P})$ for all $1 \leq i \leq |E|$. Hence, from Proposition 2, we conclude that $\sigma^*(G) \geq \frac{1}{d^+(\mathbb{P})}$. ∎

In the following section, we show how to apply Proposition 3 to a class of network graphs.

## V. Tree Network Graphs Under the $K$-hop Interference Model

We first study the efficiency ratio of *GMS* for tree networks. In [17], [20], it has been shown that *GMS* achieves the full capacity in tree networks under the 1-hop interference model. We now show how to use the result in the previous section to prove that *GMS* achieves full capacity for tree network topologies under $K$-hop interference model. (This result was shown in [21] by using a different approach.) Let $\mathbb{T}^K$ be the set of network graphs whose topology forms a tree and the interference relationship is governed by the $K$-hop

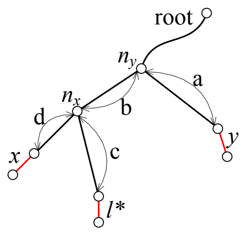



Fig. 3. Tree network graph with the deepest link $l^*$. Two links $x, y \in I_E(l^*)$ interfere with each other.

interference constraints. Recall that in the $K$-hop interference model, any two links within a $K$-hop distance cannot transmit at the same time.

*Proposition 4: GMS* achieves the full capacity for tree networks under the $K$-hop network model, i.e., $\sigma^+(\mathbb{T}^K) = 1$.

*Proof:* It is sufficient to show that $d^+(\mathbb{T}^K) = 1$ from Proposition 3.

Consider a tree network graph $G_t(V, E, I) \in \mathbb{T}^K$. We define the depth of link $l$ in $E$, denoted by $D(l)$, as the number of hops from link $l$ to the root node of the tree. Let $l^*$ be the link with the largest depth, i.e., $l^* := \mathrm{argmax}_{l \in E} D(l)$. Since $l^*$ is a leaf link of the tree, $G_t \backslash \{l^*\}$ is still a tree, and thus it belongs to $\mathbb{T}^K$.

We next show that the interference degree $d(l^*)$ of link $l^*$ is 1. It suffices to show that any two links $x, y \in I(l^*)$ interfere with each other. Let $n_x$ (or $n_y$) denote the closest common parent node of $x$ (or $y$) and $l^*$. Note that both $n_x$ and $n_y$ lie on the line from link $l^*$ to the root node. Without loss of generality, we assume that $n_y$ is a parent of $n_x$ as shown in Fig. 3. Let $a$, $b$, $c$, and $d$ denote the number of links placed between link $y$ and node $n_y$, between node $n_y$ and node $n_x$, between node $n_x$ and link $l^*$, and between node $n_x$ and link $x$, respectively. We have the following constraints.

- $a + b + c \leq K - 1$ since link $y$ interferes with link $l^*$.
- $d \leq c$ since link $l^*$ has the maximum depth.

We thus have $a+b+d \leq K-1$. In other words, any two links $x, y \in I(l^*)$ interfere with each other, and hence, $d(l^*) = 1$.

In summary, for a graph $G_t(V, E, I) \in \mathbb{T}^K$, there exists a link $l^* \in E$ with the largest depth and its interference degree is $d(l^*) = 1$. Further, $G_t \backslash \{l^*\} \in \mathbb{T}^K$. Therefore, we conclude that $\mathbb{T}^K$ has a recurrent interference degree $d^+(\mathbb{T}^K) = 1$ and the result of Proposition 4 follows. ■

Proposition 4 show that *GMS* is a throughput-optimal scheduling policy in tree networks under $K$-hop interference models. However, when the network topology is not a tree, in general *GMS* will not have an efficiency ratio of 1. In fact, whenever $K \geq 2$, we can construct network topologies, in which the efficiency ratio of *GMS* can be arbitrarily small under the $K$-hop interference model. (We refer readers to [27] for the construction of these topologies.) As the reader can see in [27], these topologies are somewhat artificial and may not exist in practice. On the other hand, in our prior work [26], we have shown that *GMS* achieves $\frac{\tilde{d}}{2\tilde{d}-1}$ under the 1-hop interference model, where $\tilde{d}$ is the largest node degree of the network graph. This suggests that we may be able to obtain improved bounds on the worst-case performance limits of *GMS* when there are additional constraints on the network topology. Therefore, in the next section, we focus on geometric unit-diskgraphs and revisit the question of the worst-case efficiency ratio of *GMS*.

## VI. Estimates of the Local-Pooling Factor for Geometric Unit-disk Network Graphs

In this section, we are interested in the performance of *GMS* for undirected unit-disknetwork graphs, in which the connectivity between nodes and the interference between links depend on their geometric locations. We assume that nodes lie on a finite two-dimensional space. We also assume that two nodes $n_i$ and $n_j$ form a link if their distance $s(n_i, n_j)$ is less than the communication range $c$, and two links $l_i(n_i^1, n_i^2)$ and $l_j(n_j^1, n_j^2)$ interfere with each other if the distance between any two nodes, one from each pair of nodes $\{n_i^1, n_i^2\}, \{n_j^1, n_j^2\}$, is less than the interference range $r$. We say that a unit-disknetwork graph operates under the $K$-*distance* (interference) model if $r = (K-1)c$, where $K$ is an integer no smaller than 2. Scheduling algorithms for these types of networks have been studied by many researchers, e.g., in [12], [14], [28], [29]. It has been shown that distributed scheduling algorithms can achieve $O(1)$ fraction of the optimal performance. More specifically, Chaporkar *et al.* [12] have shown that the efficiency ratio of Maximal Scheduling is bounded by $\frac{1}{8}$ in arbitrary unit-diskgraphs under the 2-distance model, and Sharma *et al.* [14] have shown that it is no smaller than $\frac{1}{49}$ under any $K$-distance model. In this section, we will show that *GMS* typically has better efficiency ratios than Maximal Scheduling studied in [12], [14].

Our methodology is again based on Proposition 3. Note that the edge links in a unit-diskgraph typically have a smaller interference degree than the links in the middle of the graph. If we can bound the interference degree of some edge links $l$ to a number $d$, we can then use Proposition 3 to show that the efficiency ratio is $\frac{1}{d}$. We will use the methodology first on the 2-distance model, then on $K$-distance models.

### A. Unit-disk graphs under the 2-distance model

Let $\mathbb{G}_g$ denote the set of graphs conforming to geometric unit-diskconstraints. Given a unit-disknetwork graph $G_g(V, E, I) \in \mathbb{G}_g$, we can assign a two-dimensional coordinate $(x, y)$ for each node. We say that node $A$ is to the left of node $B$ if $A$'s $x$-coordinate is less than $B$'s $x$-coordinate. Then for each link $l$, we can define the left end-point (i.e., node) $n^L(l)$ and right end-point $n^R(l)$. If the two end-points have the same $x$-coordinate, we assign them to the left or the right arbitrarily. We consider the set of all right nodes of all links $N_V^R = \{n^R(l) \in V \mid l \in E\}$. We say node $n$ in $N_V^R$ is located at the edge if there exist a line through node $n$ such that all other nodes in $N_V^R$ are in the *interior* of one of the half-planes. Note that since the graph is on a two-dimensional finite space, there always exists some right node that is on the edge. Let $n_V^R$ denote the edge node that has the smallest $x$-coordinate in $N_V^R$. Then, all other nodes of $N_V^R$ are in the interior of a half-plane (see Fig. 4) whose boundary is through $n_V^R$. We define a left-most link as a link whose right node is

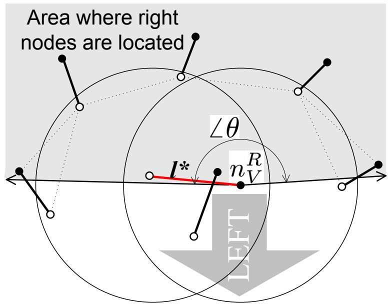



Fig. 4. Geometric network graph under the 2-distance model. Downward is the left direction of the coordinate system as indicated by a big arrow. For each link, its left node is colored in white and its right node in black. The node $n_V^R$ is the left-most right node, the link $l^*$ is the left-most link. Note that all other right nodes must be within an angle of less than $180^o$ from $n_V^R$. This figure shows how 6 other links can be placed within the interference range of $l^*$ and they do not interfere with each other. Note that each node of the 6 links must be outside an interference range of $c$ of each other, and further, their right node must be inside an angle less than $180^o$ from $n_V^R$.

$n_V^R$. Assuming that every node in $V$ is connected by some links of $E$ (otherwise, we can remove the node from $V$), we can always find at least one left-most link $l^*$ in $E$.

Let $\mathbb{G}_g^K$ denote the set of unit-disknetwork graphs under the $K$-distance model. The following lemma specifies the performance limits of *GMS* in unit-disknetwork graphs under the 2-distance model.

*Proposition 5:* The worst-case efficiency ratio of *GMS* in geometric unit-diskgraphs under the 2-distance model is $\frac{1}{6}$, i.e., $\gamma^+(\mathbb{G}_g^2) \geq \frac{1}{6}$.

*Sketch of proof:* From Proposition 1, it suffices to show that $\sigma^+(\mathbb{G}_g^2) \geq \frac{1}{6}$. Since a unit-disknetwork graph $G(V, E, I) \in \mathbb{G}_g^2$ has at least one left-most link $l^*$ and $G(V, E, I) \backslash \{l^*\} \in \mathbb{G}_g^2$, it suffices to show that $d(l^*) \leq 6$. It then follows that $\mathbb{G}_g^2$ has a recurrent interference degree $d^+(\mathbb{G}_g^2) \geq 6$, and $\sigma^+(\mathbb{G}_g^2) \leq \frac{1}{6}$ by Proposition 3. We refer the readers to Appendix B for the detailed proof of $d(l^*) \leq 6$. In Fig. 4, we show how 6 links that do not interfere with each other can be placed within the interference range of $l^*$. In Appendix B, we show that this is the largest number of non-interfering links one can put in $I(l^*)$.

*Remark:* The key step in the proof of Proposition 3 is to bound the number of neighboring links that can be activated simultaneously. Although the techniques that we used in Appendix B have some similarity to those in [12], in order to improve the bound from 8 (in [12]) to 6, we have to be much more careful in the analysis. Specifically, the left-most link must be carefully chosen (as described above), and more cases of network topology must be considered. For details, please refer to Appendix B.

Recall that Maximal Scheduling achieves an efficiency ratio of $\frac{1}{8}$ in unit-diskgraphs under the 2-distance model. Our result shows that with some increase in computational complexity[3], *GMS* indeed outperforms Maximal Scheduling. In the rest section, we show that the performance gap is even bigger for $K > 2$.

[3]The complexity of state-of-the-art distributed *GMS* algorithms [18] is $O(|E|)$, which is higher than that of distributed Maximal Scheduling algorithms [30], which is $O(\log |E|)$.

### *B. Unit-disk graphs under $K$-distance models*

It is well known in the literature that the worst-case efficiency ratio of Maximal Scheduling in unit-diskgraphs degrades when $K$ increases [12], [14]. We next show that this is not the case for *GMS*. In fact, the worst-case efficiency ratio of *GMS* tends to increase as $K$ increases. In the next lemma, we compare two graphs $G_1(V, E_1, I) \in \mathbb{G}_g^{K_1}$ and $G_2(V, E_2, I) \in \mathbb{G}_g^{K_2}$ with $K_1 > K_2$. Note that both $G_1$ and $G_2$ have the same set of nodes $V$ and have the same interference range $r$. However, the communication range of $G_1$ is $c_1 = \frac{r}{K_1 - 1}$, which is smaller than that of $G_2$, (i.e, $c_2 = \frac{r}{K_2 - 1}$).

*Proposition 6:* Given a set of nodes $V$ and their location, if $K_1 > K_2$, the local-pooling factor of the network graph $G_1(V, E_1, I) \in \mathbb{G}_g^{K_1}$ is no smaller than the local-pooling factor of the network graph $G_2(V, E_2, I) \in \mathbb{G}_g^{K_2}$, i.e., $\sigma^*(G_1) \geq \sigma^*(G_2)$.

*Proof:* Note that the set of nodes is the same and the interference range is also identical for both $G_1$ and $G_2$. Suppose that we have a subset $L \subset E_1$ in $G_1$ and two maximal scheduling vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$ such that $\sigma\vec{\mu} \succeq \vec{\nu}$. If the same vectors $\vec{\mu}, \vec{\nu}$ are also valid maximal scheduling vectors in $G_2$, then we have $\sigma^*(G_1) \geq \sigma^*(G_2)$ from the definition of the local-pooling factor. Toward this end, we first show that two maximal scheduling vectors in a subset of links in $G_1$ are also valid maximal scheduling vectors in $G_2$.

Since the interference range is fixed, $G_1$ has a smaller communication range than $G_2$. Hence, any link in $G_1$ is also a link in $G_2$, i.e., $E_1 \subset E_2$. We consider the subset $L \subset E_1$. From $E_1 \subset E_2$, we have $L \subset E_2$. Further, since the interference range is identical, the interference constraints between links in $L$ do not change. Specifically, two links in $L$ that interfere with each other under the $K_1$-distance model also interfere under the $K_2$-distance model. Hence, maximal scheduling vectors $\vec{\mu}, \vec{\nu}$ in $L$ under the $K_1$-distance model are valid maximal scheduling vectors in $L$ under the $K_2$-distance model.

Therefore, if there exist two maximal scheduling vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_L)$ satisfying $\sigma\vec{\mu} \succeq \vec{\nu}$ and a subset $L$ of links in $G_1$, the same maximal scheduling vectors and the same subset $L$ are valid for $G_2$. This implies that the local-pooling factor under the $K_2$-distance model is no greater than $\sigma$. Hence, $\sigma^*(G_1) \geq \sigma^*(G_2)$. ∎

*Remark:* Propositions 1 and 6 immediately imply that the efficiency ratio of *GMS* increases as the interference range increases. We note however that this result does not imply that the capacity region of *GMS* increases with $K$. In fact, as $K$ increases, the optimal capacity region $\Lambda$ decreases. Hence, the result suggests that as $K$ increases, the optimal capacity region $\Lambda$ decreases faster than the capacity region of *GMS*. Finally, we note that the result of Proposition 6 is also consistent with the result of [21], which shows that for a given network graph, *GMS* can achieve the optimal capacity region if the interference range $K$ is sufficiently large.

We next state Theorem 1, which is a direct consequence of Propositions 5 and 6.

*Theorem 1:* The worst-case efficiency ratio of *GMS* in geo-

metric unit-diskgraphs under $K$-distance models is no smaller than $\frac{1}{6}$, i.e., $\gamma^+(\mathbb{G}_g^K) \geq \frac{1}{6}$ for $K \geq 2$.
How tight is this bound? We next present a network graph in $\mathbb{G}_g^K$ with a local-pooling factor of $\frac{1}{3}$.

*Lemma 6:* There exists a large number $K_0$ such that for all $K > K_0$ and $\sigma$ arbitrarily close to $\frac{1}{3}$, some geometric unit-diskgraph $G(V, E, I) \in \mathbb{G}_g^K$ has the local-pooling factor no larger than $\sigma$, i.e., $\sigma^*(G) \leq \sigma$.

It suffices to construct a graph such that there exists two vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_E)$ that satisfy $\sigma\vec{\mu} \succeq \vec{\nu}$. Due to lack of space, we sketch the main idea in this paper. For the detailed proof, we refer the readers to [27].

We construct a network graph $G(V, E, I) \in \mathbb{G}_g^K$ as follows. First, when $K$ is very large, we can think of a link as a point and its interference range as a circle with radius $r$ because the communication range is close to zero. Second, we form two set of links $L_1$ and $L_2$. Suppose that $|L_1|$ and $|L_2|$ are finite but very large, and $|L_1| = |L_2|$. Remember that we approximate a link by a point. The links from $L_1$ form a circle $C_1$ with radius $R$ at origin $O$, and links from $L_2$ form another circle $C_2$ with radius $R + \frac{\sqrt{3}}{2}r$ at the same origin $O$. In Fig. 5, we show the small arcs from the two circles. Since the radius $R$ is very large, the two arcs can be approximated by two parallel lines. Since $|L_1|$ and $|L_2|$ are very large, there exists a link at almost every point of the two arcs.

We now find two maximal scheduling vectors $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_{L_1 \cup L_2})$. To form $\vec{\mu}$, take any maximal schedules of the form in Fig. 5(a), where active points (i.e., links) are colored in black. Since $|L_1|$ and $|L_2|$ are very large, there will be a large number of such maximal schedules and we produce $\vec{\mu}$ by taking the convex combination with equal weights of these schedules. Similarly, to form $\vec{\nu}$, we take maximal schedules of the form in Fig. 5(b) and produce $\vec{\nu}$ by taking the convex combination with equal weights of them. Clearly, the maximal schedules in Fig. 5(a) are more efficient than those in Fig. 5(b). We next show that the ratio of $\vec{\mu}, \vec{\nu}$ is close to $\frac{1}{3}$.

Assuming that points (i.e., links) are uniformly distributed on $C_1$ and $C_2$, then the distance between activated links in Fig. 5(a) is approximately $\frac{1}{3}$ of the distances between activated links in Fig. 5(b). Hence, the schedules that form $\vec{\mu}$ serves 3 times more links than the schedules that form $\vec{\nu}$. We thus obtain that $\frac{1}{3}\vec{\mu}$ is approximately equal to $\vec{\nu}$. In [27], we show this with a more formal proof and conclude that $\sigma\vec{\mu} \succeq \vec{\nu}$ with $\sigma$ close to $\frac{1}{3}$.

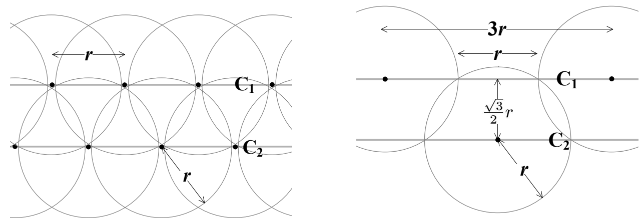
Lemma 6 leads to the following corollary.

*Corollary 1:* There exists a geometric unit-disknetwork graph $G(V, E, I) \in \mathbb{G}_g^K$ with $K \geq 2$, in which the efficiency ratio of *GMS* is no more than $\frac{1}{3}$.

*Proof:* From Lemma 6, there exist a number $K_0$ and graphs $G(V, E, I) \in \mathbb{G}_g^K$ for all $K \geq K_0$ such that $\sigma^*(G) \leq \frac{1}{3}$. By Proposition 6, we also have network graphs $G_K \in \mathbb{G}_g^K$ for all $K \leq K_0$ such that $\sigma(G_K) \leq \sigma(G_{K_0}) \leq \frac{1}{3}$. Therefore, we have $\sigma^+(\mathbb{G}_g^K) \leq \frac{1}{3}$ for all $K \geq 2$. ■

From Theorem 1 and Corollary 1, we can bound the worst-case efficiency ratio of *GMS in arbitrary geometric unit-disknetwork graphs under the $K$-distance model* as

$$\frac{1}{6} \leq \gamma^+(\mathbb{G}_g^K) \leq \frac{1}{3}. \tag{5}$$




(a) An instance of (dense) maximal schedules $\vec{M}_i$; $\vec{\mu} \in Co(\{\vec{M}_i\})$

(b) An instance of (sparse) maximal schedules $\vec{M}_j$; $\vec{\nu} \in Co(\{\vec{M}_j\})$

Fig. 5. A geometric unit-disknetwork graph $G(V, E, I) \in \mathbb{G}_g^K$ and $\vec{\mu}, \vec{\nu} \in Co(\mathcal{M}_E)$ such that $\frac{1}{3}\vec{\mu} \succeq \vec{\nu}$. With $K \to \infty$, we assume that a link is a point and its interference range is a circle with radius $r$. Figures illustrate an instance of maximal schedules from $\vec{\mu}$ and $\vec{\nu}$, respectively. Note that since links are uniformly and closely placed on circles $C_1$ and $C_2$ (a small fraction of them is shown in the figures), the interference range of active links in each maximal schedule must cover $C_1$ and $C_2$. Let $\vec{\mu}$ consist of dense maximal schedules and let $\vec{\nu}$ consist of sparse maximal schedules. From the uniform placement of (finite) links on $C_1$ and $C_2$, the time required to serve all links for a unit time is determined by the distance between two neighboring active links in $C_1$ (or $C_2$). Since the distance is $r$ in dense maximal schedules and $3r$ in sparse maximal schedules, we have $\frac{1}{3}\vec{\mu} \succeq \vec{\nu}$.

## VII. Conclusion

In this paper, we have provided new analytical results on the achievable performance of *GMS* for a large class of network topologies under general $K$-hop interference models. We first provide an equivalent characterization of the efficiency ratio of *GMS* through the local-pooling factor of the underlying graph. We then provide an iterative procedure to estimate the local-pooling factor of arbitrary graphs. This new procedure allows us to estimate the worst-case efficiency ratio of *GMS* for a large set of network graphs and interference models. In particular, we observe that *GMS* achieves the optimal capacity region in tree networks *under the $K$-hop interference model*. Further, in *geometric unit-disknetwork topologies* under the $K$-distance interference model, we show that the worst-case efficiency ratio of *GMS* increases with $K$, and is between $\frac{1}{6}$ and $\frac{1}{3}$.

For future work, there remain many interesting open problems in these directions. For example, it has been empirically shown in [2], [6] that *GMS* achieves the optimal performance in a variety of network settings. This suggests that our bounds on the worst-case efficiency ratio for unit-diskgraphs could be further improved. Further, it would be an interesting question whether these results can be extended to interference models other than the geometric unit-diskmodel, e.g., SNR-based interference model, and non-uniform disk model that incorporates the effects of varying power levels. Finally, we note that there are efforts to develop high-performance scheduling algorithms by ordering links or nodes [19], [31]. It is an interesting direction to explore since, in a certain sense, *GMS* introduces dynamic ordering of links based on the queue lengths.

## Appendix

### A. Proof of Lemma 1

Assume that $|E|$ is finite. Since for all $L \subset E$, the set of maximal schedules $\mathcal{M}_L$ has finite elements. Then its convex hull $Co(\mathcal{M}_L)$ is bounded and closed, and thus, compact.

By definition of $\sigma^*$, for any $k > 0$, there must exist a subset $L_k$, and two vectors $\vec{\mu}_k, \vec{\nu}_k \in Co(\mathcal{M}_{L_k})$ satisfying $(\sigma^*+\frac{1}{k})\vec{\mu}_k \succeq \vec{\nu}_k$. Hence, we can obtain a sequence $\{(\vec{\mu}_k, \vec{\nu}_k)\}$. Since the number of subsets of $E$ is finite, there must exists a subsequence $(\vec{\mu}_{k_n}, \vec{\nu}_{k_n}) \in Co(\mathcal{M}_L)\text{x}Co(\mathcal{M}_L)$ for some $L \subset E$, where x stands for the cartesian product of the sets. Since $Co(\mathcal{M}_L)$ is a compact set, $Co(\mathcal{M}_L)\text{x}Co(\mathcal{M}_L)$ is compact and hence, $\{(\vec{\mu}_{k_n}, \vec{\nu}_{k_n})\}$ has a convergent subsequence that converges to some element of $Co(\mathcal{M}_L)\text{x}Co(\mathcal{M}_L)$. Let $(\vec{\mu}_{k_i}, \vec{\nu}_{k_i})$ denote the subsequence converging to $(\vec{\mu}^*, \vec{\nu}^*) \in Co(\mathcal{M}_L)\text{x}Co(\mathcal{M}_L)$. Hence, from $(\sigma^* + \frac{1}{k_i})\vec{\mu}_{k_i} \succeq \vec{\nu}_{k_i}$ for all $k_i$, we obtain $\vec{\mu}^*, \vec{\nu}^* \in Co(\mathcal{M}_L)$ and $\sigma^*\vec{\mu}^* \succeq \vec{\nu}^*$.

### *B. Proof of Proposition 5*

In this section, we prove Proposition 5. Since a geometric unit-disknetwork graph $G(V, E, I) \in \mathbb{G}_g^2$ has at least one left-most link $l^*$ and $G(V, E, I)\backslash\{l^*\} \in \mathbb{G}_g^2$, it suffices to show that $d(l^*) \leq 6$.

Our strategy is basically to count the number of nodes that can transmit simultaneously in the interference area of $l^*$: i) We first divide the scenario into cases based on the placement of $l^*$, i.e., the angle between the link $l^*$ and the y-axis. ii) Then we visit the cases in turn, and in each case, we appropriately partition the interference area of $l^*$ to restrict the number of transmitters in each partition area. iii) Finally, we show that in each case, the total number of nodes that can transmit simultaneously is no greater than 6, i.e., $d(l^*) \leq 6$. To this end, we first provide some definitions following two facts that restrict the number of simultaneous transmitters in a small area. We will use them extensively in the proof.

Fig. 6(a) illustrates the neighborhood of our left-most link $l^*$. Let $L$ and $R$ denote the location of its left node and right node, respectively. The left direction in the (Cartesian) coordinate system is pointed out by a big gray arrow. The interference area of $l^*$ is a union of two unit disks $D_L$ and $D_R$ with radius $r$. We assume $r = 1$ for simplicity. Note that the distance[4] between $L$ and $R$ is less than 1, i.e., $\overline{LR} \leq 1$. We divide each disk into 6 equal-size sectors as shown in Fig. 6(a). The points $A$ through $J$ on the edge of the disks denote the boundary of these sectors. Note that in the area where the two disks intersect, some sectors also overlap, and the intersections form triangles. We name these triangles and remaining sectors as $P_1$ through $P_{10}$.

We define *indenpendently* interfering nodes or links as follows: Two nodes (or links) are said to *independently* interfere with link $l^*$ if both nodes (or links) interfere with link $l^*$, but do not interfere with each other. Then, we have the following fact.

- *Fact 1:* If two *independently* interfering nodes are located in the same disk, their angle at the center of the disk should be larger than $\frac{\pi}{3}$ because their distance has to be larger than 1.

Let us define the function $f(\cdot)$ as the number of independently interfering nodes, i.e., if there exists an independently interfering link $l_i$ such that one of its end-point is in $P_i$, and the other end point is NOT in $\cup_{k=1}^{i-1} P_k$, then we let $f(P_i) = 1$. Otherwise, if there exists no such link $l_i$, we let $f(P_i) = 0$. Clearly, there is at most one such link $l_i$ for each $P_i$ from Fact 1. If such a link $l_i$ exists, we let $N_i$ to denote the end point in $P_i$, and $\bar{N}_i$ to denote the other end point. We also define $f(\cdot)$ with multiple arguments as the number of independently interfering nodes in the union of the arguments, i.e., $f(P_i, P_j, P_k) := f(P_i \cup P_j \cup P_k) = f(P_i) + f(P_j\backslash P_i) + f(P_k\backslash\{P_i \cup P_j\})$ for $i < j < k$. If the arguments are mutually exclusive, i.e., for any two arguments $X$ and $Y$ satisfying $X \cap Y = \emptyset$, we have $f(P_i, P_j, P_k) = f(P_i) + f(P_j) + f(P_k)$.

In Fig. 6(a), since $R$ is the position of the left-most right node $n_V^R$, which is chosen such that all right nodes should be *strictly* inside the right half plane, we can draw two rays $Y_1$ and $Y_2$ from $R$ such that all other right nodes are located between these two rays, and the angle $\angle Y_1^* R Y_2^*$ of two rays are less than $\pi$, where $Y_1^*$ and $Y_2^*$ are the (infinite) end point of ray $Y_1$ and $Y_2$, respectively. We measure an angle clockwise. Let $A_R$ denote the area that other right nodes are located, which is lightly shaded in Fig. 6(a). If an interfering link $l = (N, \bar{N})$ has $N$ not in $A_R$, the other node $\bar{N}$ should be located in in $A_R$. In this case, from our choice of the left-most link $l^*$, $N$ should be a left node, $\bar{N}$ should be a right node, and one of them should be located in $D_L \cup D_R$ (for $l$ to be an interfering link of $l^*$). The following fact comes from our choice of $l^*$.

- *Fact 2:* All links interfering with $l^*$ have their right nodes in $A_R$.

From Fact 2, it is obvious that more (independently interfering) nodes can be located within two disks areas $D_L \cup D_R$ with a larger $\angle Y_1^* R Y_2^*$. In the sequel, we assume that $\angle Y_1^* R Y_2^*$ is very close to $\pi$ in order to obtain the largest interference degree. Finally, let $\phi$ denote the angle between $Y_1^*$ and the left-most link $l^*$ as shown in Fig. 6(a). In order to prove Proposition 5, it suffices to show that $d(l^*) \leq 6$ for $0 \leq \phi \leq \frac{\pi}{2}$ because of the symmetry.

#### *1) Case 1:* $0 \leq \phi \leq \frac{\pi}{6}$.

Fig. 6(b) illustrates such a case, where a couple of dotted lines indicates two bounds of $\phi$. We will first show that $d(l^*) \leq 7$. From $d(l^*) = f(P_1, \ldots, P_{10})$, we have $d(l^*) = f(P_1, P_2, P_4, P_5) + f(P_3, P_8) + f(P_6, P_7) + f(P_9, f_{10})$. Since it is clear that $f(P_1, P_2, P_4, P_5) \leq 4$, we will show in turn that

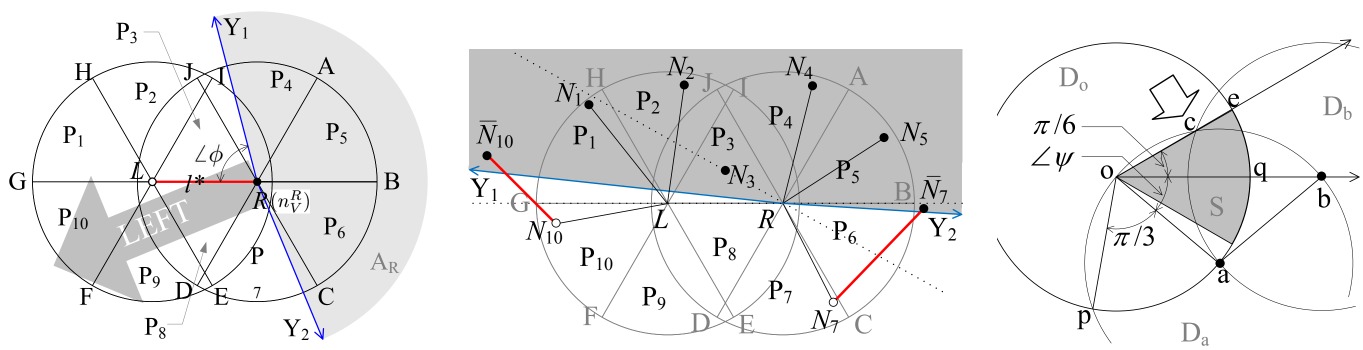
- $f(P_6, P_7) \leq 1$,
- $f(P_9, P_{10}) \leq 1$,
- $f(P_3, P_8) \leq 1$.

Then we prove that $d(l^*) < 7$ by showing that all the equalities cannot hold at the same time.

We begin with the following lemma and corollary, which will clarify the constraints between independently interfering links, in particularly when one of links has its right node outside $\{D_L \cup D_R\}$.

*Lemma 7:* For a unit disk $D_o$ at the origin $o$ (see Fig. 6(c)), assume that there are two points $a$ and $b$: point $a$ is inside $D_o$ and point $b$ is on the positive x-axis. Consider two unit disks $D_a$ and $D_b$ centered at $a$ and $b$, respectively. If the distance between $a$ and $b$ is less than 1, then the union of two disks

[4] Let $\overline{AB}$ denote the distance between two points $A$ and $B$.




(a) Left-most link $l^*$ and its neighborhood

(b) Diagrams of Case 1 ($0 \le \phi \le \frac{\pi}{6}$)

(c) Diagram of Lemma 7

Fig. 6. Neighborhood of the left-most link $l^*$, and diagrams of Case 1 and Lemma 7.

includes the shaded sector $S$ of Fig. 6(c), which is the set of points $s \in D_o$ such that $-\frac{\pi}{6} \le \angle sob \le \frac{\pi}{6}$.

*Proof:* If $b$ is within $D_o$, it is trivial. Hence, we assume that the x-coordinate of $b$ is greater than 1. Let $a$ move on arc of $D_o$ and $b$ is located at the x-axis with distance 1 from $a$ as shown in Fig. 6(c). Clearly, this is the case that the overlap of $D_a \cup D_b$ with $D_o$ is the smallest. Let $c$ denote the point in $D_o$ satisfying $\overline{ca} = \overline{cb} = 1$, which is marked by an arrow in Fig. 6(c).

We represent the coordinate of $a$ by $(\cos\psi, \sin\psi)$, where $\psi := \angle boa$ as shown in Fig. 6(c). Since $\overline{ab} = 1$ and the y-coordinate of $b$ is 0, we can also represent $b$ and $c$ as $b = (2\cos\psi, 0)$ and $c := (c_x, c_y) = (2\cos\psi - \cos(\frac{\pi}{3} - \psi), \sin(\frac{\pi}{3} - \psi))$. Hence, we obtain $c_y = \frac{1}{\sqrt{3}} c_x$. This implies that the point $c$ is on the boundary of $S$ shown in Fig. 6(c). Let $e$ denote the point that line $(o, c)$ meets $D_o$. Clearly, line segment $(o, c)$ is included in $D_a$. To conclude that $S$ is included in $D_a \cup D_b$, it suffices to show that line segment $(c, e)$ is included in $D_a \cup D_b$. Since $c$ is already included, we only need to show $e \in D_a \cup D_b$. In the following, we prove by dividing it into three sub-cases.

Note that the coordinate of $e = (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

- If $0 \le \psi \le \frac{\pi}{6}$, we have $\overline{ae} \le 1$ and thus $e \in D_a$.
- If $\frac{\pi}{6} < \psi \le \frac{\pi}{3}$, we have $1 \le 2\cos\psi < \sqrt{3}$. Hence, we obtain $(\overline{eb})^2 = (\frac{\sqrt{3}}{2} - 2\cos\psi)^2 + \frac{1}{4} \le 1$, which implies that $e \in D_b$.
- If $\psi > \frac{\pi}{3}$, we have $\overline{eb} \le 1$ and thus $e \in D_b$.

Therefore, the line segment $(c, e)$ is included in $D_a \cup D_b$, and so is $S$. ■

*Corollary 2:* Let $p$ denote the point on the edge of $D_o$ satisfying $\angle aop = \frac{\pi}{3}$, and let $q$ the point where the x-axis meets the edge of $D_o$ as shown in Fig. 6(c). Then, area $(oeqap)$ in $D_o$ is covered by the interference range from the link $(a, b)$.

*Proof:* We divide the area into three sectors: sectors $(oeq)$, $(oqa)$, and $(oap)$. Sector $(oeq)$ is covered by $D_a \cup D_b$ from Lemma 7. Sector $(oqa)$ is covered by $D_a$ if $0 \le \psi \le \frac{\pi}{3}$ and by $D_b$ if $\psi > \frac{\pi}{3}$. Finally, sector $(oap)$ is covered by $D_a$. ■

We first show $f(P_6, P_7) \le 1$ by contradiction. We divide $A_R$ into two mutually exclusive areas; $\check{A}_R := \{D_L \cup D_R\} \cap A_R$ and $\hat{A}_R := \{D_L \cup D_R\}^c \cap A_R$. Suppose that $f(P_6, P_7) = 2$. We must have $f(P_7) = 1$, and then $\bar{N}_7 \in \hat{A}_R$ because $N_7 \notin A_R$ and $\bar{N}_7 \notin \cup_{k=1}^{6} P_k$. Since $\angle BR\bar{N}_7 \le \frac{\pi}{6}$, Corollary 2 (with $a = N_7$ and $b = \bar{N}_7$) immediately implies that sector $P_6$ is covered by the interference range from link $(N_7, \bar{N}_7)$ as shown in Fig. 6(b). Hence, there cannot be an independently interfering node in $P_6$, which contradicts our assumption that $f(P_6) = 1$. Similarly, we can show $f(P_9, P_{10}) \le 1$.

Next, we show $f(P_3, P_8) \le 1$. Again suppose that $f(P_3, P_8) = 2$, we must have $f(P_8) = 1$, and $\bar{N}_8 \in \hat{A}_R$. Since $\overline{N_8\bar{N}_8} \le 1$ and $\bar{N}_8 \in \hat{A}_R$, link $l_8 := (N_8, \bar{N}_8)$ should cross either line $(L, I)$ or line $(R, J)$. If the link crosses line $(L, I)$, Corollary 2 (with $a = N_8, b = \bar{N}_8, o = L$) implies that $P_3$ is covered by the interference range from link $l_8$. The same conclusion can be drawn if link $l_8$ crosses line $(R, J)$ by using Corollary 2 with $o = R$. Hence, we obtain $f(P_3) = 0$, which contradicts our assumption that $f(P_3) = 1$.

Then we have

$$\begin{aligned} f(P_1, \ldots, P_{10}) = f(P_1, P_2, P_4, P_5) + f(P_3, P_8) \\ + f(P_6, P_7) + f(P_9, P_{10}) \le 7. \end{aligned}$$

It remains to show that $d(l^*) < 7$. For notational convenience, let $P_{i,j}$ denote $P_i \cup P_j$. If $f(P_{i,j}) = 1$, we denote the interfering node in $P_{i,j}$ and the other end-point of the corresponding link by $N_{i,j}$ and $\bar{N}_{i,j}$, respectively.

Suppose that $d(l^*) = f(P_1, P_2, P_{3,8}, P_4, P_5, P_{6,7}, P_{9,10}) = 7$. Note that we must have $f(P_1) = f(P_2) = f(P_{3,8}) = \cdots = f(P_{9,10}) = 1$. We are going to prove that this is not possible by showing $\angle Y_1^* LB + \angle GRY_2^* > 2\pi$.

$$\begin{aligned} &\angle Y_1^* LB + \angle GRY_2^* \\ &\quad = (\angle Y_1^* LN_1 + \angle N_1 LI + \angle ILB) \\ &\qquad\qquad + (\angle GRJ + \angle JRN_5 + \angle N_5 RY_2^*) \\ &\quad = \angle Y_1^* LN_1 + \angle N_5 RY_2^* + (\angle N_1 LI + \angle JRN_5) + \tfrac{2\pi}{3}. \end{aligned}$$

Let us consider each term one by one.

1) We have $\angle Y_1^* LN_1 \ge \angle \bar{N}_{9,10} LN_1 > \frac{\pi}{6}$ from $f(P_{9,10}) = 1$ and Lemma 7.
2) Similarly, we also have $\angle N_5 RY_2^* \ge \angle N_5 R\bar{N}_{6,7} > \frac{\pi}{6}$ from $f(P_{6,7}) = 1$ and Lemma 7.
3) From $\angle N_1 LN_2 > \frac{\pi}{3}$, we have $\angle N_1 LI + \angle JRN_5 > \frac{\pi}{3} + (\angle N_2 LI + \angle JRN_5)$. It is shown[5] in [12] that $(\angle N_2 LI +$

[5] If a virtual node $N^*$ is drawn in $D_R$ such that $\angle N_2 LB = \angle N^* RB$ and $\overline{N_2 L} = \overline{N^* R}$, then it has been proven that $\overline{N^* N_5} > 1$. This implies that $\angle N_2 LI + \angle JRN_5 > \frac{2\pi}{3}$. See [12] for the detailed proof.

$\angle JRN_5) > \frac{2\pi}{3}$. Then, we have $\angle N_1LI + \angle JRN_5 > \pi$. We then have $\angle Y_1^*LB + \angle GRY_2^* > 2\pi$, which leads to a contradiction because we should have $\angle Y_1^*LB + \angle GRY_2^* < \angle Y_1^*RB + GRY_2^* = \angle GRB + \angle Y_1^*RY_2^* < 2\pi$ from $\angle Y_1^*RY_2^* < \pi$. Hence, we conclude that $f(P_1, P_2, P_{3,8}, P_4, P_5, P_{6,7}, P_{9,10}) < 7$.

*2) Case 2:* $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$.

We illustrate this case in Fig. 7(a), where we divide $D_L \cup D_R$ into a different partition compared with Case 1. We first describe the new partitions before proceeding with our proof.

Let $J'$ denote the position where ray $Y_1$ meets the edge of disk $D_R$. Other points $A', B', C'$ and $D'$ are defined as the points on the edge of disk $D_R$ such that $\angle J'RA' = \angle A'RB' = \angle B'RC' = \angle C'RD' = \frac{\pi}{3}$. Similarly, we set $H'$ as the point where $Y_1$ meets the edge of disk $D_L$, and set other points $E', F', G', I'$ on the edge of disk $D_L$ such that $\angle H'LI' = \angle G'LH' = \angle F'LG' = \angle E'LF' = \frac{\pi}{3}$ as shown in Fig. 7(a). We also let $K'$ denote the point where the edges of $D_L$ and $D_R$ meet. Note that since $\overline{LR} \leq 1$, we have $\angle K'LR = \angle LRK' > \frac{\pi}{3}$. Hence, two dotted lines, which present the bounds of $\angle\phi$, go below $K'$.

We introduce two additional arcs in the figure. Arc $(L, R)$ corresponds to points $x$ between $L$ and $R$ with $\overline{K'x} = 1$ and arc $(R, D')$ corresponds to points $x$ between $R$ and $D'$ with $\overline{C'x} = 1$. We name each area in $D_L \cup D_R$ as follows:

- $Q_1$: sector $(H'LI')$, • $Q_2$: sector $(J'RA')$,
- $Q_3$: sector $(A'RB')$, • $Q_4$: sector $(B'RC')$,
- $Q_5^+$: the shaded area in $(C'RD')$,
- $Q_6^+$: the shaded area in the middle including line $(L, R)$,
- $Q_8$: the shaded sector $(F'LG')$, • $Q_9$: sector $(G'LH')$,
- $Q_7^+$: the remaining area in $D_L \cup D_R$, i.e., area $(F'LRD'E')$, surrounded by shaded areas $Q_5^+, Q_6^+, Q_8$.

Note that $Q_1$ and $Q_2$ are the only two regions that overlap. We define function $f(\cdot)$ and nodes $N_i, \bar{N}_i$ as before using $Q_i$, i.e., $f(Q_i) = 1$ implies that there exists an independently interfering link such that one end-point $N_i \in Q_i$ and the other end node $\bar{N}_i \notin \cup_{k=1}^{i-1} Q_k$.

From $d(l^*) = f(Q_1, \ldots, Q_9)$, we will prove $d(l^*) \leq 6$ by showing

- $f(Q_3) \leq 1$,
- $f(Q_8, Q_9) \leq 1$,
- $f(Q_1, Q_2, Q_6^+) \leq 2$,
- $f(Q_4, Q_5^+, Q_7^+) \leq 2$.

Since it is clear that $f(Q_3) \leq 1$, we focus on the rest.

We first show that $f(Q_8, Q_9) \leq 1$. Specifically, we will show that $f(Q_9) = 0$ if $f(Q_8) = 1$. To this end, we show that if $f(Q_8) = 1$, then $\angle \bar{N}_8LH' \leq \frac{\pi}{6}$. Then using Corollary 2 (with $a = N_8, b = \bar{N}_8, o = L$), we can conclude that $f(Q_9) = 0$. In the following, we prove that $\angle \bar{N}_8LH' \leq \frac{\pi}{6}$ when $\phi \leq \frac{\pi}{3}$.

Let us consider area $Q_9$ in detail. Fig. 7(b) illustrates the zoomed diagram. Let $Z$ denote the point on $Y_1$ satisfying $\angle ZLH' = \angle G'LZ = \frac{\pi}{6}$. Clearly, if link $(N_8, \bar{N}_8)$ crosses line segment $(Z, H')$, we have $\angle \bar{N}_8LH' \leq \frac{\pi}{6}$. We are going to show that $G'$ is the closest point in $Q_8$ to $Z$, and that $\overline{G'Z} \geq 1$. Then from $\overline{N_8\bar{N}_8} \leq 1$ and $N_8 \in Q_8$, link $(N_8, \bar{N}_8)$ should cross line segment $(Z, H')$.

Note that we have $\overline{LR} \leq 1$, $\overline{LH'} = 1$, and $\overline{H'R} \geq 1$, which imply that $\angle RH'L \leq \frac{\pi}{3}$. Then, from $\angle LH'G' = \frac{\pi}{3}$, $\overline{G'H'} = 1$, and the fact that lines $(G', H')$ and $(L, Z)$ are perpendicular, it follows that $\overline{LZ} \geq \sqrt{3}$. Since $\angle G'LZ = \frac{\pi}{6}$, we have $\overline{LZ}(\cos\angle G'LZ) > 1 = \overline{G'L}$, and conclude that $G'$ is the closest point in $Q_8$ to $Z$. Hence, it suffices to show $\overline{G'Z} \geq 1$.

Since $\angle RH'L \leq \frac{\pi}{3}$, it follows $\angle H'ZL = \angle RH'L - \angle ZLH' \leq \frac{\pi}{6}$. From $\angle H'ZL \leq \frac{\pi}{6} = \angle ZLH'$, we obtain $\overline{H'Z} \geq \overline{H'L} = 1$. Therefore, we conclude $\overline{G'Z} = \overline{H'Z} \geq 1$, which results in $f(Q_9) = 0$ from Corollary 2, and we have $f(Q_8, Q_9) \leq 1$.

Next, we prove $f(Q_1, Q_2, Q_6^+) \leq 2$ using the following lemmas.

*Lemma 8:* If three nodes $x, y, z$ satisfy that $\overline{xz} \leq 1$ and $\overline{yz} \leq 1$, then all points $t$ in triangle area $(xyz)$ satisfy $\overline{xt} \leq 1$ or $\overline{yt} \leq 1$.

Lemma 8 is obvious and we omit the proof.

*Lemma 9:* If two nodes $x$ and $y$ satisfy $\overline{xy} \leq \sqrt{3}$, and link $l = (a, b)$ with $\overline{ab} \leq 1$ intersects line segment $(x, y)$, then the link interferes with either $x$ or $y$.

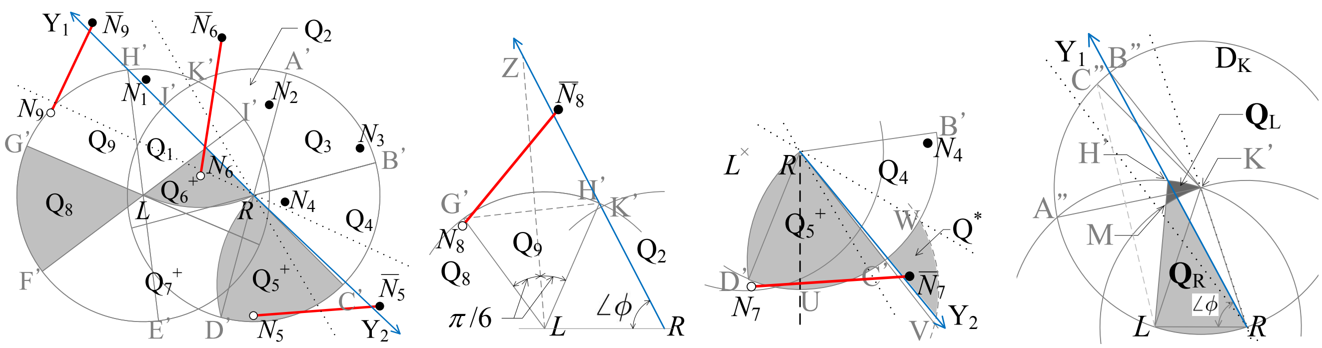
We also omit the proof of Lemma 9. It can be easily shown by drawing two nodes $x, y$ with $\overline{xy} \leq \sqrt{3}$ and a link $(a, b)$ with $\overline{ab} \leq 1$. From $\overline{ab} \leq 1$ and $\overline{xy} \leq \sqrt{3}$, we should have $\overline{ax} \leq 1$, $\overline{ay} \leq 1$, $\overline{bx} \leq 1$, or $\overline{by} \leq 1$.

We prove $f(Q_1, Q_2, Q_6^+) \leq 2$ by contradiction. Supposing $f(Q_1, Q_2, Q_6^+) = 3$, we must have $f(Q_1) = f(Q_2) = f(Q_6^+) = 1$. Note that $Q_6^+ \cap A_R = \emptyset$. Then, $f(Q_6^+) = 1$ implies that there must exists $\bar{N}_6 \in \hat{A}_R$. Note that link $l_6 := (N_6, \bar{N}_6)$ will divide $Q_1$ or $Q_2$ (or both of them) into two parts. If it divides $Q_1$ (or $Q_2$, correspondingly), from Corollary 2 node $N_1$ cannot be in the right part of $Q_1$ (or node $N_2$ cannot be in the left part of $Q_2$, correspondingly). This implies that link $l_6$ intersects either line segment $(N_1, K')$ or $(N_2, K')$. Moreover, since $\overline{N_1K'} \leq 1$ and $\overline{N_2K'} \leq 1$, from Lemma 8 node $N_6$ cannot be in the triangle area of $(N_1N_2K')$. This immediately implies that $N_6$ should be located below line segment $(N_1, N_2)$. Hence link $l_6$ must intersect line segment $(N_1, N_2)$. Now we use Lemma 9 to prove that this is not possible. In the following, we show that the distance between any two points within $Q_1 \cup Q_2$ is no greater than $\sqrt{3}$ (i.e., $\overline{N_1N_2} \leq \sqrt{3}$). Then, Lemma 9 implies that independently interfering link $l_6$ cannot exist, i.e., $f(Q_6^+) = 0$.

It is easy to see that $\max_{X \in Q_1, Y \in Q_2} \overline{XY}$ is equal[6] to $\max\{1, \overline{H'R}, \overline{LA'}, \overline{H'A'}\}$. We can show that each length is no greater than $\sqrt{3}$. (Details are available in [27].) Note that $\max_{X \in Q_1, Y \in Q_2} \overline{XY} \leq \sqrt{3}$ immediately implies that $\overline{N_1N_2} \leq \sqrt{3}$. From Lemma 9, it leads to $f(Q_6^+) = 0$, which contradicts our assumption that $f(Q_1, Q_2, Q_6^+) = 3$ and we conclude $f(Q_1, Q_2, Q_6^+) \leq 2$.

Finally, we show $f(Q_4, Q_5^+, Q_7^+) \leq 2$. To begin with, we note that if $f(Q_7^+) \geq 1$, an interfering link must cross ray $Y_2$. The link cannot pass through $Y_1$ to reach $\check{A}_R$ because

[6]For a point $x \in Q_1$, let $D_1$ and $D_2$ denote two disks centered at $x$ with radius $\max\{1, \overline{xR}\}$ and $\max\{1, \overline{xA'}\}$ respectively. Then, clearly $Q_2 \subset D_1 \cup D_2$, and thus $\overline{xY} \leq \max\{1, \overline{xR}, \overline{xA'}\}$ for all $Y \in Q_2$. Similarly, for a point $y \in Q_2$, we have $\overline{Xy} \leq \max\{1, \overline{Ly}, \overline{H'y}\}$ for all $X \in Q_1$. Therefore, we obtain $\max_{X \in Q_1, Y \in Q_2} \overline{XY} = \max\{1, \overline{H'R}, \overline{LA'}, \overline{H'A'}\}$.




(a) Diagram of Case 2 ($\frac{\pi}{6} \leq \phi \leq \frac{\pi}{3}$)

(b) Diagram of Case 2: Zoomed near sector $Q_9$

(c) Diagram of Case 2: Areas of $Q_5^+$ and $Q^*$

(d) Diagram of Case 3 ($\phi_1 < \phi \leq \frac{\pi}{2}$)

Fig. 7. Diagrams of Cases 2 and 3.

its distance would have been larger than 1. However, we can show that at most one such link $l_7 := (N_7, \bar{N}_7)$ can cross $Y_2$ as shown in Fig. 7(c), which implies that $f(Q_7^+) \leq 1$ [27].

Fig. 7(c) illustrates a scenario with $f(Q_7^+) = 1$. Let $U$ denote a point on the edge of $D_R$ with $\angle URL = \frac{\pi}{2}$. Let the coordinate of $R$ be $(0, 0)$, then the coordinate of $U$ is $(0, -1)$, and clearly node $N_7$ has its coordination $(n_x, n_y)$ with $n_x \leq 0$ and $n_y \geq -1$. Let $V$ denote the point on the extension of line $(R, C')$ satisfying $\overline{UV} = 1$ and let $W$ denote the point on the edge of $D_R$ with $\overline{UW} = 1$, as shown in the figure. Points $v$ with $\overline{Uv} = 1$ are drawn as a dashed arc. The points $V$ and $W$ are where the extended line $(R, C')$ and the disk $D_R$, respectively, meet the dashed arc. Since $N_7$ has $n_x \leq 0$ and $n_y \geq -1$, its other end node $\bar{N}_7$ should be located in the area i) above line $(C', V)$, ii) within the dashed arc centered at $U$, and iii) the exterior of $D_R$. Let $Q^*$ denote this area as shown in Fig. 7(c).

Now we prove $f(Q_4, Q_5^+, Q_7^+) \leq 2$ by contradiction. Suppose $f(Q_4, Q_5^+, Q_7^+) = 3$. Since each area can hold only one independently interfering node, we must have $f(Q_4) = 1, f(Q_5^+) = 1$ and $f(Q_7^+) = 1$. We prove that this is not possible by showing $f(Q_5^+) = 0$ when $f(Q_4) = 1$ and $f(Q_7^+) = 1$.

Let us consider a scenario with $f(Q_4) = 1$ and $f(Q_7^+) = 1$ as shown in Fig. 7(c). Note that link $l_5 := (N_5, \bar{N}_5)$ and $l_7$ cannot intersect with each other from Lemma 9, and the interference range from link $l_7$ covers lower part of $Q_5^+$. Hence, link $l_5$ can only be placed above link $l_7$. Link $l_5$ also has to go below $N_4$ (i.e., cannot cross line segment $(R, N_4)$) because otherwise we should have $f(Q_4) = 0$ from Corollary 2, which contradicts our assumption that $f(Q_4) = 1$. Also note that $\bar{N}_5 \notin Q_4$ and $\bar{N}_5 \notin Q^*$ due to the interference range from $N_4$ and $\bar{N}_7$ respectively, and $\bar{N}_5 \notin$ triangle area of $(N_4 \bar{N}_7 C')$ from Lemma 8. These along with the facts that link $l_5$ goes below $(R, N_4)$ and above $l_7$ imply that $\bar{N}_5$ cannot be located to the left of line $(N_4, \bar{N}_7)$ and hence, link $l_5$ must intersect line segment $(N_4, \bar{N}_7)$.

It remains to show that $\overline{XY} \leq \sqrt{3}$ for all $X \in Q_4$ and $Y \in Q^*$, which implies that $\overline{N_4 \bar{N}_7} \leq \sqrt{3}$. Then from Lemma 9, any link intersecting line segment $(N_4, \bar{N}_7)$ will interfere with either $N_4$ or $\bar{N}_7$, which will lead to $f(Q_5^+) = 0$. From Fig. 7(c), it is clear that $\max_{X \in Q_4, Y \in Q^*} \overline{XY} < \max\{\overline{RV}, \overline{B'V}\}$ when $\frac{2\pi}{3} \leq (\angle VRL = \pi - \phi) \leq \frac{5\pi}{6}$. From $\overline{RU} = 1$, $\overline{UV} = 1$ and $\angle VRU \geq \frac{\pi}{6}$, we have $\overline{RV} \leq \sqrt{3}$. Moreover, from $\overline{RB'} = 1$, $\angle B'RV = \frac{\pi}{3}$ and $\overline{RV} \leq \sqrt{3}$, we also have $\overline{B'V} < \sqrt{3}$. Hence, we conclude that $\max_{X \in Q_4, Y \in Q^*} \overline{XY} \leq \sqrt{3}$, and $\overline{N_4 \bar{N}_7} \leq \sqrt{3}$.

Summing up all results, we obtain $d(l^*) \leq f(Q_3) + f(Q_8, Q_9) + f(Q_1, Q_2, Q_6^+) + f(Q_4, Q_5^+, Q_7^+) \leq 6$.

*3) Case 3:* $\frac{\pi}{3} < \phi \leq \phi_1$, where $\phi_1 := \angle LRK'$ in Fig. 7(d).

Using the same partitioning approach and the techniques as in Case 2, we can show the following.

- $f(Q_3) \leq 1$.
- $f(Q_8, Q_9) \leq 1$: To prove this, we need $\angle RH'L \leq \frac{\pi}{3}$, which comes from $\overline{LR} \leq 1$, $\overline{LH'} = 1$, and $\overline{H'R} \geq 1$.
- $f(Q_1, Q_2, Q_6^+) \leq 2$: To prove this, we need to show that $\overline{H'R} \leq \sqrt{3}$ and $\overline{LA'} \leq \sqrt{3}$, which can be proven using a similar approach as in Case 2 [27].
- $f(Q_7^+) \leq 1$, and thus $f(Q_4, Q_5^+, Q_7^+) \leq 3$.

Then we prove that $d(l^*) \leq 6$ by showing that for the above equations, the equalities cannot hold all at the same time.

Suppose that $d(l^*) = 7$. Clearly, we must have $f(Q_3) = 1, f(Q_8, Q_9) = 1, f(Q_1, Q_2, Q_6^+) = 2, f(Q_4, Q_5^+, Q_7^+) = 3$. Let $l_{8,9} := (N_{8,9}, \bar{N}_{8,9})$ denote the independently interfering link in $Q_8 \cup Q_9$. Also let $\mathbf{Q}_L := Q_1 \backslash D_R$ and $\mathbf{Q}_R := \{Q_1 \cup Q_2 \cup Q_6^+\} \backslash \{Q_2 \cup \mathbf{Q}_L\}$, which are presented as dimly and lightly shaded areas, respectively, in Fig. 7(d). The two dotted lines are the bounds of $\phi$. We can show that the interference range from $l_{8,9}$ covers $\mathbf{Q}_L$. (We refer to [27] for the detailed proof.) Once the interference range from link $l_{8,9}$ includes $\mathbf{Q}_L$, we must have $f(Q_2) = 1$ and $f(\mathbf{Q}_R) = 1$ from $f(Q_1, Q_2, Q_6^+) = 2$, it follows $f(\mathbf{Q}_R, Q_2, Q_3, Q_4, Q_5^+) = 5$. However, this leads to contradiction to $Y_1^* R Y_2^* < \pi$. Let $\mathbf{N}_R$ denote the independently interfering node in $\mathbf{Q}_R$, and let $\bar{\mathbf{N}}_R$ denote the corresponding other end-point. Since $\mathbf{N}_R \notin A_R$, we must have $\bar{\mathbf{N}}_R \in \hat{A}_R$, and thus we obtain

$$\begin{aligned}\angle Y_1^* R Y_2^* &\geq \angle \bar{\mathbf{N}}_R R N_2 + \angle N_2 R N_4 + \angle N_4 R \bar{N}_5 \\ &> \frac{\pi}{6} + \frac{2\pi}{3} + \frac{\pi}{6} = \pi,\end{aligned}$$

from Lemma 7 and Fact 1. Then, we can conclude that $d(l^*) \leq 6$.

*4) Case 4:* $\phi_1 < \phi \leq \frac{\pi}{2}$.

The procedure is similar as in Case 3. We partition the interference area of $l^*$ such that the interference constraints lead to multiple inequalities, and show that all the equalities cannot hold at the same time. Due to the limited space, we refer readers to [27] for the detailed proof.

Considering all Cases 1 through 4, we conclude that $d(l^*) \leq 6$ for the left-most link $l^*$ in a geometric unit-disknetwork graph under the 2-hop interference model.

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