On the Connection-Level Stability of Congestion-Controlled Communication Networks

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***Abstract*—We are interested in the connection-level stability of a network employing congestion control. In particular, we study how the stability region of the network (i.e., the set of offered loads for which the number of active users in the network remains finite) is affected by congestion control. Previous works in the literature typically adopt a time-scale separation assumption, which assumes that, whenever the number of users in the system changes, the data rates of the users are adjusted instantaneously to the optimal and fair rate allocation. Under this assumption, it has been shown that such rate assignment policies can achieve the largest possible stability region. In this paper, this time-scale separation assumption is removed and it is shown that the largest possible stability region can still be achieved by a large class of congestion control algorithms. A second assumption often made in prior work is that the packets of a source (or user) are offered to each link along its path instantaneously, rather than passing through one queue at a time. We show that connection-level stability is again maintained when this assumption is removed, provided that a back-pressure scheduling algorithm is used jointly with the appropriate congestion controller.**

***Index Terms*—Communication networks, congestion control, connection-level dynamics, stability, time-scale separation.**

I. Introduction

CONGESTION control is a key functionality in modern communication networks. The objective of congestion control is to regulate the data rate at which each user injects data into the network such that (a) the network capacity is fully utilized, (b) excessive congestion inside the network is avoided, and (c) some form of fairness (in terms of the amount of service that each user receives) is ensured. Since the seminal work by Kelly *et al* [3], it is clear that these objectives can be mapped to a global optimization problem that maximizes the total system utility, where different fairness objectives can be achieved by appropriately choosing the utility functions. Congestion control can then be viewed as a distributed iterative solution to the afore-mentioned global optimization problem [3]–[8].

Significant advances in the understanding of congestion control have been made under this optimization framework. The results can roughly be categorized into two groups. In the first body of work, it is assumed that the number of users in the network is fixed and each user has infinite data to transfer. This research focuses on developing distributed iterative algorithms that converge to a fair rate allocation, which corresponds to the solution of the global optimization problem. Various issues have been addressed in this body of work, including global convergence of the congestion control algorithm, local stability of the equilibrium rate allocation under feedback delays, the impact of random noise, and the asymptotic behavior of the system when the number of users is large.

The second body of work studies a network with *connection-level dynamics*, i.e., when the users randomly enter and leave the network. When the users are in service, their data rates are dynamically controlled according to the congestion level in the network. A basic question in studying the connection-level dynamics is the *stability region* of the system employing congestion control. Here, by *stability*, we mean that the number of active users in the system and the queue lengths at each link in the network remain finite. The *stability region* of the system under a given congestion control algorithm is the set of offered loads under which the system is stable. This body of work typically assumes that, whenever the number of users in the system changes, the data rates of the users are adjusted *instantaneously* to the *optimal (and fair) rate allocation* computed by the global optimization problem. This model essentially assumes a *time-scale separation*, i.e., the time scale of the arrivals and departures of the users is much slower than that of the dynamics determined by the congestion control algorithms derived in the first body of work. Under this time-scale separation assumption, it has been shown that the *largest possible* stability region can be achieved by allocating data rates among the users according to certain fairness criteria [9]–[12]. Further, it has been shown in [10] that "unfair" resource allocations such as priority scheduling may not achieve the largest possible stability region. Thus, "fairness" is not merely an *aesthetic* property, but it actually has a strong global *performance* implication, i.e., in achieving the *largest possible* stability region.

In this paper, we study the connection-level stability region of congestion controlled communication networks without using the time-scale separation assumption. Note that the time-scale separation assumption arises from the following well-known observation: most of the traffic in the Internet is due

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to a small fraction of files which are extremely large. Since these files only form a small fraction of the traffic and these are ones for which congestion control can be assumed to reach a steady-state, it would seem reasonable to assume that these files arrive, stay in the system for a long time and then depart, thus leading to two time-scale behavior. While this intuition is reasonable, in reality, the distinction between small files and large files is not as precise as this model might suggest and therefore, it is hard to rigorously justify the time-scale separation assumption. We will show that, even when we remove the time-scale separation assumption, the *largest possible* stability region can still be achieved by a large class of congestion control algorithms that are derived from the optimization framework. Hence, our result reinforces the performance benefit of congestion control in a stronger sense.

A second assumption that is usually made in the congestion control literature is that the packets of a source (or user) are offered to each link along its path instantaneously, rather than going through one queue at a time. While we adopt this assumption in Sections 2-4, we will study a different model in Section 5 that directly takes into account the packet dynamics of going through one queue at a time. We will show that connection-level stability is again maintained when the "user-rates-applied-simultaneously-to-all-links" assumption is removed, provided that a back-pressure scheduling algorithm is used jointly with the appropriate congestion controller.

The rest of the paper is structured as follows. In Section II, we present the system model and review some related results in the literature. Our main result is presented in Section III, where we show that the largest stability region can be achieved even without the time-scale separation assumption. In Section IV, we extend our main result to the case with multi-path routing and the case with time-varying capacity. In Section V, we extend our main result to explicitly account for the packet dynamics of going through one queue at a time. We conclude in Section VI.

II. The System Model and Related Results

In this section, we describe our system model and review several related works. We consider a network with L links and S classes of users. Let $\mathcal{E} = \{1, 2, ..., L\}$ denote the set of links. For now, we assume that the capacity of each link $l \in \mathcal{E}$ is R^l, and users of each class s have one path through the network. (The extensions to the case with time-varying capacity and multi-path routing are treated in Section IV.) Let $H_s^l = 1$, if the path of users of class s uses link l, and $H_s^l = 0$, otherwise.

A. Congestion Control

We first motivate the congestion controller used in the paper. Note that congestion control is a key functionality in modern communication networks to efficiently utilize the network resources while avoiding excessive congestion, and to provide fair rate-allocation to the users. Without proper congestion control, the network can run into "congestion-collapse" [13], a state where the useful throughput of the network drops by orders of magnitude due to excessive congestion inside the network.

Since the seminal work [3] by Kelly *et al*, it is clear that congestion control can be modeled as a distributed solution to the following network utility maximization problem. We first consider the case when the number of users n_s of each class s is fixed, and each user has an infinite backlog to transfer. Let x_s denote the rate at which each user of class s sends data into the network, and let $U_s(x_s)$ be the utility received by the user of class s when it sends data at rate x_s. The utility function $U_s(\cdot)$ characterizes the "satisfaction level" of a user of class s when it sends data at a certain rate, and as we will soon discuss, it also corresponds to a certain fairness objective. As is typically assumed in the literature, we assume that each user of class s has a maximum data-rate limit of M_s, and the utility function $U_s(\cdot)$ is increasing, strictly concave, and twice continuously differentiable on $(0, M_s]$ [4]. Let $\vec{n} = [n_1, ..., n_S]$ and $\vec{x} = [x_1, ..., x_S]$. Congestion control can then be formulated as the following global optimization problem [3]:

$$\begin{aligned} \max_{\vec{x}:0\le x_s\le M_s, s=1,...,S} \quad & \sum_{s=1}^{S} n_s U_s(x_s) \\ \text{subject to} \quad & \sum_{s=1}^{S} H_s^l n_s x_s \le R^l \\ & \text{for all } l = 1, ..., L. \end{aligned} \tag{1}$$

1) Fair and Efficient Rate Allocation: The constraint in (1) ensures that the network utilization never exceeds the capacity, while the increasing utility functions ensure that the capacity will be utilized as much as possible. Further, the fairness objectives can be achieved by appropriately choosing the utility functions [10], [14]. For example, utility functions of the form

$$U_s(x_s) = w_s \log x_s \tag{2}$$

correspond to *weighted proportional fairness*, where $w_s, s = 1, ..., S$ are the weights. A more general form of the utility function is

$$U_s(x_s) = w_s \frac{x_s^{1-\beta}}{1-\beta}, \text{ for some } \beta > 0 \text{ and } \beta \neq 1. \tag{3}$$

Maximizing the total system utility will correspond to *maximizing weighted throughput* as $\beta \to 0$, *weighted proportional fairness* as $\beta \to 1$, and *max-min fainess* as $\beta \to \infty$.

2) A Distributed Congestion-Controller and its Convergence: Congestion control can then be viewed as a distributed iterative solution to the above global optimization problem [3]–[8]. The type of congestion controller that is most related to this paper is the so-called "dual solution" given below [4]. We associate an implicit cost q^l with each link l and let $\vec{q} = [q^1, ..., q^L]$. The following iterative algorithm can solve problem (1) with an appropriate choice of the step-size.

Algorithm $\mathcal{A}$:

At each time instant t,

- The data rate of each user of class s is determined by:

$$x_s(t) = \operatorname*{argmax}_{0\le x_s\le M_s} U_s(x_s) - x_s \sum_{l=1}^{L} H_s^l q^l(t). \tag{4}$$

- The implicit cost at each link l is updated by:

$$q^l(t+1) = \left[q^l(t) + \alpha_l \left(\sum_{s=1}^{S} H_s^l n_s x_s(t) - R^l\right)\right]^+, \quad (5)$$

where $[\cdot]^+$ denotes the projection to $[0, \infty)$ and α_l is a positive step-size for each link l.

The following proposition was shown in [4] with slightly different notation.

Proposition 1: Assume that the number of users in the system is fixed. Further, assume that the curvatures of $U_s(\cdot)$ are bounded away from zero on $(0, M_s]$, i.e., there exists a positive number γ_s for each class s such that

$$-U_s''(x_s) \geq \gamma_s > 0 \text{ for all } x_s \in (0, M_s]. \quad (6)$$

Let $\vec{x}^*$ denote the optimal solution to problem (1). Let $\mathcal{S} = \max_l \sum_{s=1}^{S} H_s^l n_s$ denote the maximum number of **users** using any link, and let $\mathcal{L} = \max_s \sum_{l=1}^{L} H_s^l$ denote the maximum number of links used by any **user**. If

$$\max_l \alpha_l \leq \frac{2}{\mathcal{S}\mathcal{L}} \min_s \gamma_s, \quad (7)$$

then Algorithm $\mathcal{A}$ converges, i.e., $\vec{x}(t) \to \vec{x}^*$ as $t \to \infty$.

Hence, when the step-sizes α_l are sufficiently small, the "dual solution" can solve the problem (1) as $t \to \infty$.

B. Connection-Level Stability

The discussion of congestion control in Section II-A has focused on the *static setting*, i.e., we have assumed that the number of users of each class is fixed and each user has an infinite amount of data to transfer. In real networks, users also exhibit *connection-level dynamics*, i.e., they randomly enter the network, have only a finite amount of data to transfer, and leave the network after the data transfer is completed. When the users are in service, their data rates are assumed to be dynamically controlled according to the congestion controller presented earlier. However, because the number of active users constantly changes, the congestion control algorithm may not be able to converge before the next time instant when a user enters or leaves the system. Hence, although convergence is an important goal for the static setting, we would be interested in other performance measures (such as stability and completion time) when there are connection-level dynamics. In particular, *Stability* is often a first-order and important performance measure to study [9]–[12], [15]. Here, by *stability*, we mean that the number of active users in the system and the queue lengths at each link in the network remain finite. To be precise, we assume that users of class s arrive to the network according to a Poisson process with rate λ_s and that each user brings with it a file for transfer whose size is exponentially distributed with mean $1/\mu_s$. The load brought by users of class s is then $\rho_s = \lambda_s/\mu_s$. Let $\vec{\rho} = [\rho_1, ..., \rho_S]$. Let $n_s(t)$ denote the number of active users of class s that still have data to inject into the system at time t and let $\vec{n}(t) = [n_1(t), ..., n_S(t)]$. Note that according to this definition of $\vec{n}(t)$, once a user injects all of its data (from the file that it brings) into the network, even though the data may not have reached the destination yet, we assume that the user will immediately leave the system. We assume that users of the same class s send data into the network at the same rate. Let $x_s(t)$ denote the data injection rate of users of class s at time t and let $\vec{x}(t) = [x_1(t), ..., x_S(t)]$.[1] The data injection rate $x_s(t)$ will be determined by the congestion control algorithm. Due to the assumptions on Poisson arrivals and exponential file size distributions, the evolution of $\vec{n}(t)$ will be governed by a Markov process. Its transition rates are given by:

$$\begin{aligned} n_s(t) &\to n_s(t) + 1, && \text{with rate } \lambda_s, \\ n_s(t) &\to n_s(t) - 1, && \text{with rate } \mu_s n_s(t) x_s(t) \\ & && \text{if } n_s(t) > 0. \end{aligned} \quad (8)$$

The data injected into the network create queue-backlogs at the links. We assume that the queue length are updated every time slot of length T. Assuming that the data injection rate of each user is applied instantaneously over all links that the user's route traverses, the evolution of the queue length Q^l at link l is then governed by:[2]

$$\begin{aligned} Q^l((k+1)T) = \Big[Q^l(kT) \\ + \left(\sum_{s=1}^{S} H_s^l \int_{kT}^{(k+1)T} n_s(t) x_s(t) dt - TR^l\right)\Big]^+. \end{aligned} \quad (9)$$

Let $\vec{Q}(t) = [Q^1(t), ..., Q^L(t)]$.

We can now mathematically define the notion of stability that we are interested in. Note that here the stability of the system must be defined jointly for $\vec{n}(t)$ and $\vec{Q}(t)$. To see this, consider a rate allocation policy that allocates a constant rate $x_s(t) = x_s$ for each user of class s. At an arbitrarily large offered load $\vec{\rho}$, if we can choose x_s sufficiently large, then the number of active users $\vec{n}(t)$ in the system can be easily made stable. (Recall that in our definition of $\vec{n}(t)$, a user is considered to have left the system once it injects all of its data into the network.) However, the links inside the network may be severely congested and the queue lengths $\vec{Q}(t)$ may approach infinity. On the other hand, if we choose x_s sufficiently small, then the queue length $\vec{Q}(t)$ can be easily made stable. However, each user will then take a long time to complete service, and hence the number of active users $\vec{n}(t)$ may approach infinity. Clearly, for the system to operate correctly, we should avoid both of the above two cases. Hence, it is necessary to define the notion of stability jointly for $[\vec{n}(t), \vec{Q}(t)]$.

We say that the Markov process $[\vec{n}(t), \vec{Q}(t)]$ is *stable-in-the-mean* or simply *stable* [16] if

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{E}\left[\sum_{s=1}^{S} n_s(t) + \sum_{l=1}^{L} Q^l(t)\right] dt < \infty. \quad (10)$$

[1] Although we do not specify the units of these quantities, we do assume that their units are consistent. For example, if the file size is in *bits*, and time is in *seconds*, then the units of λ_s and $1/\mu_s$ are *per second* and *bits*, respectively, and the units of ρ_s, x_s and the link capacity R^l are all *bits per second*.

[2] This assumption will be removed in Section V.

By the Markov inequality, this implies that, as $M \to \infty$,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{P}\left[\sum_{s=1}^{S} n_s(t) + \sum_{l=1}^{L} Q^l(t) > M\right] dt \to 0.$$

Roughly speaking, the system is stable if both the number of users in the system and the queue lengths at each link in the network remain finite. Note that the queue length Q^l in our model can be any real positive number. One could have used the techniques of [2] to discretize Q^l, and modeled the system as a countable state-space Markov process. If in addition this countable state-space Markov process is irreducible, then the above notion of stability also implies positive recurrence. However, in this paper we do not pursue this approach, and we simply use stability-in-the-mean to be our notion of stability.

We define the *stability region* Θ of the system under a given congestion controller to be the set of offered loads $\vec{\rho}$ such that the system is stable for any $\vec{\rho} \in \Theta$. We say that the stability region achieved by a congestion controller is *the largest possible* when the following holds: for any offered load, if this congestion controller cannot stabilize the system, no other congestion controller can. Trivially, the capacity constraint determines an *upper bound*[3] on the stability region achieved by *any* congestion controller, i.e.,

$$\Theta \subset \Theta_0 \triangleq \left\{\vec{\rho}\,|\sum_{s=1}^{S} H_s^l \rho_s \leq R^l \text{ for all } l\right\}. \tag{11}$$

In the rest of the paper, we will be interested in studying whether the congestion controller developed through a static setting (such as the one in Section II-A) can also achieve the largest possible stability region when there are connection-level dynamics. As we reviewed related work in Section II-D, there are cases where the congestion controller (or more precisely, the rate allocation algorithm) is not properly designed, and can lead to a strictly smaller stability region. Hence, the connection-level stability of a system employing congestion control is not a trivial problem. In prior work, the connection stability problem has typically been studied under the following *time-scale separation assumption*.

The Time-Scale Separation Assumption:

- The data rates $\vec{x}(t)$ of the users at each time instant t are adjusted *instantaneously* to the optimal rate allocation computed by the global optimization problem (1) with $\vec{n} = \vec{n}(t)$.

We refer to a congestion controller that allocates data rates according to the above time-scale separation assumption as the *idealized congestion controller*. Note that in this case, since the congestion controller chooses the data rates $\vec{x}(t)$ such that the aggregate data arrival rate at each link is no greater than the link capacity (see the constraints in (1)), the queues $Q^l(t)$ are always stable according to (9). Hence, we can model the system as a countable state-space Markov chain by denoting the number of active users $\vec{n}(t)$, as the system state. The stability of the system is then equivalent to the positive recurrence of the Markov chain $\vec{n}(t)$.

[3]This upper bound can be established along the lines of Lemma 3.3 in [17] or Section III.A of [18], i.e., it can be shown that the Markov process will not be stable if $\vec{\rho}$ lies outside the set Θ_0.

The next proposition from [10] shows that the stability region achieved by the *idealized congestion controller* is indeed *the largest possible* found on the right hand side of (11).

Proposition 2: Under the time-scale separation assumption, if the utility functions are of the form in (2) or (3) for some $\beta > 0$, then for any offered load $\vec{\rho}$ that resides strictly inside Θ_0, the Markov process $\vec{n}(t)$ is positive recurrent and hence,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{\{\sum_{s=1}^{S} n_s(t) > M\}} dt \to 0, \text{ as } M \to \infty.$$

Proofs of the above result using fluid limits and the Foster-Lyapunov theorem can also be found in [19] and [20], respectively.

C. Problem Statement

As discussed in the Introduction, time-scale separation is difficult to verify in real networks. When the number of users in the network changes rapidly, the data rates of the users employing a congestion control algorithm (such as algorithm $\mathcal{A}$) may never converge. Further, note that the step-size condition (7) in Proposition 1 becomes more stringent as the number of users in the system increases. As the offered load $\vec{\rho}$ approaches the boundary of the stability region Θ_0, the number of users in the system will approach infinity. Hence, given a chosen set of step-sizes, algorithm $\mathcal{A}$ will fail to converge when the offered load is close to the boundary of Θ_0. The time-scale separation assumption will not hold in either of these two cases.

In this paper, we will study the connection-level stability region of congestion controlled communication networks without using the time-scale separation assumption. In particular, we are interested in the following question: will the class of congestion control algorithms introduced in Section II-A be able to achieve the largest stability region Θ_0, even when the time-scale separation assumption is removed?

D. Related Work

Stability is an important subject for many stochastic systems. The stability region for arbitrary networks has been studied in [17], [21]. The original models in [17], [21] do not have congestion control or connection-level dynamics. Instead, they assume that packets arrive to the system according to a given stochastic process, and hence the packet injection rate cannot be controlled. The role of the network is to route these packets through the system, and, if the link capacity can also be controlled, to determine the optimal link transmission patterns. Optimal control schemes are developed in [17], [21] that achieve the largest stability region, which is in a similar form as (11) for the case when the link capacity is fixed. It would be instructive to map the results in [17], [21] to the model in Section II-B. We can either take $x_s(t)$ as being infinitely large, in which case when a user arrives to the system the entire file is injected immediately into the network; or, we can take $x_s(t)$ to be at a fixed value. In both cases, the packet arrival process is completely determined once the value of $x_s(t)$ is fixed. Hence, we can apply the control schemes and the results from [17], [21]. In particular, since the average packet arrival rate by users of class s is exactly equal to the offered load ρ_s when $x_s(t)$ is a constant, if we apply the control schemes in [17], [21], the results there imply that the system will be stable as long as $\vec{\rho}$ falls strictly inside the set Θ_0 given by (11). Note that when the rates $x_s(t)$ are fixed, we only need to consider the stability of the queue length $\vec{Q}(t)$, because at any fixed $x_s(t)$ the number of active users $\vec{n}(t)$ is always finite by standard results from $M/M/\infty/\infty$ queues.

However, choosing a fixed $x_s(t)$ means that the system does not have the capability to dynamically control the data rates based on the current congestion levels. Hence, we will refer to the above stability result (that is directly derived from [17], [21]) as *stability without congestion control*. Historically, the Internet without proper congestion control has suffered from "congestion collapse" [13], a state where the useful throughput of the network drops by orders of magnitude due to excessive congestion inside the system. Hence, most modern communication networks do employ some form of congestion control mechanism. In this paper, we are interested in the stability of the system when congestion control algorithms are employed. In order to differentiate with the notion of "stability without congestion control" [17], [21], we refer to the stability of a system employing congestion control as *connection-level stability*. Note that connection-level stability cannot be inferred from the results of [17], [21]. Specifically, in [17], [21], although the control policy itself does not require any knowledge of the statistics of the packet arrival process, the stability result does require certain assumptions on its correlation structure. When one employs a dynamic congestion controller, it is unclear whether these assumptions hold for the stochastic process with which users inject data into the network. Hence, we cannot use the stability results of [17], [21].

The authors in [9]–[12], [15] study the connection-level stability problem under the *time-scale separation assumption* that, at each time-instant t, the data injection rates $\vec{x}(t)$ are chosen as a function of the current number of active users $\vec{n}(t)$. In particular, the data injection rates $x_s(t)$ may be chosen to maximize the total system utility as in (1), where n_s is replaced by the instantaneous number of users $n_s(t)$ in the system [10], [12]. Alternatively, the rates $\vec{x}(t)$ can be chosen according to other fairness and optimality criteria (e.g., according to max-min fairness [9]). Under this time-scale separation assumption, it has been pointed out that the introduction of certain congestion-control mechanisms may prevent link capacities from being fully utilized, which might reduce the stability region of the system [9]. In fact, examples have been provided in [10] that, under certain "unfair" rate-allocation mechanisms (e.g., those using priority-scheduling), the connection-level stability region of the system is indeed strictly smaller than the upper bound Θ_0 given in (11). Therefore, the question of connection-level stability is not a trivial one. On the positive side, it has been shown that certain types of "fair" congestion controllers can indeed achieve the largest stability region (e.g., in Proposition 2), when time-scale separation is assumed. Thus, the "fairness" objective of congestion control is not merely an *aesthetic* property, but it actually has a strong global *performance* implication, i.e., in achieving the *largest possible* stability region.

III. Connection-Level Stability without Time-Scale Separation

In this section, we study the connection-level stability of systems employing congestion control, without the aforementioned time-scale separation assumption. We first describe some more details on the dynamics of the system. We assume that time is divided into slots of length T. We use the following congestion controller similar to Algorithm $\mathcal{A}$ in Section II-A. We still assign an implicit cost q^l with each link l. The implicit cost q^l corresponds to the congestion level at link l (and we will soon see that q^l is simply a scalar multiple of the real queue length at link l). We assume that the implicit costs are updated only at the end of each time slot. However, users may arrive and depart at any point within a time slot. Let $\vec{q}(kT)$ denote the implicit costs at time slot k. At any time t, the data rate of users of class s is given by

$$x_s(t) = x_s(kT) = \min\left\{\left(\frac{w_s}{\sum\limits_{l=1}^{L} H_s^l q^l(kT)}\right)^{1/\beta}, M_s\right\}, \quad (12)$$

for $kT \leq t < (k+1)T$, where $\beta > 0$. Note that Equation (12) is the solution to (4) when the utility function is of the form (2) or (3). (We use the convention $\beta = 1$ when the utility function is of the form (2).) At the end of each time slot k, the implicit costs are updated by

$$q^l((k+1)T) = \Big[q^l(kT) + \alpha_l\left(\sum_{s=1}^{S} H_s^l \int_{kT}^{(k+1)T} n_s(t)x_s(kT)dt - TR^l\right)\Big]^+. \quad (13)$$

Note that both equations are closely related to Algorithm $\mathcal{A}$. However, unlike the case in Proposition 2 where the rate allocation $\vec{x}(t)$ is determined by the solution to (1), which by itself requires Algorithm $\mathcal{A}$ to converge instantaneously at each time, now the rate allocation $\vec{x}(t)$ is determined by the current implicit costs. Hence, we have removed the time-scale separation assumption.

Remark: By comparing (13) with (9), we observe that the implicit cost $q^l(t)$ is simply a scaled version of the queue length $Q^l(t)$, i.e., $q^l(t) = \alpha_l Q^l(t)$ assuming $q^l(0) = Q^l(0) = 0$. Hence, in the sequel we will focus on the stability of the Markov process $[\vec{n}(kT), \vec{q}(kT)]$ because it is equivalent to the stability of $[\vec{n}(kT), \vec{Q}(kT)]$ as defined in (10).

The following proposition shows that, even when the time-scale separation assumption is removed, the above congestion control algorithm can still achieve *the largest possible* stability region.

Proposition 3: Assume that the utility functions are either of the form in (2) (in which case we use the convention that $\beta = 1$), or of the form in (3) for some $\beta > 1$, and that the data rates of the users are controlled by (12) and (13). Let $\bar{\mathcal{S}} = \max_l \sum\limits_{s=1}^{S} H_s^l$ denote the maximum number of **classes** using any link, and let $\bar{\mathcal{L}} = \max_s \sum_{l=1}^{L} H_s^l$ denote the maximum number of links used by any **class**, If

$$\max_l \alpha_l \leq \frac{1}{T\bar{\mathcal{S}}\bar{\mathcal{L}}} \frac{2^\beta - 1}{16} \min_s \frac{w_s}{\rho_s M_s^\beta}, \quad \beta \geq 1, \tag{14}$$

then for any offered load $\vec{\rho}$ that resides strictly inside Θ_0, the system described by the Markov process $[\vec{n}(kT), \vec{q}(kT)]$ is stable.

Remark: Since Θ_0 in (11) is the upper bound on the stability region achieved by any congestion controller, Proposition 3 implies that the congestion controller in (12) and (13) can achieve the largest possible stability region.

Before we state the proof for Proposition 3, we would like to highlight the difference between the results in Propositions 2 and 3. First, no time-scale separation assumption is required in Proposition 3. Hence, we do not require the data rates of the users to be chosen at each time according to the solution to (1), which by itself requires an iterative procedure. Second, a step-size rule that is *independent* of the instantaneous number of users in the system is provided in (14) (note the difference between $\mathcal{S}$ and $\bar{\mathcal{S}}$). Given our discussion at the beginning of this section, it is quite surprising that we do not need to reduce the step-sizes even when the offered load is close to the boundary of the stability region. In fact, since the set Θ_0 is bounded, the feasible value of ρ_s is also upper-bounded by $\max_{\vec{\rho}\in\Theta_0} \rho_s$. Hence, the step-sizes can be chosen independently of the offered load. The step-size rule (14) is also dependent on M_s, which is the maximum data rate of users belonging to class s. This dependence is not surprising. Recall that Equation (12) is the solution to (4) when the utility function $U_s(\cdot)$ is of the form (2) or (3). We thus have,

$$U_s''(x_s) = -\beta \frac{w_s}{x_s^{\beta+1}}.$$

Hence, the minimum curvature of $U_s(\cdot)$ is

$$\gamma_s = \frac{\beta w_s}{M_s^{\beta+1}}.$$

Let $\tilde{n}_s = \rho_s / M_s$, which can be interpreted as the average number of users of class s in a (fictitious) $M/M/\infty/\infty$ system, *where each user of class s is served at its maximum data rate M_s*. The step-size condition (14) then becomes

$$\max_l \alpha_l \leq \frac{1}{T\bar{\mathcal{S}}\bar{\mathcal{L}}} \frac{2^\beta - 1}{16\beta} \min_s \frac{\gamma_s}{\tilde{n}_s},$$

which is comparable to (7). However, note that $\tilde{n}_s$ is *quite different* from $\mathbf{E}[n_s(t)]$, the average number of users of class s in the *real* system. Again, in our model without time-scale separation, since $\tilde{n}_s$ is always bounded, the step-sizes can be chosen independently of the offered load.

Sketch of the Proof of Proposition 3: The precise proof is quite technical. Hence, here we will use a heuristic fluid-model argument to illustrate the main idea of the proof and relegate the full proof to the Appendix. This fluid-model argument will be used again in Sections IV and V to treat the extensions of Proposition 3. We will look at the following fluid model that approximates the dynamics of the original system in (8), (12) and (13):

$$\frac{d}{dt} n_s(t) = \lambda_s - \mu_s n_s(t) x_s(t), \tag{15}$$

$$x_s(t) = \left(\frac{w_s}{\sum_{l=1}^{L} H_s^l q^l(t)} \right)^{1/\beta}, \tag{16}$$

$$\frac{d}{dt} q_l(t) = \tag{17}$$

$$\begin{cases} \alpha_l \left(\sum_{s=1}^{S} H_s^l n_s(t) x_s(t) - R^l \right), \\ \quad \text{when } \sum_{s=1}^{S} H_s^l n_s(t) x_s(t) \geq R^l \text{ or } q^l > 0, \\ 0, \quad \text{otherwise.} \end{cases}$$

Roughly speaking, this fluid model becomes a good approximation of the original system when the quantities $\vec{n}(t)$ and $\vec{q}(t)$ are large, in which case the effect of the randomness in (8) and the "jumps" in the discrete-time update (13) are dominated by the "overall trend" that is described by the continuous-time differential equations (15)-(17).

We consider the following Lyapunov function

$$\mathcal{V}(\vec{n}, \vec{q}) = V_n(\vec{n}) + V_q(\vec{q}), \tag{18}$$

where

$$\begin{aligned} V_n(\vec{n}) &= \frac{1}{1+\beta} \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta+1}}{\mu_s \rho_s^\beta} \\ V_q(\vec{q}) &= \frac{1}{2} \sum_{l=1}^{L} \frac{K_{l2} (q^l)^2}{\alpha_l}, \end{aligned}$$

and the positive parameters K_{s1} and K_{l2} will be determined later on. (Recall that the implicit cost q^l is equal to α_l multiplied by the queue length Q^l. Hence, in order to show stability we can focus on q^l instead of Q^l.) The rationale for choosing such a Lyapunov function is as follows: the first term $V_n(\vec{n})$ is the Lyapunov function used in [10] to prove connection-level stability with the time-scale separation assumption. The second term $V_q(\vec{q})$ is a natural Lyapunov function which can be used to prove the stability of (17) if the arrival rates at the links are fixed. Thus, we hope that some linear combination of these two terms would serve as the Lyapunov function to establish the connection-level stability of the fluid model system (15)-(17). Note that, according to (15), we have,

$$\begin{aligned} \frac{d}{dt} n_s^{\beta+1}(t) &= (\beta+1) n_s^\beta(t) \frac{d}{dt} n_s(t) \\ &= (\beta+1) n_s^\beta(t) (\lambda_s - \mu_s n_s(t) x_s(t)). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{dV_n(\vec{n}(t))}{dt} &= \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta(t)}{\mu_s \rho_s^\beta} (\lambda_s - \mu_s n_s(t) x_s(t)) \\ &= \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta(t)}{\rho_s^\beta} (\rho_s - n_s(t) x_s(t)). \end{aligned} \tag{19}$$

From the assumption that $\vec{\rho}$ resides strictly inside Θ_0, there exists an $\epsilon > 0$ such that

$$(1+2\epsilon)\sum_{s=1}^{S} H_s^l \rho_s \le R^l, \text{ for all } l. \tag{20}$$

Adding and subtracting $\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta(t)}{\rho_s^{\beta-1}}$, we have (dropping the variable t for ease of exposition),

$$\begin{aligned}
&\frac{dV_n}{dt} \\
= \;& -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} + \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^\beta}[(1+\epsilon)\rho_s \\
& \qquad -n_s x_s] \qquad (21)\\
= \;& -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} + \sum_{s=1}^{S} \frac{K_{s1}(1+\epsilon)^\beta w_s}{x_s^\beta}[(1+\epsilon)\rho_s \\
& \qquad -n_s x_s] + (A) \qquad (22)\\
= \;& -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} \\
& + \sum_{s=1}^{S} K_{s1}(1+\epsilon)^\beta \left(\sum_{l=1}^{L} H_s^l q^l\right)[(1+\epsilon)\rho_s \\
& \qquad -n_s x_s] + (A) \qquad (23)
\end{aligned}$$

where in (22) we have denoted

$$(A) = -\sum_{s=1}^{S} K_{s1} w_s \left[\frac{(1+\epsilon)^\beta}{x_s^\beta} - \frac{n_s^\beta}{\rho_s^\beta}\right][(1+\epsilon)\rho_s - n_s x_s], \tag{24}$$

and in (23) we have used Equation (16). Further, according to (17), we have

$$q^l(t)\frac{d}{dt}q^l(t) = \alpha_l q^l(t)\left[\sum_{s=1}^{S} H_s^l n_s(t) x_s(t) - R^l\right]$$

under both conditions in Equation (17). Hence,

$$\frac{dV_q(\vec{q}(t))}{dt} = \sum_{l=1}^{L} K_{l2} q^l(t)\left[\sum_{s=1}^{S} H_s^l n_s(t) x_s(t) - R^l\right]. \tag{25}$$

Combining (23) and (25), we can compute the derivative of $\mathcal{V}(\vec{n}(t), \vec{q}(t))$ with respect to t as (again we drop the variable t for ease of exposition):

$$\begin{aligned}
&\frac{d\mathcal{V}}{dt} \\
= \;& -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} \\
& + \sum_{s=1}^{S} K_{s1}(1+\epsilon)^\beta \left(\sum_{l=1}^{L} H_s^l q^l\right)[(1+\epsilon)\rho_s - n_s x_s] \\
& + \sum_{l=1}^{L} K_{l2} q^l \left[\sum_{s=1}^{S} H_s^l n_s x_s - R^l\right] + (A) \\
= \;& -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} + \sum_{l=1}^{L} q^l \left[\sum_{s=1}^{S} K_{s1}(1+\epsilon)^{\beta+1} H_s^l \rho_s \right. \\
& \left. - \sum_{s=1}^{S} K_{s1}(1+\epsilon)^\beta H_s^l n_s x_s\right] \\
& + \sum_{l=1}^{L} K_{l2} q^l \left[\sum_{s=1}^{S} H_s^l n_s x_s - R^l\right] + (A), \qquad (26)
\end{aligned}$$

where in the last step we have interchanged the order of the summation over l and over s. Note that by construction (24),

$$\begin{aligned}
& (A) \\
= \;& -\sum_{s=1}^{S} K_{s1} w_s \frac{[(1+\epsilon)^\beta \rho_s^\beta - n_s^\beta x_s^\beta]}{x_s^\beta \rho_s^\beta}[(1+\epsilon)\rho_s - n_s x_s] \\
\le \;& 0. \qquad (27)
\end{aligned}$$

Further, if we choose $K_{s1} = 1/(1+\epsilon)^\beta$ and $K_{l2} = 1$, then

$$\begin{aligned}
& \sum_{l=1}^{L} q^l \left[\sum_{s=1}^{S} K_{s1}(1+\epsilon)^\beta H_s^l n_s x_s\right] \\
= \;& \sum_{l=1}^{L} K_{l2} q^l \left[\sum_{s=1}^{S} H_s^l n_s x_s\right] \qquad (28)
\end{aligned}$$

and they cancel each other in (26). We thus have,

$$\begin{aligned}
\frac{d\mathcal{V}}{dt} = \;& -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} \\
& + \sum_{l=1}^{L} q^l \left[(1+\epsilon)\sum_{s=1}^{S} H_s^l \rho_s - R^l\right] + (A) \qquad (29)\\
\le \;& -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} \\
& - \epsilon \sum_{l=1}^{L} q^l \sum_{s=1}^{S} H_s^l \rho_s + (A), \qquad (30)
\end{aligned}$$

where in (30) we have used Inequality (20). This provides the negative drift [22] necessary to establish the stability of the system governed by (15)-(17).

For the original system governed by (8), (12) and (13), the above fluid-model argument may serve as a *plausible* explanation for its connection-level stability. Note that if we could establish that the fluid model (15)-(17) is indeed a *fluid limit* of the original system, we could have used the result of [23] to show the stability of the original system. However, there are some technical difficulties in pursuing this approach, namely, in establishing that the fluid model system (15)-(17) is indeed the fluid limit of the original system under some limiting regime. Typically, we need some form of uniform convergence over compact intervals to construct such a fluid limit [23], which usually requires that the rates of the state-transition in (8) and the rates of change in the update (13) are bounded. Unfortunately, the state-transition rates in (8) and the rates of change in (13) can both be arbitrarily large when $n_s(t)x_s(t)$ is large, or, in other words, when $\vec{n}(t)$ is large and $\vec{q}(t)$ is small. This unboundedness becomes an obstacle for constructing the appropriate fluid limit. (Note that this problem does not arise when we use the time-scale separation assumption [10], since in that case there is no $\vec{q}(t)$, and

$n_s(t)x_s(t)$ is upper-bounded by the capacity of the largest link in the network.)

There are two approaches to resolving this difficulty. The first approach would be to impose an additional constraint on $n_s(t)x_s(t)$, i.e., one can assume that the aggregate data rate of users of each class s is bounded by a number $\hat{M}_s$. Thus, the data rate of each class-s user is governed by the following equation, which replaced (12):

$$x_s(t) = \min\left\{\left(\frac{w_s}{\sum_{l=1}^{L} H_s^l q^l(t)}\right)^{1/\beta}, \frac{\hat{M}}{n_s}\right\}. \tag{31}$$

In [2], we have used this approach to establish the connection-level stability for all $\beta > 0$. Further, the stepsize rule in (14) is also not needed if the above additional constraint is imposed. The weakness of this approach is that equation (31) requires some additional congestion control mechanisms to share the bandwidth between n_s flows when the upper bound $\hat{M}_s$ is met.

An alternative approach is to search for a Lyapunov function for the original system directly. In fact, the fluid-model argument has already disclosed to us the Lyapunov function (18) and the main idea to obtain the negative drifts. The key step is in (28) where the terms containing $n_s(t)x_s(t)$ cancel each other. In the Appendix, we will establish the negative drift of the Lyapunov function (18) for the original system directly. Readers will find that the line of argument there closely resembles that of the fluid-model argument in this section. The main difference in the full proof in the Appendix is that we will carefully bound the additional terms due to the randomness and the jumps in the discrete-time updates. The weakness of this second approach, however, is that it cannot be used to establish the connection-level stability for $\beta < 1$. We briefly outline the main difference in the full technical proof. Lemma 5 in Appendix A can be compared to Equation (23). Note that when we compute the derivative of $V_n(\vec{n}(t))$ in Equation (23), we have ignored all second-order variations. However, to compute the difference of $V_n(\vec{n}((k+1)T)) - V_n(\vec{n}(kT))$ in Lemma 5, the second-order terms must be carefully accounted for. Further, the effect of M_s (the upper bound on the data injection rate) was ignored in our heuristic fluid-model argument, and must also be accounted for. In Lemma 5, we will show that the term (A) in (27) is non-positive and is on the order of $-n_s^{\beta+1}(t)x_s(t)$, and the effect of all second-order changes can be bounded by $O(n_s^{\beta}(t)x_s(t))$. Hence, the effect of the second-order changes can be bounded by $|(A)|/2$ plus a constant. Further, we will show that the effect of M_s can also be bounded by a constant. Hence, the net effect of these additional terms plus the term (A) is bounded by $(A)/2$, which is non-positive and is on the order of $-n_s^{\beta+1}(t)x_s(t)/2$. When $\beta \geq 1$, since $x_s(t) \leq M_s$, the net effect $(A)/2$ can be further upper bounded by $-n_s^2(t)x_s^2(t)/(2M_s)$ times a constant[4]. Then, in Appendix C, we show Lemma 6, which can be compared to Equation (25). We again show that those second-order variations ignored in our heuristic fluid-model argument is on the order of $O(n_s^2(t)x_s^2(t))$. Note that this second-order term can thus be dominated by $(A)/2$ with appropriate choice of the stepsizes. Finally, in Appendix D, the steps are comparable to Equation (30), when the terms containing $n_s(t)x_s(t)$ cancel each other (readers can compare Equation (28) with Equation (74)), and all second-order terms can be bounded by a constant provided that the stepsize condition (14) holds. We thus establish the negative drift for proving the stability of the original system.

[4]It is in this step that we need the condition $\beta \geq 1$.

IV. Extensions to Systems with Multipath Routing and Time-Varying Capacity

In this section, we extend our main result to systems with multipath routing and time-varying capacity. For ease of exposition, we will mainly use the type of heuristic fluid-model argument as in Section III to illustrate these results. However, rigorous proofs of these results can be readily obtained by following the techniques in the Appendix.

A. Multipath Routing

We now assume that each user of class s can use $\theta(s)$ alternate paths through the network. Let $H_{sj}^l = 1$ if path j of class s uses link l, $H_{sj}^l = 0$ otherwise. Let x_{sj} denote the data rate on path j of each class-s user. Let $\vec{x}_s = [x_{s1}, ..., x_{s,\theta(s)}]$. The existence of multiple alternate paths affects our network model in a number of ways. Firstly, for connection-level dynamics, the transition rates of the number of active users are now given by:

$$\begin{aligned} n_s(t) &\to n_s(t)+1, && \text{with rate } \lambda_s, \\ n_s(t) &\to n_s(t)-1, && \text{with rate } \mu_s n_s(t)\textstyle\sum_{j=1}^{\theta(s)} x_{sj}(t) \\ & && \text{if } n_s(t) > 0. \end{aligned} \tag{32}$$

Secondly, to determine an upper bound on the stability region of the system, note that for any offered load $\vec{\rho}$ that the system can stabilize, there must exist $\rho_{sj}, j = 1, ..., \theta(s), s = 1, ..., S$, such that

$$\begin{aligned} &\sum_{j=1}^{\theta(s)} \rho_{sj} = \rho_s, \text{ for all } s \text{ and} \\ &\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l \rho_{sj} \leq R^l, \text{ for all } l. \end{aligned} \tag{33}$$

Hence, an upper bound on the stability region is the set Θ_0 given by:

$$\Theta_0 = \{\vec{\rho} | (33) \text{ can be satisfied}\}. \tag{34}$$

Finally, the congestion controller is also affected by multi-path. We consider the following multipath congestion control algorithm: For some positive constants $c_s, s = 1, ..., S$, and non-negative constants $y_{sj}, j = 1, ..., \theta(s), s = 1, ..., S$,

- At each time slot k, the data rate of users of class s is given by

$$\begin{aligned}\vec{x}_s(t) &= \vec{x}_s(kT) \qquad (35)\\ &= \underset{\vec{x}_s\geq 0,\sum_{j=1}^{\theta(s)} x_{sj}\leq M_s}{\operatorname{argmax}} \Bigg\{ U_s(\sum_{j=1}^{\theta(s)} x_{sj}) \\ &\quad -\frac{c_s}{2}\sum_{j=1}^{\theta(s)}(x_{sj}-y_{sj})^2 - \sum_{j=1}^{\theta(s)} x_{sj}\sum_{l=1}^{L} H_{sj}^l q^l(kT)\Bigg\},\end{aligned}$$

  for $kT \leq t < (k+1)T$, where M_s is again the maximum data rate of each user of class s.
- At the end of time slot k, the implicit costs are updated by:

$$\begin{aligned}q^l((k+1)T) &= \Big[q^l(kT)\\ &+\alpha_l\left(\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l \int_{kT}^{(k+1)T} n_s(t)x_{sj}(kT)dt\right.\\ &\left. - TR^l\right)\Big]^+. \qquad (36)\end{aligned}$$

Remark: When the number of users n_s in the system is fixed, the update in (35) and (36) solves the following optimization problem [24]:

$$\begin{aligned}\max_{\vec{x}\geq 0} \quad & \sum_{s=1}^{S} n_s\left\{U_s(\sum_{j=1}^{\theta(s)} x_{sj}) - \frac{c_s}{2}\sum_{j=1}^{\theta(s)}(x_{sj}-y_{sj})^2\right\}\\ \text{subject to} \quad & \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l n_s x_{sj} \leq R^l \text{ for all } l\\ & \sum_{j=1}^{\theta(s)} x_{sj} \leq M_s \text{ for all } s. \qquad (37)\end{aligned}$$

We can imagine that when c_s is small, the above problem approximates the multipath utility maximization problem given below [24]:

$$\begin{aligned}\max_{\vec{x}\geq 0} \quad & \sum_{s=1}^{S} n_s U_s(\sum_{j=1}^{\theta(s)} x_{sj})\\ \text{subject to} \quad & \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l n_s x_{sj} \leq R^l \text{ for all } l\\ & \sum_{j=1}^{\theta(s)} x_{sj} \leq M_s \text{ for all } s,\end{aligned}$$

which is comparable to the single-path utility-maximization problem (1). The constants c_s and y_{sj} in the approximate problem (37) are used to alleviate the "oscillation" problem in systems with multipath routing [24]. This oscillation problem will occur when $c_s = 0$, because then the maximization in (35) will only use the path with the smallest "cost" to transmit data. (The cost q_{sj} of a path j of class s is the sum of the implicit costs of all links on the path, i.e., $q_{sj} = \sum_{l=1}^{L} H_{sj}^l q^l$.) Hence, when the costs of two paths are close to each other, a small perturbation on the implicit costs will trigger the entire load offered by class s to be switched to another path. When c_i is positive, the maximizer of (35) becomes continuous with respect to the implicit costs, and such oscillation will not occur any more.

We now provide a heuristic fluid-model argument similar to Section III that the system governed by (32), (35) and (36) will achieve the largest possible stability region Θ_0 given in (34). Again, we assume that the utility function is of the form (2) or (3). We can write down the following fluid model that approximates the dynamics of the original system:

$$\begin{aligned}\frac{d}{dt}n_s(t) &= \lambda_s - \mu_s n_s(t)\sum_{j=1}^{\theta(s)} x_{sj}(t),\\ \vec{x}_s(t) &= \underset{\vec{x}_s\geq 0}{\operatorname{argmax}}\Bigg\{U_s(\sum_{j=1}^{\theta(s)} x_{sj}) - \frac{c_s}{2}\sum_{j=1}^{\theta(s)}(x_{sj}-y_{sj})^2\\ &\quad -\sum_{j=1}^{\theta(s)} x_{sj}\sum_{l=1}^{L} H_{sj}^l q^l(t)\Bigg\},\\ \frac{d}{dt}q_l(t) &= \begin{cases}\alpha_l\left(\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l n_s(t)x_{sj}(t) - R^l\right)\\ \quad \text{when } \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l n_s(t)x_{sj}(t) \geq R^l\\ \quad \text{or } q^l > 0\ ,\\ 0, \ \text{otherwise.}\end{cases}\end{aligned}$$

We will use the following Lyapunov function

$$\mathcal{V}(\vec{n},\vec{q}) = V_n(\vec{n}) + V_q(\vec{q}),$$

where

$$V_n(\vec{n}) = \frac{1}{1+\beta}\sum_{s=1}^{S}\frac{K_{s1}w_s n_s^{\beta+1}}{\mu_s\rho_s^{\beta}} + \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)}\frac{c_s y_{sj} n_s}{\mu_s},$$

and

$$V_q(\vec{q}) = \sum_{l=1}^{L}\frac{(q^l)^2}{2\alpha_l}.$$

Then, using the same type of fluid-model argument as in Section III, we have,

$$\begin{aligned}\frac{dV_n(\vec{n}(t))}{dt} &= \sum_{s=1}^{S}\frac{K_{s1}w_s n_s^{\beta}(t)}{\rho_s^{\beta}}[\rho_s - n_s(t)\sum_{j=1}^{\theta(s)} x_{sj}(t)]\\ &\quad +\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s y_{sj}[\rho_s - n_s(t)\sum_{j=1}^{\theta(s)} x_{sj}(t)].\end{aligned}$$

This equation is comparable to (19). From the stability condition, if $\vec{\rho}$ lies strictly inside Θ_0, there exists an $\epsilon > 0$ such that $(1+2\epsilon)\vec{\rho} \in \Theta_0$. This implies that there exists ρ_{sj} such that $\rho_s = \sum_{j=1}^{\theta(s)} \rho_{sj}$ for all s and

$$(1+2\epsilon)\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l \rho_{sj} \leq R^l \text{ for all } l.$$

Adding and subtracting $\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}}$ as in Section III, we have (dropping the variable t for ease of exposition),

$$\begin{aligned}
&\frac{dV_n}{dt}\\
=\ & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}}\\
&+\sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta}} \left[(1+\epsilon)\rho_s - n_s \sum_{j=1}^{\theta(s)} x_{sj}\right]\\
&+\sum_{s=1}^{S} c_s \left[\sum_{j=1}^{\theta(s)} y_{sj}\right] \left[(1+\epsilon)\rho_s - n_s \sum_{j=1}^{\theta(s)} x_{sj}\right]\\
&-\epsilon \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s y_{sj} \rho_s \qquad (38)\\
=\ & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}}\\
&+\sum_{s=1}^{S} \frac{K_{s1}(1+\epsilon)^\beta w_s}{(\sum_{j=1}^{\theta(s)} x_{sj})^\beta} \left[(1+\epsilon)\rho_s - n_s \sum_{j=1}^{\theta(s)} x_{sj}\right]\\
&+\sum_{s=1}^{S} c_s \left[\sum_{j=1}^{\theta(s)} y_{sj}\right] \left[(1+\epsilon)\rho_s - n_s \sum_{j=1}^{\theta(s)} x_{sj}\right]\\
&-\epsilon \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s y_{sj} \rho_s + (A), \qquad (39)
\end{aligned}$$

where

$$\begin{aligned}
(A) \ =\ & -\sum_{s=1}^{S} \left\{ K_{s1} w_s \left[\frac{(1+\epsilon)^\beta}{(\sum_{j=1}^{\theta(s)} x_{sj})^\beta} - \frac{n_s^\beta}{\rho_s^\beta}\right]\right.\\
&\left. \times \left[(1+\epsilon)\rho_s - n_s \sum_{j=1}^{\theta(s)} x_{sj}\right]\right\} \leq 0. \qquad (40)
\end{aligned}$$

Using $\rho_s = \sum_{j=1}^{\theta(s)} \rho_{sj}$, we have

$$\begin{aligned}
&\sum_{s=1}^{S} \frac{K_{s1}(1+\epsilon)^\beta w_s}{(\sum_{j=1}^{\theta(s)} x_{sj})^\beta} \left[(1+\epsilon)\rho_s - n_s \sum_{j=1}^{\theta(s)} x_{sj}\right]\\
=\ & \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} \frac{K_{s1}(1+\epsilon)^\beta w_s}{(\sum_{h=1}^{\theta(s)} x_{sh})^\beta} \left[(1+\epsilon)\rho_{sj} - n_s x_{sj}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{s=1}^{S} c_s \left[\sum_{j=1}^{\theta(s)} y_{sj}\right] \left[(1+\epsilon)\rho_s - n_s \sum_{j=1}^{\theta(s)} x_{sj}\right]\\
=\ & \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s y_{sj} \left[(1+\epsilon)\rho_{sj} - n_s x_{sj}\right]\\
&+\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s y_{sj} \sum_{h\neq j} \left[(1+\epsilon)\rho_{sh} - n_s x_{sh}\right].
\end{aligned}$$

Hence, adding and subtracting $\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s x_{sj}[(1+\epsilon)\rho_{sj} - n_s x_{sj}]$, we have,

$$\begin{aligned}
&\frac{dV_n}{dt}\\
\leq\ & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} + \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} \left[\frac{K_{s1}(1+\epsilon)^\beta w_s}{(\sum_{h=1}^{\theta(s)} x_{sh})^\beta}\right.\\
&\left. - c_s(x_{sj} - y_{sj})\right] \left[(1+\epsilon)\rho_{sj} - n_s x_{sj}\right]\\
&+\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s x_{sj} [(1+\epsilon)\rho_{sj} - n_s x_{sj}] + F_0 \qquad (41)
\end{aligned}$$

where

$$F_0 \leq (1+\epsilon)\sum_{s=1}^{S} c_s \sum_{j=1}^{\theta(s)} y_{sj} \sum_{h\neq j} \rho_{sh} - \epsilon \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s y_{sj} \rho_s + (A).$$

Note that Equation (41) is comparable to Equation (22). Since $x_{sj} \leq M_s$ for all s, j, we have

$$\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s x_{sj} [(1+\epsilon)\rho_{sj} - n_s x_{sj}] \leq F_1, \qquad (42)$$

where $F_1 = (1+\epsilon)\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} c_s M_s \rho_{sj}$. Further, assuming that the utility function $U_s(\cdot)$ is of the form (2) or (3), since $\vec{x}_s$ solves (35), there must exist some $\delta_{sj} \geq 0, j = 1, ..., \theta(s)$ and $\delta_{s0} \geq 0$ such that

$$\frac{w_s}{(\sum_{h=1}^{\theta(s)} x_{sh})^\beta} - c_s(x_{sj} - y_{sj}) + \delta_{sj} - \delta_{s0} = q_{sj}, \quad \text{for all } s, j, \qquad (43)$$

where $q_{sj} = \sum_{l=1}^{L} H_{sj}^l q^l$. Here, $\delta_{sj}, j = 1, ..., \theta(s)$ are the Lagrange multiplers for the constraint $x_{sj} \geq 0$, and δ_{s0} is the Lagrange multiplier for the constraint $\sum_{j=1}^{\theta(s)} x_{sj} \leq M_s$. Further, $\delta_{sj} x_{sj} = 0$ and $\delta_{s0}(\sum_{j=1}^{\theta(s)} x_{sj} - M_s) = 0$ due to the complementary slackness condition. Hence, if we let $K_{s1} = 1/(1+\epsilon)^\beta$, we have

$$\begin{aligned}
&\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} \left[\frac{K_{s1}(1+\epsilon)^\beta w_s}{(\sum_{h=1}^{\theta(s)} x_{sh})^\beta} - c_s(x_{sj} - y_{sj})\right]\\
&\times [(1+\epsilon)\rho_{sj} - n_s x_{sj}]\\
=\ & \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} (q_{sj} - \delta_{sj} + \delta_{s0}) [(1+\epsilon)\rho_{sj} - n_s x_{sj}]\\
\leq\ & \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} q_{sj} [(1+\epsilon)\rho_{sj} - n_s x_{sj}]\\
&+\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} (1+\epsilon)\delta_{s0}\rho_{sj} \quad (\text{using } \delta_{sj} x_{sj} = 0)\\
=\ & \sum_{l=1}^{L} q^l \left[(1+\epsilon)\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l \rho_{sj} - \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l n_s x_{sj}\right]\\
&+\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} (1+\epsilon)\delta_{s0}\rho_{sj}. \qquad (44)
\end{aligned}$$

It is easy to show that δ_{s0} is bounded for all s. Specifically, note that when $\delta_{s0} > 0$, we must have $\sum_{j=1}^{\theta(s)} x_{sj} = M_s$. Summing (43) over all $x_{sj} > 0$, and noting that $\delta_{sj} = 0$ when $x_{sj} > 0$, we have, for all s,

$$J_s \frac{w_s}{M_s^\beta} - c_s(M_s - \sum_{j:x_{sj}>0} y_{sj}) - J_s\delta_{s0} = \sum_{j:x_{sj}>0} q_{sj},$$

where $J_s \leq \theta(s)$ is the cardinality of the set $\{j : x_{sj} > 0\}$. Since $q_{sj} \geq 0$, δ_{s0} is therefore bounded by

$$\delta_{s0} \leq \frac{w_s}{M_s^\beta} + c_s \sum_{j=1}^{\theta(s)} y_{sj}/J_s \leq F'_s,$$

where $F'_s = \frac{w_s}{M_s^\beta} + c_s \sum_{j=1}^{\theta(s)} y_{sj}$. Finally, substituting (42) and (44) into (41), we have,

$$\begin{aligned}\frac{dV_n}{dt} \quad \leq \quad & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} \\ & + \sum_{l=1}^{L} q^l \left[(1+\epsilon) \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l \rho_{sj} \right. \\ & \left. - \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l n_s x_{sj} \right] + F_0 + F_1 + F_2,\end{aligned}$$

where $F_2 = (1+\epsilon)\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} F'_s \rho_{sj}$. This equation is comparable to (23). Now, using the argument in Section III again (see Equation (25)), we have,

$$\frac{dV_q}{dt} \quad = \quad \sum_{l=1}^{L} q^l \left[\sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l n_s x_{sj} - R^l \right].$$

The terms containing $n_s x_{sj}$ in the expression of $\frac{dV_n}{dt}$ and $\frac{dV_q}{dt}$ cancel each other. We thus have,

$$\begin{aligned}\frac{d\mathcal{V}}{dt} \quad \leq \quad & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} \\ & + \sum_{l=1}^{L} q^l \left[(1+\epsilon) \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l \rho_{sj} - R^l \right] \\ & + F_0 + F_1 + F_2 \\ \leq \quad & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta}{\rho_s^{\beta-1}} - \epsilon \sum_{l=1}^{L} q^l \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^l \rho_{sj} \\ & + F_0 + F_1 + F_2.\end{aligned}$$

This thus provides the negative drift for stability. Similar to the discussions at the end of Section III, the full proof that takes into account the second-order variations can be obtained along the line of proof in the Appendix. Specifically, all second-order variations (due to randomness and discrete-time updates) can be bounded by $|(A)|/2$, where the quantity (A) is given by (40) and is non-positive.

B. Time-Varying Capacity

Our main result can also be readily extended to the case with time-varying capacity. Time-varying capacity arises naturally in wireless networks. When there is channel fading, the propagation gain between the transmitter and the receiver varies over time. Hence, even when the transmission power is fixed, the capacity at each link is still time-varying [25]. Further, even if there is no fading in a wireless network, it is usually preferable to activate different sets of links alternately [17], [21], [26], [27]. Otherwise, if all links are activated at the same time, they may create so much mutual interference such that none of the links can carry data at an acceptable rate.

Through the above discussion on wireless networks, we observe two factors that lead to time-varying capacity. The first factor is the change in the environment (i.e., channel fading). The second factor is that different configurations (i.e., activation patterns of links) can be chosen at different times even if the outside environment is identical. We now describe a general model that captures both of these two factors. We use κ to denote the state of the environment. In the case of wireless networks, the state κ summarizes the current set of propagation gains between transmitters and receivers. We assume that time is again divided into slots of length T, and the state of the environment is fixed within each time slot. Let $\kappa(kT)$ denote the state of the environment at time slot k. We assume that, at each time slot, the state $\kappa(kT)$ is chosen independently and identically from a set $\mathcal{K}$, following the distribution $\pi_\kappa, \kappa \in \mathcal{K}$. Next, let $\vec{R} = [R^1, ..., R^l]$ denote the vector of link capacities on all links. We assume that, when the state of the environment is κ, the vector $\vec{R}$ can be chosen within a set $\mathcal{R}_\kappa$. In other words, this set $\mathcal{R}_\kappa$ models the possible configurations of the link capacities at state κ. For technical reasons, we assume that $\mathcal{R}_\kappa$ is closed and bounded for each $\kappa \in \mathcal{K}$.

Consider the model of Section II-A, where n_s users of class $s = 1, ..., S$ are sharing the network capacity. Each user of class s may still traverse multiple links, where routing is defined by H_s^l (for simplicity we assume single-path routing here). The goal of congestion control is again to maximize the total system utility, subject to the constraint that the aggregate rates allocated to the users are no greater than the links' service capacity supported by choosing appropriate schedules $\vec{R}(t)$. With time-varying capacity, the congestion controller must be designed jointly with the schedules $\vec{R}(t)$ for each time slot [28]–[32]. In the literature, the following congestion controller has often been used to achieve this goal. Assuming that the rate-assignments and the scheduling decision are also updated at the beginning of every time-slot of length T. The rate-control component of the congestion controller is in fact nearly identical to the one in Section III. We still associate an implicit cost q^l for each link l. At each time slot k, the data rate of users of class s is given by

$$x_s(t) = x_s(kT) = \min \left\{ \left(\frac{w_s}{\sum_{l=1}^{L} H_s^l q^l(kT)} \right)^{1/\beta}, M_s \right\}, \quad (45)$$

for $kT \leq t < (k+1)T$, (use $\beta = 1$ when the utility functions are of the form (2)). At the end of each time slot, the implicit costs are updated by

$$q^l((k+1)T) = \left[q^l(kT) + \alpha_l \left(\sum_{s=1}^{S} H_s^l n_s(kT) x_s(kT) dt - R^l(kT) \right) \right]^+ . \quad (46)$$

(Note that the capacity of each link $R^l(t)$ is now time-varying.) The scheduling policy is chosen as follows: at each time slot k, the schedule $\vec{R}(kT)$ is chosen to be the vector $\vec{R} \in \mathcal{R}_{\kappa(kT)}$ that maximizes $\sum_{l=1}^{L} q^l(kT) R^l$, i.e.,

$$\vec{R}(t) = \vec{R}(kT) = \operatorname*{argmax}_{\vec{R} \in \mathcal{R}_{\kappa(kT)}} \sum_{l=1}^{L} q^l(kT) R^l, \quad (47)$$

for $kT \leq t < (k+1)T$. This scheduling policy is in fact identical to the scheduling policies in [17], [21], [27]. It can be shown that, when the number of users n_s of each class s is fixed, and the positive stepsizes α_l are sufficiently small, the above joint congestion-control and scheduling algorithm will converge to a neighborhood of the rate-allocation that maximizes the total system utility [28], [29].

Next, consider the model with connection-level user-dynamics (as in Section II-B), where users of class s arrive according to a Poisson process with rate λ_s, that each user brings with it a file for transfer whose size is exponentially distributed with mean $1/\mu_s$. We ask the question again: without the time-scale separation assumption, does the above congestion controller achieve the largest possible stability region? We first answer the following question: what is the largest possible stability region for such systems with time-varying capacity. Consider the following system which is generated from the original system. For each node i that can act as the source node of a certain class s, we divide node i into two nodes i_A and i_B. The new system has the same user arrival process as the original system. However, in the new system users of class s arrive at node i_A first. Further, whenever a user of class s arrives to the new system, the entire file is injected into node i_A, and the user leaves the system immediately afterwards. Node i_A is connected to node i_B with a link with infinite capacity, and node i_B will be connected to the rest of the network according to the original network topology. We now argue that if the original network with congestion control can be stabilized, then this new system can also be stabilized. Specifically, the congestion control mechanism simply regulates the rate between each pair of node i_A and node i_B. Hence, the stability region of the original system with congestion control must be *no larger than* the stability region of the new system where we do not impose any restriction on whether or not to use congestion control. Now, let us focus on the new system: the statistics of the data arrival process at each source node i_A is known, and the average data injection rate by class-s users is equal to ρ_s. Hence, we can now use some of the stability results from [17], [21]. Let

$$\Theta_0 = \left\{ \vec{\rho} \,\middle|\, [\sum_{s=1}^{S} H_s^l \rho_s] \in \sum_{\kappa \in \mathcal{K}} \pi_\kappa \mathrm{Co}(\mathcal{R}_\kappa) \right\}. \quad (48)$$

It has been shown in [17], [21] that no control scheme can stabilize the new system if the average rate $\vec{\rho}$ of the data injection process is outside the set Θ_0. Therefore, we can conclude that Θ_0 is also an upper bound on the stability region of our original system. On the other hand, although [17], [21] provide control schemes to stabilize the new system for any offered load $\vec{\rho}$ that lies strictly inside Θ_0, as we discussed in Section II-D, these results correspond to the case with no congestion control. Hence, we cannot use them to deduce the connection-level stability of the original system when congestion control is enforced.

We next show that the congestion controller in (45)-(47) indeed achieves the largest possible *stability region* Θ_0, without the time-scale separation assumption. We first specify some more details of the system dynamics. The congestion control and scheduling decision are still assumed to be updated every time-slot of length T according to (45) and (47). Thus, the value of $x_s(t)$ and $\vec{R}(t)$ are equal to $x_s(kT)$ and $\vec{R}(kT)$, respectively, when $kT \leq t < (k+1)T$. However, users may enter or leave the system within a time-slot (according to the transition rates in (8)). Therefore, we change the implicit-cost update at the end of the time-slot to

$$q^l((k+1)T) = \left[q^l(kT) + \alpha_l \left(\sum_{s=1}^{S} H_s^l \int_{kT}^{(k+1)T} n_s(t) x_s(kT) dt - TR^l(kT) \right) \right]^+ . \quad (49)$$

Note that $q^l(kT)$ can again be viewed as a scalar-multiple of the real queue length $Q^l(kT)$ at link l.

We now provide a heuristic fluid-model argument similar to Section III that the above joint congestion-control and scheduling policy achieves the largest possible *stability region* Θ_0. We use the Lyapunov function $\mathcal{V}(\cdot, \cdot)$ in (18). Following the heuristic fluid-model argument in Section III, we can show that, given $\vec{n}(t), \vec{q}(t)$ and $\kappa(t)$,

$$\frac{d\mathcal{V}(\vec{n}(t), \vec{q}(t))}{dt} = -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^\beta(t)}{\rho_s^{\beta-1}} + \sum_{l=1}^{L} q^l(t) \left[(1+\epsilon) \sum_{s=1}^{S} H_s^l \rho_s - R^l(t) \right] + (A).$$

Note that this equation is comparable to Equation (29), where the only change is in the time-varying capacity $R^l(t)$. The quantity (A) is again given by (24) and is non-positive. Taking expectation with respect to $\kappa(t)$, we have

$$\begin{aligned}
&\mathbf{E}\left[\frac{d\mathcal{V}}{dt}|\vec{n}(t),\vec{q}(t)\right] \\
=\quad & -\epsilon\sum_{s=1}^{S}\frac{K_{s1}w_s n_s^{\beta}(t)}{\rho_s^{\beta-1}}+\sum_{l=1}^{L}q^l(t)\left\{(1+\epsilon)\sum_{s=1}^{S}H_s^l\rho_s\right. \\
& \left.-\mathbf{E}[R^l(t)|\vec{q}(t)]\right\}+(A). \qquad (50)
\end{aligned}$$

If $\vec{\rho}\in\Theta_0$, there must exist an $\epsilon>0$ such that

$$(1+2\epsilon)\vec{\rho}\in\Theta_0.$$

Since Θ_0 is given by (48), there must exist R_κ^l, $l=1,...,L$, $\kappa\in\mathcal{K}$, such that the vectors $[R_\kappa^1,...,R_\kappa^L]$ satisfy

$$[R_\kappa^1,...,R_\kappa^L]\in\mathrm{Co}(\mathcal{R}_\kappa),\qquad \text{for all } \kappa\in\mathcal{K},$$

and

$$(1+2\epsilon)\sum_{s=1}^{S}H_s^l\rho_s\le\sum_{\kappa\in\mathcal{K}}\pi_\kappa R_\kappa^l,\qquad\text{for all } l.$$

By the definition of the scheduling policy (47), given $\vec{q}(t)$,

$$\sum_{l=1}^{L}q^l(t)R^l(t)\ge\sum_{l=1}^{L}q^l(t)R_\kappa^l \text{ if } \kappa(t)=\kappa.$$

Hence,

$$\begin{aligned}
\sum_{l=1}^{L}q^l(t)\mathbf{E}[R^l(t)|\vec{q}(t)] \quad&\ge\quad \sum_{l=1}^{L}q^l(t)\sum_{\kappa\in\mathcal{K}}\pi_\kappa R_\kappa^l \\
&\ge\quad (1+2\epsilon)\sum_{l=1}^{L}q^l(t)\sum_{s=1}^{S}H_s^l\rho_s.
\end{aligned}$$

Substituting into (50), we have

$$\begin{aligned}
\mathbf{E}[\frac{d\mathcal{V}}{dt}|\vec{n}(t),\vec{q}(t)] \quad\le\quad & -\epsilon\sum_{s=1}^{S}\frac{K_{s1}w_s n_s(t)^{\beta}}{\rho_s^{\beta-1}} \\
& -\epsilon\sum_{l=1}^{L}q^l(t)\sum_{s=1}^{S}H_s^l\rho_s+(A).
\end{aligned}$$

Noting that the quantity (A) is non-positive, the above inequality then provides the negative drift required to establish stability. A rigorous proof without using the heuristic fluid-model can also be readily obtained by following the techniques in the Appendix, and by showing that all second-order variations in the original discrete-time system is bounded by $|(A)|/2$.

V. Joint Congestion Control with Back-Pressure Algorithms

A key assumption that is made in the congestion control algorithms considered in the previous sections is that: when the implicit costs are updated (Equations (5), (13), (36) and (49)), the data rate x_s of each user of class s is applied simultaneously to all links along its path. While this assumption is widely adopted in the optimization flow-control literature [3]–[8], it can have some subtle and undesirable implications for stability. Note that there are two ways the congestion control algorithms in the previous sections can be implemented. The user could communicate its data rate x_s to each link along its path using a dedicated control channel. Alternatively, the link l can count the total number of packets (or fluid) that go through itself, and use this measurement to estimate the aggregate data rate $\sum_{s=1}^{S}H_s^l n_s(t)x_s(t)$ directly. It can then use this estimate to update the implicit cost $q^l(t)$. The latter approach does not require a separate control channel, and hence may be preferable. However, note that in a real system packets have to traverse the links one-by-one. Hence, when a user of class s changes its rate x_s at time t, it will take a certain delay Δt before a downstream link can observe this change. Since this delay Δt consists of the queueing delay at each upstream link, it can potentially be large, especially when the offered load is high. It is then unclear whether the stability results in the previous sections will still hold when the implicit-cost updates are based on arbitrarily-delayed versions of the users' rates. In fact, for queueing networks without congestion control, instability problems have been observed when this "user-rates-applied-simultaneously-to-all-links" assumption is removed. Specifically, examples have been constructed where a queueing network appears to be stable under this "user-rates-applied-simultaneously-to-all-links" assumption, but is actually unstable when packets traverse the network link-by-link [33]–[35].

In this section, we will use a different congestion control algorithm that does not require the above assumption that user rates are applied simultaneously to all links. This congestion controller is motivated by the recent results in joint congestion control and scheduling in multihop wireless networks [28], [30]–[32], and the so-called "back-pressure" algorithm in the literature [17], [21]. Unlike the models in the previous sections, we now explicitly model the process of the data traversing the network, link by link. We will then show that this congestion-controller achieves the largest stability region of the system even without the time-scale separation assumption.

We first introduce some additional notation. We use a node-pair (i,m) to denote a link $l\in\mathcal{E}$, where i is the transmitter node and m is the receiving node. The routing matrix $[H_{sj}^l]$ and the link-rate R^l is rewritten as $[H_{sj}^{im}]$ and R^{im}, respectively. For each class s, let f_s and d_s denote the source node and destination node, respectively, of users of class s. At each node i, let q_i^s denote the implicit cost for users of class s. As will become clear soon, the implicit cost q_i^s corresponds to a scalar-multiple of the length of the queue at node i that contains data from class-s users. (Note: it is possible to define the implicits costs at each node, as one cost for *each destination* rather than one cost for *each class*. Here we adopt the per-class definition for ease of exposition.) Let $q_i^s=0$ if the node i is the destination node for class s, i.e., $i=d_s$.

The new congestion controller with the back-pressure algorithm is given by the following set of equations. Note that the congestion controller takes into account the possibility of multiple paths from the source and destination, and the possibility of time-varying capacity (the models are identical to that in Section IV). Assuming that the channel state changes every time-slot of duration T, and both the congestion control decision and the scheduling decision are updated at the beginning of every time-slot.

- At each time slot k, the data rate of users of class s is given by

$$x_s(t) = x_s(kT) = \operatorname*{argmax}_{0 \le x_s \le M_s} U_s(x_s) - x_s q^s_{f_s}(kT), \quad (51)$$

  for $kT \le t < (k+1)T$, where M_s is again the maximum data rate of each user of class s, and $q^s_{f_s}$ is the class-s implicit-cost at the source node f_s for class s.
- At each time slot k, each link (i,m) picks the class s^*_{im} with the largest differential backlog, i.e.,

$$s^*_{im}(t) = \operatorname*{argmax}_{s: H^{im}_{sj}=1 \text{ for some } j} (q^s_i(kT) - q^s_m(kT)),$$

  for $kT \le t < (k+1)T$. Let the weight of link (i,m) be the largest differential backlog, i.e.,

$$w_{im}(t) = \max\{q_i^{s^*_{im}(t)}(kT) - q_m^{s^*_{im}(t)}(kT), 0\}.$$

  The schedule at time slot k is then determined by

$$\vec{R}(t) = \operatorname*{argmax}_{\vec{R} \in \mathcal{R}_{\kappa(t)}} \sum_{l=1}^{L} w_{im}(t) R^{im}, \quad (52)$$

  for $kT \le t < (k+1)T$. Note that $s^*_{im}(t), w_{im}(t)$ and $\vec{R}(t)$ are all fixed over $kT \le t < (k+1)T$.
- At each time slot k (i.e., $kT \le t \le (k+1)T$), each link (i,m) forwards data for users of class $s^*_{im}(t)$ at the rate of $R^{im}(t)$. Let r^{im}_s denote the forwarding rate of class-s data at link (i,m), then for $kT \le t < (k+1)T$,

$$r^{im}_s(t) = \begin{cases} R^{im}(t) & \text{if } s = s^*_{im}(t) \\ 0 & \text{otherwise.} \end{cases}$$

  At the end of time slot k, the implicit costs are updated by:

$$\begin{aligned} q^s_i(k+1)T) = [q^s_i(kT) \\ +a^s_i \Bigg(\int_{kT}^{(k+1)T} n_s(t) x_s(kT) \mathbf{1}_{\{f_s=i\}} dt \\ +T \sum_{\substack{m\,:\,(m,i)\in\mathcal{E}, \\ H^{mi}_{sj}=1 \text{ for some } j}} r^{mi}_s(t) \\ -T \sum_{\substack{m\,:\,(i,m)\in\mathcal{E}, \\ H^{im}_{sj}=1 \text{ for some } j}} r^{im}_s(t) \Bigg) \Bigg]^+ \\ \text{for all } s \text{ and } i \text{ such that } i \neq d_s. \end{aligned} \quad (53)$$

Remark: We can assume that each node maintains a separate queue for the data from each class s. Then q^s_i is obviously a^s_i multiplied by the length of the class-s queue at node i.

When the number of users n_s is the system is fixed, the above algorithm can be shown to maximize the system utility subject to capacity constraints [28], [30]. When there are connection-level user-dynamics, the number of users $n_s(t)$ can change within a time-slot. Again, we are interested the stability of the system employing the above congestion control algorithm. The following proposition shows that the above congestion controller achieves the largest connection-level stability region, even without using the time-scale separation assumption.

Proposition 4: Assume that the utility functions are either of the form in (2) (in which case we use the convention that $\beta = 1$), or of the form in (3) for some $\beta > 1$, and that the data rates of the users and the scheduling decision are governed by the equations above. There exists a constant α_0 such that if the stepsizes $\alpha^i_s \le \alpha_0$ for all s and i, then for any offered load $\vec{\rho}$ that resides strictly inside Θ_0, the system described by the Markov process $[\vec{n}(kT), \vec{q}(kT)]$ is stable.

Again, we now only provide a heuristic fluid-model argument for this proposition (similar to Section III). When the utility function is of the form (2) or (3), the heuristic fluid-model is given by:

$$\begin{aligned} \frac{dn_s(t)}{dt} &= \lambda_s - \mu_s n_s(t) x_s(t), \\ x_s(t) &= \left(\frac{w_s}{q^s_{f_s}(t)} \right)^{1/\beta}, \\ s^*_{im}(t) &= \operatorname*{argmax}_{s: H^{im}_{sj}=1 \text{ for some } j} (q^s_i(t) - q^s_m(t)) \\ w_{im}(t) &= \max\{q_i^{s^*_{im}}(t) - q_m^{s^*_{im}}(t), 0\} \\ \vec{R}(t) &= \operatorname*{argmax}_{\vec{R} \in \mathcal{R}_{\kappa(t)}} \sum_{l=1}^{L} w_{im}(t) R^{im}, \\ r^{im}_s(t) &= \begin{cases} R^{im}(t) & \text{if } s = s^*_{im}(t) \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

and finally,

$$\begin{aligned} \frac{dq^s_i(t)}{dt} = a^s_i \Bigg(& n_s(t) x_s(t) \mathbf{1}_{\{f_s=i\}} \\ & - \sum_{\substack{m\,:\,(m,i)\in\mathcal{E}, \\ H^{mi}_{sj}=1 \text{ for some } j}} r^{mi}_s(t) \\ & + \sum_{\substack{m\,:\,(i,m)\in\mathcal{E}, \\ H^{im}_{sj}=1 \text{ for some } j}} r^{im}_s(t) \Bigg) \end{aligned}$$

when $q^s_i(t) > 0$ or when the term in parenthesis is non-negative, and

$$\frac{dq^s_i(t)}{dt} = 0, \qquad \text{otherwise.}$$

We use the following Lyapunov function

$$\mathcal{V}(\vec{n}, \vec{q}) = V_n(\vec{n}) + V_q(\vec{q}),$$

where

$$\begin{aligned} V_n(\vec{n}) &= \frac{1}{1+\beta} \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta+1}}{\mu_s \rho_s^\beta} \\ V_q(\vec{q}) &= \sum_{s=1}^{S} \sum_{i=1}^{I} \frac{(q^s_i)^2}{2a^s_i}. \end{aligned}$$

Following the heuristic fluid-model argument in Section III, we can show that, given $\vec{n}(t), \vec{q}(t)$ and $\kappa(t)$,

$$\begin{aligned} & \frac{dV_n(\vec{n}(t))}{dt} \\ = & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta}(t)}{\rho_s^{\beta-1}} \\ & + \sum_{s=1}^{S} K_{s1}(1+\epsilon)^{\beta} q_{f_s}^{s}(t)[(1+\epsilon)\rho_s - n_s(t)x_s(t)] \\ & +(A). \end{aligned}$$

This equation is comparable to Equation (23), and the quantity (A) is still given by (24). Further,

$$\begin{aligned} & \frac{dV_q(\vec{q}(t))}{dt} \\ = & \sum_{s=1}^{S}\sum_{i=1}^{I} q_i^s(t) \left[n_s(t)x_s(t)\mathbf{1}_{\{f_s=i\}} \right. \\ & \quad + \sum_{\substack{m\,:\,(m,i)\,\in\,\mathcal{E}, \\ H_{sj}^{mi}\,=\,1 \text{ for some } j}} r_s^{mi}(t) \\ & \quad \left. - \sum_{\substack{m\,:\,(i,m)\,\in\,\mathcal{E}, \\ H_{sj}^{im}\,=\,1 \text{ for some } j}} r_s^{im}(t) \right], \end{aligned}$$

which is comparable to Equation (25). Let $K_{s1} = 1/(1+\epsilon)^{\beta}$. Noting that

$$\sum_{s=1}^{S} q_{f_s}^s(t) n_s(t) x_s(t) = \sum_{s=1}^{S}\sum_{i=1}^{I} q_i^s(t) n_s(t) x_s(t) \mathbf{1}_{\{f_s=i\}}$$

and the corresponding terms in $\frac{dV_n}{dt}$ and $\frac{dV_q}{dt}$ cancel each other, we thus have (dropping the variable t for ease of exposition):

$$\begin{aligned} & \frac{d\mathcal{V}}{dt} \\ = & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta}}{\rho_s^{\beta-1}} + \sum_{s=1}^{S} q_{f_s}^s (1+\epsilon)\rho_s \\ & \quad + \sum_{s=1}^{S}\sum_{i=1}^{I} q_i^s \left[\sum_{\substack{m\,:\,(m,i)\,\in\,\mathcal{E}, \\ H_{sj}^{mi}\,=\,1 \text{ for some } j}} r_s^{mi} \right. \\ & \quad \left. - \sum_{\substack{m\,:\,(i,m)\,\in\,\mathcal{E}, \\ H_{sj}^{im}\,=\,1 \text{ for some } j}} r_s^{im} \right] + (A) \\ = & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta}}{\rho_s^{\beta-1}} + \sum_{s=1}^{S} q_{f_s}^s (1+\epsilon)\rho_s \\ & - \sum_{(i,m)\in\mathcal{E}} \sum_{s:H_{sj}^{im}=1 \text{ for some } j} (q_i^s - q_m^s) r_s^{im} + (A) \\ = & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta}}{\rho_s^{\beta-1}} \\ & + \sum_{s=1}^{S} q_{f_s}^s (1+\epsilon)\rho_s - \sum_{(i,m)\in\mathcal{E}} w_{im} R^{im} + (A). \end{aligned}$$

Taking expectation with respect to $\kappa(t)$, we have

$$\begin{aligned} & \mathbf{E}\left[\frac{d\mathcal{V}}{dt} | \vec{n}(t), \vec{q}(t) \right] \\ = & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta}(t)}{\rho_s^{\beta-1}} + \sum_{s=1}^{S} q_{f_s}^s (1+\epsilon)\rho_s \\ & - \sum_{(i,m)\in\mathcal{E}} w_{im} \mathbf{E}[R^{im} | \vec{q}(t)] + (A). \end{aligned} \quad (54)$$

If $\vec{\rho} \in \Theta_0$, where Θ_0 is given by (48), there must exist an $\epsilon > 0$ such that

$$(1+2\epsilon)\vec{\rho} \in \Theta_0.$$

Therefore, there must exist R_{κ}^{im}, $(i,m) \in L$, $\kappa \in \mathcal{K}$, such that

$$[R_{\kappa}^{im}] \in \mathrm{Co}(\mathcal{R}_{\kappa}), \quad \text{for all } \kappa \in \mathcal{K},$$

and there must exist $\rho_{sj}, s = 1, ..., S, j = 1, ..., \theta(s)$, such that $\rho_s = \sum_{j=1}^{\theta(s)} \rho_{sj}$, and $[R_{\kappa}^{im}]$ and $[\rho_{sj}]$ satisfy

$$(1+2\epsilon) \sum_{s=1}^{S} \sum_{j=1}^{\theta(s)} H_{sj}^{im} \rho_{sj} \le \sum_{\kappa\in\mathcal{K}} \pi_{\kappa} R_{\kappa}^{im}, \quad \text{for all } (i,m) \in L.$$

By the definition of the scheduling policy (52), given $\vec{q}(t)$,

$$\sum_{(i,m)\in\mathcal{E}} w_{im} R^{im}(t) \ge \sum_{l=1}^{L} w_{im} R_{\kappa}^{im}, \text{ if } \kappa(t) = \kappa.$$

Hence,

$$\begin{aligned} & \sum_{(i,m)\in\mathcal{E}} w_{im} \mathbf{E}[R^{im}(t) | \vec{q}(t)] \\ \ge & \sum_{l=1}^{L} w_{im} \sum_{\kappa\in\mathcal{K}} \pi_{\kappa} R_{\kappa}^{im} \\ \ge & (1+2\epsilon) \sum_{(i,m)\in\mathcal{E}} w_{im} \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} H_{sj}^{im} \rho_{sj} \\ \ge & (1+2\epsilon) \sum_{(i,m)\in\mathcal{E}} \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} (q_i^s - q_m^s) H_{sj}^{im} \rho_{sj} \\ = & (1+2\epsilon) \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} \rho_{sj} \sum_{(i,m)\in\mathcal{E}} (q_i^s - q_m^s) H_{sj}^{im} \\ = & (1+2\epsilon) \sum_{s=1}^{S}\sum_{j=1}^{\theta(s)} q_{f_s}^s \rho_{sj} = (1+2\epsilon) \sum_{s=1}^{S} q_{f_s}^s \rho_s. \end{aligned}$$

Substituting into (54), we have

$$\begin{aligned} & \mathbf{E}\left[\frac{d\mathcal{V}}{dt} | \vec{n}(t), \vec{q}(t) \right] \\ \le & -\epsilon \sum_{s=1}^{S} \frac{K_{s1} w_s n_s^{\beta}}{\rho_s^{\beta-1}} - \epsilon \sum_{s=1}^{S} q_{f_s}^s \rho_s + (A). \end{aligned}$$

Noting that the quantity (A) is non-positive, the above inequality then provides the negative drift required to establish stability. A rigorous proof without using the heuristic fluid-model can also be readily obtained by following the techniques in the Appendix, and by showing that all second-order variations in the original discrete-time system are bounded by $|(A)|/2$. In addition, the precise bound α_0 on the stepsize can be obtained as a function of the network topology and the parameters of the utility function.

VI. Conclusion

In this paper, we have studied the connection-level stability of a network employing congestion control. The type of congestion control algorithms that we used in this paper are similar to the so-called "dual solutions" in the congestion control literature [3], which can be viewed as an iterative solution to an optimization problem that maximizes the total network utility. Prior works that study the connection-level stability problem typically adopt a *time-scale separation assumption*, which assumes that, whenever the number of users in the system changes, the data rates of the users are adjusted *instantaneously* to the optimal rate allocation computed by the global utility maximization problem. In this paper, we have removed the *time-scale separation assumption*, and established that a large class of congestion control algorithms based on the so-called "dual solutions" can still achieve the *largest possible* stability region.

There could be several directions for future work. First, it would be interesting to see whether our main result holds for the so-called "primal solutions" in the congestion control literature [3]. Second, we have assumed that user inter-arrival times and file-size distributions are exponential. Removing this assumption is well-known to be a difficult problem, even for models where time-scale separation is assumed. Third, we have been unable to establish Proposition 3 for the case $\beta < 1$. It would be interesting to prove stability (or prove that the system may not be stable in general) for this case.

Appendix
Proof of Proposition 3

We now establish directly the stability of the original stochastic system governed by Equations (8), (12) and (13). As readers will see soon, our proof here follows closely along the lines of the heuristic fluid-model argument in Section III. We will still use the Lyapunov function found in Section III, i.e, we define

$$\mathcal{V}(\vec{n},\vec{q}) = V_n(\vec{n}) + V_q(\vec{q}),$$

where

$$\begin{aligned} V_n(\vec{n}) &= \frac{1}{(1+\epsilon)^\beta}\sum_{s=1}^{S}\frac{w_s n_s^{\beta+1}}{(1+\beta)\mu_s\rho_s^\beta}, \\ V_q(\vec{q}) &= \sum_{l=1}^{L}\frac{(q^l)^2}{2\alpha_l}, \end{aligned}$$

and ϵ is a positive constant in $(0,1]$ to be chosen later (in fact, ϵ will be chosen in the same way as in (20)). We begin with a few lemmas.

A. The Bound on the Changes of $V_n(\cdot)$

The first lemma bounds the change in $V_n(\cdot)$ at each time slot.

Lemma 5:

$$\begin{aligned} &\mathbf{E}[V_n(\vec{n}((k+1)T) - V_n(\vec{n}(kT))|\vec{n}(kT),\vec{q}(kT)] \\ \le\ & -\epsilon\sum_{s=1}^{S}E_0(s)\int_{kT}^{(k+1)T}\mathbf{E}[n_s^\beta(t)|\vec{n}(kT),\vec{q}(kT)]dt \\ & +\sum_{s=1}^{S}\left[\sum_{l=1}^{L}H_s^l q^l(kT)\right][(1+\epsilon)\rho_s T \\ & \qquad -\int_{t=kT}^{(k+1)T}\mathbf{E}[n_s(t)x_s(t)|\vec{n}(kT),\vec{q}(kT)]dt\Bigg] \\ & -\sum_{s=1}^{S}\frac{2^\beta-1}{8(1+\epsilon)}\frac{w_s}{\rho_s M_s^\beta}\int_{kT}^{(k+1)T} \\ & \qquad \mathbf{E}[n_s^2(t)x_s^2(t)|\vec{n}(kT),\vec{q}(kT)]dt \\ & +E_1, \end{aligned} \tag{55}$$

where $E_0(s)$ and E_1 are finite positive constants.

Proof: Fix $t\in[kT,(k+1)T)$. Over a small time interval δt, we have

$$\begin{aligned} &\mathbf{E}\left[n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t)|\vec{n}(t),\vec{q}(t)\right] \\ =\ & [(n_s(t)+1)^{\beta+1} - n_s^{\beta+1}(t)]\lambda_s\delta t \\ & +[(n_s(t)-1)^{\beta+1} - n_s^{\beta+1}(t)]\mu_s n_s(t)x_s(t)\delta t + o(\delta t). \end{aligned}$$

By the Mean-Value Theorem,

$$\begin{aligned} &(n+\Delta n)^{\beta+1} - n^{\beta+1} \\ =\ & (\beta+1)n^\beta\Delta n + \frac{\beta(\beta+1)}{2}(n+\nu\Delta n)^{\beta-1}(\Delta n)^2 \end{aligned}$$

for some $\nu\in(0,1)$. Hence, letting $\Delta n = \pm 1$, and using the relationship that

$$\begin{aligned} &(n\pm\nu)^{\beta-1} \le (2n)^{\beta-1} \text{ when } n\ge 1 \text{ and } \nu\in(0,1)\ , \\ &(n+\nu)^{\beta-1} \le 1 \text{ when } n=0 \text{ and } \nu\in(0,1), \end{aligned}$$

we have

$$\begin{aligned} &\mathbf{E}\left[n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t)|\vec{n}(t),\vec{q}(t)\right] \\ \le\ & (\beta+1)n_s^\beta(t)[\lambda_s\delta t - \mu_s n_s(t)x_s(t)\delta t] \\ & \quad +2^{\beta-2}\beta(\beta+1)n_s^{\beta-1}(t)\left[\lambda_s\delta t + \mu_s n_s(t)x_s(t)\delta t\right] \\ & \quad +N_1(s)\delta t + o(\delta t) \end{aligned}$$

for some positive constant $N_1(s)$. We then have,

$$\begin{aligned} &\frac{\mathbf{E}[V_n(\vec{n}(t+\delta t)) - V_n(\vec{n}(t))|\vec{n}(t),\vec{q}(t)]}{\delta t} \\ \le\ & \frac{1}{(1+\epsilon)^\beta}\sum_{s=1}^{S}\left\{\frac{w_s n_s^\beta(t)}{\mu_s\rho_s^\beta}[\lambda_s - \mu_s n_s(t)x_s(t)]\right. \\ & \quad +\frac{\beta 2^{\beta-2}w_s n_s^{\beta-1}(t)}{\mu_s\rho_s^\beta}\left[\lambda_s + \mu_s n_s(t)x_s(t)\right] \\ & \quad \left. +N_1(s)\right\} + o(1) \\ =\ & \frac{1}{(1+\epsilon)^\beta}\sum_{s=1}^{S}\left\{\frac{w_s n_s^\beta(t)}{\rho_s^\beta}[\rho_s - n_s(t)x_s(t)]\right. \end{aligned}$$

$$
\begin{aligned}
&\quad +\frac{\beta 2^{\beta-2} w_s n_s^{\beta-1}(t)}{\rho_s^\beta}\left[\rho_s+n_s(t)x_s(t)\right]+N_1(s)\Bigg\} \\
&\quad +o(1) \\
=&\quad -\frac{\epsilon}{(1+\epsilon)^\beta}\sum_{s=1}^{S}\frac{w_s n_s^\beta(t)}{\rho_s^{\beta-1}} \\
&\quad +\frac{1}{(1+\epsilon)^\beta}\sum_{s=1}^{S}\Bigg\{\frac{w_s n_s^\beta(t)}{\rho_s^\beta}[(1+\epsilon)\rho_s-n_s(t)x_s(t)] \\
&\quad +\frac{\beta 2^{\beta-2} w_s n_s^{\beta-1}(t)}{\rho_s^\beta}\left[\rho_s+n_s(t)x_s(t)\right]+N_1(s)\Bigg\} \\
&\quad +o(1) \\
\leq&\quad -\epsilon\sum_{s=1}^{S}N_0(s)n_s^\beta(t)+\sum_{s=1}^{S}\Bigg\{\left[\sum_{l=1}^{L}H_s^l q^l(t)\right][(1+\epsilon)\rho_s \\
&\qquad -n_s(t)x_s(t)] \qquad (56) \\
&\quad +\left[\frac{w_s}{x_s^\beta(t)}-\sum_{l=1}^{L}H_s^l q^l(t)\right][(1+\epsilon)\rho_s \\
&\qquad -n_s(t)x_s(t)] \qquad (57) \\
&\quad +w_s\left[\frac{n_s^\beta(t)}{((1+\epsilon)\rho_s)^\beta}-\frac{1}{x_s^\beta(t)}\right][(1+\epsilon)\rho_s \\
&\qquad -n_s(t)x_s(t)] \qquad (58) \\
&\quad +\frac{\beta 2^{\beta-2}}{(1+\epsilon)^\beta}\frac{w_s n_s^{\beta-1}(t)}{\rho_s^\beta}\left[\rho_s+n_s(t)x_s(t)\right] \qquad (59) \\
&\quad +\frac{N_1(s)}{(1+\epsilon)^\beta}\Bigg\}+o(1),
\end{aligned}
$$

where

$$N_0(s)=\frac{1}{(1+\epsilon)^\beta}\frac{w_s}{\rho_s^{\beta-1}}.$$

We now draw the connections to the fluid-model argument in Section III. The first two terms in (56) correspond to the first two terms in the fluid-model expression (23). They will eventually lead to the first two terms in (55) of Lemma 5. The term in (58) is the same as (A) in (24). The rest of the terms (57) and (59) did not appear in the fluid-model expression. The terms in (57) and (59) are due to, respectively: (a) the bound M_s on the maximum data rate when Equation (12) replaces Equation (16) of the fluid model; and (b) randomness when Equation (8) replaces Equation (15) of the fluid model.

Let (A) and (B) denote the terms (58) and (59), respectively. Our strategy is:

- Bound (57) by a constant.
- Show that $(A)\leq 0$.
- Show that (B) is no greater than $-(A)/2$ plus some second order terms.

Hence, the extra terms (57)-(59) will be dominated by $(A)/2$, which is non-positive.

To carry out this strategy, first note that, by (12),

$$\frac{w_s}{x_s^\beta(t)}=\max\left\{\sum_{l=1}^{L}H_s^l q^l(t),\frac{w_s}{M_s^\beta}\right\}.$$

Hence, the term (57) can be bounded by

$$
\begin{aligned}
&\left[\frac{w_s}{x_s^\beta(t)}-\sum_{l=1}^{L}H_s^l q^l(t)\right][(1+\epsilon)\rho_s-n_s(t)x_s(t)] \\
\leq&\quad \left[\frac{w_s}{x_s^\beta(t)}-\sum_{l=1}^{L}H_s^l q^l(t)\right](1+\epsilon)\rho_s \\
\leq&\quad \left[\frac{w_s}{M_s^\beta}-\sum_{l=1}^{L}H_s^l q^l(t)\right]^+(1+\epsilon)\rho_s \\
\leq&\quad N_2(s)\triangleq\frac{(1+\epsilon)w_s\rho_s}{M_s^\beta}. \qquad (60)
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
(A) &= w_s\left[\frac{n_s^\beta(t)}{((1+\epsilon)\rho_s)^\beta}-\frac{1}{x_s^\beta(t)}\right][(1+\epsilon)\rho_s \\
&\qquad -n_s(t)x_s(t)] \\
&= -w_s\frac{\left[((1+\epsilon)\rho_s)^\beta-n_s^\beta(t)x_s^\beta(t)\right]}{((1+\epsilon)\rho_s)^\beta x_s^\beta(t)} \\
&\qquad \times[(1+\epsilon)\rho_s-n_s(t)x_s(t)] \\
&\leq 0. \qquad (61)
\end{aligned}
$$

Lastly, in order to show that (B) is dominated by $-(A)/2$ plus some second-order terms, we note the following. If $n_s(t)x_s(t)\geq 2(1+\epsilon)\rho_s$, then

$$
\begin{aligned}
&[(1+\epsilon)\rho_s-n_s(t)x_s(t)]\left[((1+\epsilon)\rho_s)^\beta-n_s^\beta(t)x_s^\beta(t)\right] \\
\geq&\quad \left[\frac{n_s(t)x_s(t)}{2}\right]\left[\frac{2^\beta-1}{2^\beta}n_s^\beta(t)x_s^\beta(t)\right]. \qquad (62)
\end{aligned}
$$

Hence,

$$(A)\quad\leq\quad -\frac{2^\beta-1}{2^{\beta+1}}\frac{w_s n_s^{\beta+1}x_s(t)}{((1+\epsilon)\rho_s)^\beta},$$

and

$$(B)\quad\leq\quad \frac{\beta 2^{\beta-1}}{(1+\epsilon)^\beta}\frac{w_s n_s^\beta(t)x_s(t)}{\rho_s^\beta}.$$

Since

$$\beta 2^{\beta-1}n_s^\beta(t)\leq\frac{2^\beta-1}{2^{\beta+2}}n_s^{\beta+1}(t)+N_3(s)$$

for some positive constant $N_3(s)$, we have,

$$
\begin{aligned}
(B) &\leq -\frac{(A)}{2}+\frac{w_s N_3(s)x_s(t)}{((1+\epsilon)\rho_s)^\beta} \\
&\leq -\frac{(A)}{2}+N_4(s),
\end{aligned}
$$

where

$$N_4(s)=\frac{w_s N_3(s)M_s}{((1+\epsilon)\rho_s)^\beta}.$$

On the other hand, if $n_s(t)x_s(t)<2(1+\epsilon)\rho_s\leq 4\rho_s$, then

$$
\begin{aligned}
(B) &\leq \frac{5\beta 2^{\beta-2}}{(1+\epsilon)^\beta}\frac{w_s n_s^{\beta-1}(t)}{\rho_s^{\beta-1}} \\
&= N_5(s)n_s^{\beta-1}(t),
\end{aligned}
$$

where

$$N_5(s)=\frac{5\beta 2^{\beta-2}}{(1+\epsilon)^\beta}\frac{w_s}{\rho_s^{\beta-1}}.$$

Hence, in both cases, we have,

$$(B) \quad \leq \quad -\frac{(A)}{2} + N_5(s)n_s^{\beta-1}(t) + N_4(s). \tag{63}$$

Substituting (60) and (63) back to (57-59), we thus have,

$$\begin{aligned}
&\frac{\mathbf{E}[V_n(\vec{n}(t+\delta t)) - V_n(\vec{n}(t))|\vec{n}(t),\vec{q}(t)]}{\delta t} \\
\leq \quad & -\sum_{s=1}^{S}\left[\epsilon N_0(s)n_s^{\beta}(t) - N_5(s)n_s^{\beta-1}(t)\right] \\
& +\sum_{s=1}^{S}\left[\sum_{l=1}^{L} H_s^l q^l(t)\right][(1+\epsilon)\rho_s - n_s(t)x_s(t)] \\
& +\sum_{s=1}^{S}\frac{(A)}{2} + \sum_{s=1}^{S}[N_1(s)+N_2(s)+N_4(s)] \\
& +o(1).
\end{aligned} \tag{64}$$

It remains to bound $(A)/2$ and the second order term $N_5(s)n_s^{\beta-1}(t)$. We will use (61) and (62) again. Since $x_s(t) \leq M_s$, if $n_s(t)x_s(t) \geq 2(1+\epsilon)\rho_s$, we have,

$$\begin{aligned}
\frac{(A)}{2} \quad & \leq \quad -w_s\frac{2^{\beta}-1}{2^{\beta+2}}\frac{n_s^{\beta+1}(t)x_s^{\beta+1}(t)}{((1+\epsilon)\rho_s)^{\beta}M_s^{\beta}} \\
& \leq \quad -w_s\frac{2^{\beta}-1}{2^{\beta+2}}\frac{2^{\beta-1}}{(1+\epsilon)\rho_s}\frac{n_s^2(t)x_s^2(t)}{M_s^{\beta}} \\
& \leq \quad -w_s\frac{2^{\beta}-1}{8(1+\epsilon)}\frac{n_s^2(t)x_s^2(t)}{\rho_s M_s^{\beta}}.
\end{aligned} \tag{65}$$

On the other hand, if $n_s(t)x_s(t) < 2(1+\epsilon)\rho_s$, we still have $(A)/2 \leq 0$. Hence, in both cases,

$$\frac{(A)}{2} \leq -w_s\frac{2^{\beta}-1}{8(1+\epsilon)}\frac{n_s^2(t)x_s^2(t)}{\rho_s M_s^{\beta}} + N_6(s), \tag{66}$$

where

$$N_6(s) = w_s\frac{2^{\beta}-1}{8(1+\epsilon)}\frac{(2(1+\epsilon)\rho_s)^2}{\rho_s M_s^{\beta}}.$$

Further, note that

$$N_5(s)n_s^{\beta-1}(t) \leq \frac{\epsilon N_0(s)}{2}n_s^{\beta}(t) + N_7(s) \tag{67}$$

for some positive constant $N_7(s)$. Substituting (66) and (67) into (64), we have,

$$\begin{aligned}
&\lim_{\delta t\to 0^+}\frac{\mathbf{E}[V_n(\vec{n}(t+\delta t)) - V_n(\vec{n}(t))|\vec{n}(t),\vec{q}(t)]}{\delta t} \\
\leq \quad & -\epsilon\sum_{s=1}^{S}E_0(s)n_s^{\beta}(t) \\
& +\sum_{s=1}^{S}\left[\sum_{l=1}^{L} H_s^l q^l(t)\right][(1+\epsilon)\rho_s - n_s(t)x_s(t)] \\
& -\sum_{s=1}^{S}\frac{2^{\beta}-1}{8(1+\epsilon)}\frac{w_s}{\rho_s M_s^{\beta}}n_s^2(t)x_s^2(t) + E_1(s),
\end{aligned} \tag{68}$$

where $E_0(s) = N_0(s)/2$ and

$$E_1(s) = [N_1(s)+N_2(s)+N_4(s)+N_6(s)+N_7(s)].$$

Recall that $kT \leq t < (k+1)T$. We now take expectation at both sides of (68) with respect to the distribution of $[\vec{n}(t),\vec{q}(t)]$ conditioned on $[\vec{n}(kT),\vec{q}(kT)]$. The left hand side of (68) then becomes

$$\mathbf{E}\left\{\lim_{\delta t\to 0^+}\frac{\mathbf{E}[V_n(\vec{n}(t+\delta t)) - V_n(\vec{n}(t))|\vec{n}(t),\vec{q}(t)]}{\delta t}\middle|\vec{n}(kT),\vec{q}(kT)\right\}.$$

We can show (in Appendix B) that the order of the outer expectation and the limit can be switched, i.e.,

$$\begin{aligned}
&\mathbf{E}\left\{\lim_{\delta t\to 0^+}\frac{\mathbf{E}[V_n(\vec{n}(t+\delta t)) - V_n(\vec{n}(t))|\vec{n}(t),\vec{q}(t)]}{\delta t}\middle|\vec{n}(kT),\vec{q}(kT)\right\} \\
&= \lim_{\delta t\to 0^+}\mathbf{E}\left\{\frac{\mathbf{E}[V_n(\vec{n}(t+\delta t)) - V_n(\vec{n}(t))|\vec{n}(t),\vec{q}(t)]}{\delta t}\middle|\vec{n}(kT),\vec{q}(kT)\right\} \\
&= \lim_{\delta t\to 0^+}\frac{\mathbf{E}[V_n(\vec{n}(t+\delta t)) - V_n(\vec{n}(t))|\vec{n}(kT),\vec{q}(kT)]}{\delta t}.
\end{aligned} \tag{69}$$

Using (68) and (69), and integrating over $t \in [kT,(k+1)T)$. The result of (55) then follows. ■

B. Validity of the Switch of Order in (69)

For completeness, we next provide the proof that it is valid to switch the order of the expectation and the limit in (69). (Readers may skip this subsection and go directly to Appendix C if this part of proof is of less interest.) We will use the Dominated Convergence Theorem [36, p468] to establish this validity. Towards this end, it is sufficient to show that, there exist functions $g_1(\vec{n}(t),\vec{q}(t))$, $g_2(\vec{n}(t),\vec{q}(t))$ and some $\delta_0 > 0$ such that

- The functions $g_1(\vec{n}(t),\vec{q}(t))$ and $g_2(\vec{n}(t),\vec{q}(t))$ are both integrable conditioned on $(\vec{n}(kT),\vec{q}(kT)$, i.e.,

$$\begin{aligned}
\mathbf{E}[g_1(\vec{n}(t),\vec{q}(t))|\vec{n}(kT),\vec{q}(kT)] < +\infty, \\
\mathbf{E}[g_2(\vec{n}(t),\vec{q}(t))|\vec{n}(kT),\vec{q}(kT)] > -\infty.
\end{aligned}$$

- For all $0 < \delta t < \delta_0$,

$$\begin{aligned}
& g_2(\vec{n}(t),\vec{q}(t)) \\
\leq \quad & \frac{\mathbf{E}[V_n(\vec{n}(t+\delta t) - V_n(\vec{n}(t))|\vec{n}(t),\vec{q}(t)]}{\delta t} \\
\leq \quad & g_1(\vec{n}(t),\vec{q}(t)).
\end{aligned} \tag{70}$$

We next show that there exist such $g_1(\cdot)$, $g_2(\cdot)$ and δ_0. The basic idea is that $n_s(t)$ cannot change faster than some Poisson process. Firstly, using a simple sample-path argument, we can easily show that, for any $\delta t > 0$,

$$n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t) \leq (n_s(t)+Y_1)^{\beta+1} - n_s^{\beta+1}(t),$$

where Y_1 is a Poisson random variable with mean $\lambda_s\delta t$, and Y_1 is independent of $[\vec{n}(t),\vec{q}(t)]$. Using the mean-value theorem, we have, for some $h \in [0,1]$,

$$(n_s(t)+Y_1)^{\beta+1} = n_s^{\beta+1}(t) + (\beta+1)(n_s(t)+hY_1)^{\beta}Y_1,$$

Hence,

$$\begin{aligned} & n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t) \\ \leq\ & (\beta+1)(n_s(t)+Y_1)^\beta Y_1 \\ \leq\ & 2^\beta(\beta+1)(n_s^\beta(t)+Y_1^\beta)Y_1 \\ =\ & 2^\beta(\beta+1)n_s^\beta(t)Y_1 + 2^\beta(\beta+1)Y_1^{\beta+1}. \end{aligned}$$

Dividing by δt and taking conditional expectation with respect to $[\vec{n}(t), \vec{q}(t)]$, we have

$$\begin{aligned} & \frac{\mathbf{E}[n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t)|\vec{n}(t), \vec{q}(t)]}{\delta t} \\ \leq\ & 2^\beta(\beta+1)n_s^\beta(t)\frac{\mathbf{E}[Y_1]}{\delta t} + 2^\beta(\beta+1)\frac{\mathbf{E}[Y_1^{\beta+1}]}{\delta t}. \end{aligned}$$

Since Y_1 is a Poisson random variable with mean $\lambda_s \delta t$, we have,

$$\frac{\mathbf{E}[Y_1]}{\delta t} = \lambda_s.$$

Further,

$$\begin{aligned} \frac{\mathbf{E}[Y_1^{\beta+1}]}{\delta t} & \leq \frac{\mathbf{E}[\exp\{(\beta+1)Y_1\} - 1]}{\delta t} \\ & = \frac{\exp[\lambda_s \delta t(e^{\beta+1} - 1)] - 1}{\delta t}. \end{aligned}$$

Using the derivative of the right hand side at $\delta t = 0$, we can show that there exist a constant δ_0, which does not depend on $[\vec{n}(t), \vec{q}(t)]$, such that for any positive $\delta t \leq \delta_0$,

$$\frac{\mathbf{E}[Y_1^{\beta+1}]}{\delta t} \leq 2\lambda_s(e^{\beta+1} - 1).$$

Hence, for all positive $\delta t \leq \delta_0$, we have,

$$\begin{aligned} & \frac{\mathbf{E}[n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t)|\vec{n}(t), \vec{q}(t)]}{\delta t} \\ \leq\ & 2^\beta(\beta+1)\lambda_s n_s^\beta(t) + 2^{\beta+1}(\beta+1)(e^{\beta+1}-1)\lambda_s. \end{aligned}$$

If we let

$$\begin{aligned} & g_1(\vec{n}(t), \vec{q}(t)) \\ \triangleq\ & \frac{1}{(1+\epsilon)^\beta}\sum_{s=1}^{S}\frac{w_s}{(1+\beta)\mu_s\rho_s^\beta}\left[2^\beta(\beta+1)\lambda_s n_s^\beta(t)\right. \\ & \left. +2^{\beta+1}(\beta+1)(e^{\beta+1}-1)\lambda_s\right], \end{aligned}$$

it then provides one side of the bound in (70) needed for the Dominated Convergence Theorem to hold. To obtain the other side of the bound, using a simple sample-path argument again, we can easily show that, for any $\delta t > 0$, conditioned on $[\vec{n}(t), \vec{q}(t)]$,

$$\begin{aligned} & n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t) \\ \geq\ & (\max\{0, n_s(t) - Y_2\})^{\beta+1} - n_s^{\beta+1}(t) \\ \geq\ & n_s^{\beta+1}(t) - (n_s(t)+Y_2)^{\beta+1}, \end{aligned}$$

where Y_2 is a Poisson random variable with mean $\mu_s n_s(t) x_s(t) \delta t$. Using the mean-value theorem again, we have,

$$(n_s(t)+Y_2)^{\beta+1} = n_s^{\beta+1}(t) + (\beta+1)(n_s(t)+hY_2)^\beta Y_2$$

for some $h \in [0,1]$. Hence,

$$\begin{aligned} & n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t) \\ \geq\ & -(\beta+1)(n_s(t)+Y_2)^\beta Y_2 \\ \geq\ & -2^\beta(\beta+1)(n_s^\beta(t)+Y_2^\beta)Y_2 \\ =\ & -2^\beta(\beta+1)n_s^\beta(t)Y_2 - 2^\beta(\beta+1)Y_2^{\beta+1}. \end{aligned}$$

Dividing by δt and taking conditional expectation with respect to $[\vec{n}(t), \vec{q}(t)]$, we have

$$\begin{aligned} & \frac{\mathbf{E}[n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t)|\vec{n}(t), \vec{q}(t)]}{\delta t} \\ \geq\ & -2^\beta(\beta+1)n_s^\beta(t)\frac{\mathbf{E}[Y_2|\vec{n}(t), \vec{q}(t)]}{\delta t} \\ & -2^\beta(\beta+1)\frac{\mathbf{E}[Y_2^{\beta+1}|\vec{n}(t), \vec{q}(t)]}{\delta t}. \end{aligned}$$

Conditioned on $[\vec{n}(t), \vec{q}(t)]$, since Y_2 is a Poisson random variable with mean $\mu_s n_s(t) x_s(t) \delta t$, we have,

$$\frac{\mathbf{E}[Y_2|\vec{n}(t), \vec{q}(t)]}{\delta t} = \mu_s n_s(t) x_s(t).$$

Further,

$$\begin{aligned} & \frac{\mathbf{E}[Y_2^{\beta+1}|\vec{n}(t), \vec{q}(t)]}{\delta t} \\ \leq\ & \frac{\mathbf{E}[\exp\{(\beta+1)Y_2\} - 1|\vec{n}(t), \vec{q}(t)]}{\delta t} \\ =\ & \frac{\exp[\mu_s n_s(t) x_s(t) \delta t(e^{\beta+1}-1)] - 1}{\delta t}. \end{aligned}$$

Note that the right hand side is increasing in δt. Hence, there exists function

$$g_0(\vec{n}(t), \vec{q}(t)) \triangleq \frac{\exp[\mu_s n_s(t) x_s(t) \delta_0(e^{\beta+1}-1)] - 1}{\delta_0},$$

such that for any positive $\delta t < \delta_0$,

$$\frac{\mathbf{E}[Y_2^{\beta+1}|\vec{n}(t), \vec{q}(t)]}{\delta t} \leq g_0(\vec{n}(t), \vec{q}(t)).$$

Hence, for all positive $\delta t < \delta_0$,

$$\begin{aligned} & \frac{\mathbf{E}[n_s^{\beta+1}(t+\delta t) - n_s^{\beta+1}(t)|\vec{n}(t), \vec{q}(t)]}{\delta t} \\ \geq\ & -2^\beta(\beta+1)\mu_s x_s(t) n_s^{\beta+1}(t) \\ & -2^\beta(\beta+1)g_0(\vec{n}(t), \vec{q}(t)). \end{aligned}$$

If we let

$$\begin{aligned} & g_2(\vec{n}(t), \vec{q}(t)) \\ \triangleq\ & -\frac{1}{(1+\epsilon)^\beta}\sum_{s=1}^{S}\left\{\frac{w_s}{(1+\beta)\mu_s\rho_s^\beta}\right. \\ & \times\left[2^\beta(\beta+1)\mu_s x_s(t) n_s^{\beta+1}(t)\right. \\ & \left.\left. +2^\beta(\beta+1)g_0(\vec{n}(t), \vec{q}(t))\right]\right\}, \end{aligned}$$

it provides another side of the bound needed in (70) for the Dominated Convergence Theorem to hold. Finally, it is easy to show that both $g_1(\vec{n}(t), \vec{q}(t))$ and $g_2(\vec{n}(t), \vec{q}(t))$ are integrable conditioned on $[\vec{n}(kT), \vec{q}(kT)]$, because $n_s(t)$ at most increases by a Poisson random variable with mean $\lambda_s(t-kT)$), and $x_s(t)$ is bounded by M_s. Therefore, according to the Dominated Convergence Theorem [36, p468], the switch of the expectation and the limit in (69) is correct.

C. Bound on the Changes of $V_q(\cdot)$

The next lemma bounds the change in $V_q(\cdot)$. For simplicity, we use the following matrix notation. Let A denote the $L\times L$ diagonal matrix whose l-th diagonal element is α_l. Let H denote the $L\times S$ matrix whose (l,s)-element is H_s^l. Let $\vec{R}=[R^1,...,R^l]^{\text{tr}}$, where $[\cdot]^{\text{tr}}$ denotes the transpose. Further let $X_s(t)=n_s(t)x_s(t)$ and let $\vec{X}(t)=[X_1(t),...,X_S(t)]^{\text{tr}}$. Then

$$V_q(\vec{q})=\frac{\vec{q}^{\,\text{tr}}A^{-1}\vec{q}}{2},$$

and the update on the implicit costs (13) can be written as

$$\begin{aligned}&\vec{q}((k+1)T)\\ =&\left[\vec{q}(kT)+A\left(H\int_{kT}^{(k+1)T}\vec{X}(t)dt-\vec{R}T\right)\right]^+. \quad (71)\end{aligned}$$

Lemma 6:

$$\begin{aligned}&\mathbf{E}[V_q(\vec{q}((k+1)T)-V_q(\vec{q}(kT))|\vec{n}(kT),\vec{q}(kT)]\\ \le\ &\vec{q}^{\,\text{tr}}(kT)\left[H\int_{kT}^{(k+1)T}\mathbf{E}[\vec{X}(t)|\vec{n}(kT),\vec{q}(kT)]dt\right.\\ &\left.\qquad -\vec{R}T\right]\\ &+T\alpha_{\max}\bar{\mathcal{S}}\bar{\mathcal{L}}\\ &\quad\times\sum_{s=1}^{S}\left[\int_{kT}^{(k+1)T}\mathbf{E}[n_s^2(t)x_s^2(t)|\vec{n}(kT),\vec{q}(kT)]dt\right]\\ &+E_2, \quad (72)\end{aligned}$$

where $\alpha_{\max}=\max_l\alpha^l$, $\bar{\mathcal{L}}$ and $\bar{\mathcal{S}}$ are defined as in Proposition 3, and E_2 is a finite positive constant.

Proof: By (71),

$$\begin{aligned}&V_q(\vec{q}((k+1)T)-V_q(\vec{q}(kT))\\ \le\ &\vec{q}^{\,\text{tr}}(kT)\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt-\vec{R}T\right]\\ &+\frac{1}{2}\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt-\vec{R}T\right]^{\text{tr}}\\ &\quad\times A\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt-\vec{R}T\right].\end{aligned}$$

We now draw the connections to the fluid-model argument in Section III. Note that the first term on the right hand side corresponds to that of (25). The other term on the right hand side is due to the “jumps” in the discrete-time implicit-cost update when Equation (13) replaces Equation (17) of the fluid model. Using Cauchy-Schwartz Inequality, we have,

$$\begin{aligned}&\frac{1}{2}\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt-\vec{R}T\right]^{\text{tr}}\\ &\quad\times A\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt-\vec{R}T\right]\\ \le\ &\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt\right]^{\text{tr}}A\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt\right]\\ &+T^2\vec{R}^{\,\text{tr}}A\vec{R}.\end{aligned}$$

Further,

$$\begin{aligned}&\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt\right]^{\text{tr}}A\left[H\int_{kT}^{(k+1)T}\vec{X}(t)dt\right]\\ =\ &\sum_{l=1}^{L}\alpha_l\left[\sum_{s=1}^{S}H_s^l\int_{kT}^{(k+1)T}n_s(t)x_s(t)dt\right]^2\\ \le\ &\sum_{l=1}^{L}\alpha_l\left[\sum_{s=1}^{S}H_s^l\right]\sum_{s=1}^{S}H_s^l\left(\int_{kT}^{(k+1)T}n_s(t)x_s(t)dt\right)^2\\ \le\ &\bar{\mathcal{S}}\sum_{l=1}^{L}\alpha_l\left[\sum_{s=1}^{S}H_s^l\left(\int_{kT}^{(k+1)T}n_s(t)x_s(t)dt\right)^2\right]\\ \le\ &T\bar{\mathcal{S}}\sum_{l=1}^{L}\alpha_l\sum_{s=1}^{S}H_s^l\int_{kT}^{(k+1)T}n_s^2(t)x_s^2(t)dt\\ =\ &T\bar{\mathcal{S}}\sum_{s=1}^{S}\left[\int_{kT}^{(k+1)T}n_s^2(t)x_s^2(t)dt\right]\left[\sum_{l=1}^{L}\alpha_lH_s^l\right]\\ \le\ &T\alpha_{\max}\bar{\mathcal{S}}\bar{\mathcal{L}}\sum_{s=1}^{S}\left[\int_{kT}^{(k+1)T}n_s^2(t)x_s^2(t)dt\right]. \quad (73)\end{aligned}$$

Letting $E_2=T^2\vec{R}^{\,\text{tr}}A\vec{R}$, the result of (72) then follows. ■

D. Proof of Proposition 3

Adding (55) to (72), and noting that

$$\begin{aligned}&\sum_{s=1}^{S}\left\{\left[\sum_{l=1}^{L}H_s^lq^l(kT)\right]\right.\\ &\quad\left.\times\int_{kT}^{(k+1)T}\mathbf{E}[n_s(t)x_s(t)|\vec{n}(kT),\vec{q}(kT)]dt\right\}\\ =\ &\sum_{l=1}^{L}q^l(kT)\\ &\quad\times\sum_{s=1}^{S}H_s^l\int_{kT}^{(k+1)T}\mathbf{E}[n_s(t)x_s(t)|\vec{n}(kT),\vec{q}(kT)]dt\\ =\ &\vec{q}^{\,\text{tr}}(kT)\left[H\int_{kT}^{(k+1)T}\mathbf{E}[\vec{X}(t)|\vec{n}(kT),\vec{q}(kT)]dt\right], \quad (74)\end{aligned}$$

we have,

$$\begin{aligned}&\mathbf{E}[\mathcal{V}(\vec{n}((k+1)T),\vec{q}((k+1)T))\\ &\quad-\mathcal{V}(\vec{n}(kT),\vec{q}(kT))|\vec{n}(kT),\vec{q}(kT)]\\ \le\ &-\epsilon\sum_{s=1}^{S}E_0(s)\int_{kT}^{(k+1)T}\mathbf{E}[n_s^\beta(t)|\vec{n}(kT),\vec{q}(kT)]dt\\ &+\sum_{s=1}^{S}\left[\sum_{l=1}^{L}H_s^lq^l(kT)\right](1+\epsilon)\rho_sT-\vec{q}^{\,\text{tr}}(kT)\vec{R}T\\ &-\sum_{s=1}^{S}\left\{\left[\frac{2^\beta-1}{8(1+\epsilon)}\frac{w_s}{\rho_sM_s^\beta}-T\alpha_{\max}\bar{\mathcal{S}}\bar{\mathcal{L}}\right]\right.\\ &\quad\left.\times\int_{kT}^{(k+1)T}\mathbf{E}[n_s^2(t)x_s^2(t)|\vec{n}(kT),\vec{q}(kT)]dt\right\} \quad (75)\\ &+E_3,\end{aligned}$$

where $E_3 = E_1 + E_2$. If the condition (14) is satisfied, the product term in (75) is non-positive. Hence, by a rearrangement of the order of the summations, we have,

$$\begin{aligned}
&\mathbf{E}[\mathcal{V}(\vec{n}((k+1)T), \vec{q}((k+1)T)) \\
&\qquad -\mathcal{V}(\vec{n}(kT), \vec{q}(kT)) | \vec{n}(kT), \vec{q}(kT)] \\
\leq\ & -\epsilon \sum_{s=1}^{S} E_0(s) \int_{kT}^{(k+1)T} \mathbf{E}[n_s^{\beta}(t) | \vec{n}(kT), \vec{q}(kT)] dt \\
& + T \vec{q}^{\,\mathrm{tr}}(kT) \left[(1+\epsilon) H \vec{\rho} - \vec{R} \right] + E_3.
\end{aligned}$$

By assumption, $\vec{\rho}$ lies strictly inside Θ_0. Hence, there exists some $\epsilon \in (0,1]$ such that $(1+2\epsilon)H\vec{\rho} \leq \vec{R}$. Use this value of ϵ in the definition of $\mathcal{V}(\cdot,\cdot)$. we then have,

$$\begin{aligned}
&\mathbf{E}[\mathcal{V}(\vec{n}((k+1)T), \vec{q}((k+1)T)) \\
&\qquad -\mathcal{V}(\vec{n}(kT), \vec{q}(kT)) | \vec{n}(kT), \vec{q}(kT)] \\
\leq\ & -\epsilon \sum_{s=1}^{S} E_0(s) \int_{kT}^{(k+1)T} \mathbf{E}[n_s^{\beta}(t) | \vec{n}(kT), \vec{q}(kT)] dt \\
& - \epsilon T \vec{q}^{\,\mathrm{tr}}(kT) H \vec{\rho} + E_3 \\
\leq\ & -\epsilon' \left[\sum_{s=1}^{S} n_s^{\beta}(kT) + \sum_{l=1}^{L} q^l(kT) \right] + E_3
\end{aligned}$$

for some $\epsilon' > 0$. Following the telescoping argument in [16], we have

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t E \left(\sum_{s=1}^{S} n_s^{\beta}(t) + \sum_{l=1}^{L} Q^l(t) \right) dt < \infty.$$

Since $\beta \geq 1$, this implies stability in the mean.

Remark: This proof will not work for $\beta < 1$, in which case the relationship (65) will fail to hold. (We need (65) to cancel the second term in (72) of the change in $V_q(\cdot)$.) We have not been able to either prove or disprove the result of Proposition 3 for the case $\beta < 1$. We will leave it for future work.

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