On the Queue-Overflow Probabilities of Distributed Scheduling Algorithms

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***Abstract*— In this paper, we are interested in using large-deviations theory to characterize the asymptotic decay-rate of the queue-overflow probability for distributed wireless scheduling algorithms, as the overflow threshold approaches infinity. We consider ad-hoc wireless networks where each link interferes with a given set of other links, and we focus on a distributed scheduling algorithm called Q-SCHED, which is introduced by Gupta et al. First, we derive a lower bound on the asymptotic decay rate of the queue-overflow probability for Q-SCHED. We then present an upper bound on the decay rate for all possible algorithms operating on the same network. Finally, using these bounds, we are able to conclude that, subject to a given constraint on the asymptotic decay rate of the queue-overflow probability, Q-SCHED can support a provable fraction of the offered loads achievable by any algorithms.**

I. Introduction

Link scheduling is an important problem for ad-hoc wireless networks. In wireless networks the transmissions at neighboring links can interfere with each other. Hence, in order to maximize the capacity of the system, it is critical to schedule only a subset of non-interfering links at each time. There have been many studies on designing and analyzing scheduling algorithms for wireless network. A notable result is the well-known maximum-weight scheduling algorithm, which has been shown to be throughput-optimal, i.e., it can stabilize the network at the largest set of offered loads [1]. However, this algorithm is centralized and with high computational complexity. Therefore, many researchers have proposed low-complexity and distributed scheduling algorithms, (see, e.g. [2], [3]). Often, the goal is to be able to stabilize the network for a provable fraction of the capacity region. For example, the low-complexity algorithm in [2] has been shown to sustain close to 1/2 of the capacity region under the node-exclusive interference model.

To date most studies of wireless scheduling algorithms have mainly focused on stabilities. In other words, they ensure that the queues do not grow to infinity. Although stability is an important criterion, for many real-time applications stability is far from being sufficient. For example, when watching streaming video or listening to streaming audio, the user would expect that the delay of every packet can be upper bounded with high probability. As stability only ensures that the queue-length of each link remains finite, it cannot guarantee such type of stringent quality-of-service (QoS) requirements.

In certain cases, the probability of delay violation can be mapped to the probability of queue overflow. Unfortunately, both problems have been known to be very difficult. First, the exact probability distribution is usually mathematically intractable. Hence, one often has to turn to asymptotic techniques, such as large-deviations. For wireline networks, many results have been obtained using large-deviations techniques [4], based on the assumption that the packet arrival process is known and the service rate of each link is time-invariant. However, in wireless networks, the service rate process is time-varying. Some progress has been made for the case when the scheduling decision is based only on the channel state, which means that the service rate process has known statistics [5]. However, for many wireless scheduling algorithms, even the statistics of the service rate process are unknown.

Recently, the delay-violation or queue-overflow probability for a number of queue-length based scheduling algorithms, for which the statistics of the service rate process are unknown, have been studied in [6] - [9] using sample-path large-deviations. In these works, the algorithms are centralized and are for a single cell. Further, the algorithms are deterministic in the sense that the scheduling decision is a deterministic function of the system state.

In this paper, we will develop techniques to estimate and control the QoS of distributed scheduling algorithms for ad hoc networks. We will focus on a random access algorithm for ad hoc wireless networks called Q-SCHED [2] . Note that due to the distributed and random nature of Q-SCHED, the techniques in prior works [6] - [9] do not apply directly. As in [6] - [9], the questions that we are interested in are: a) how to estimate the decay rate of the queue overflow probability of this algorithm, and b) given an overflow constraint, how to calculate the set of offer-load vectors that this algorithm can support. To answer these questions, we will first obtain a lower bound on the decay rate of the overflow probability for Q-SCHED. Then, based on this bound, we provide a lower bound on the set of offer-load vectors that this algorithm could support at a given queue-overflow constraint. To the best of our knowledge, this is the first work that characterizes the queue-overflow probability of distributed scheduling algorithms for ad-hoc networks in a large-deviations setting. Finally, we show that subject to a given queue-overflow constraint, the offer load supported by Q-SCHED is at least a provable fraction of the offered load supported by any other algorithms.

II. System Model

We use the model from [2]. We consider a wireless network of N nodes. Let V be the set of nodes, E be the set of directed links between nodes, and $G(V,E)$ be the directed connectivity graph of the network. Each link $l \in E$ interferes with a set of other links in E, which we denote as $\mathcal{E}_l$. We assume that if $k \in \mathcal{E}_l$ then $l \in \mathcal{E}_k$, i.e., the interference relationship is symmetric. We also let $l \in \mathcal{E}_l$, i.e.,

$$\mathcal{E}_l = \{l\} \cup \{l' \in E : l' \text{ interferes with } l\}.$$

This interference set varies when different communication techniques are used. For example, for bluetooth, we use the node-exclusive interference model, also known as the primary interference model or the one-hop interference model, where $\mathcal{E}_l$ is the set of all links that are connected to either end-point of l. In IEEE 802.11 WLAN, the interference set $\mathcal{E}_l$ will be the two-hop neighbors of l, including l.

We assume a slotted system. Let $a_l(n)$ denotes the number of packets that arrive at link l in time-slot n. We assume that for each l, $a_l(1), a_l(2), ...$ are *i.i.d.* and $\lambda_l = \mathbf{E}[a_l(1)]$. Moreover, we assume that $a_l(n)$ is upper bounded by A_M for all $n > 0$ and all $l \in E$, i.e., $0 \le a_l(n) < A_M$, which means that the number of arrival packets is finite in each time slot.

Let $d_l(n)$ denote the number of packets that can be served by link l in time-slot n. Assume that the capacity of each link is a fixed number c_l. Let $s_l(n) = 1$ indicates that link l is scheduled in time-slot n, $s_l(n) = 0$ otherwise. Clearly, $d_l(n) = c_l s_l(n)$. We assume a single-hop system, i.e., packets served at link l immediately leave the system. Let $q_l(n)$ denote the backlog of link l in slot n, and $\vec{q}(n) = \big(q_1(n), q_2(n), ..., q_{|E|}(n)\big)$. Then the evolution of each $q_l(n)$ is given by $q_l(n+1) = [q_l(n) + a_l(n) - d_l(n)]^+$, where $[\cdot]^+$ denote the projection to $[0, \infty)$.

We consider the algorithm Q-SCHED that was introduced in [2]. In this algorithm, it is assumed that at the beginning of each time-slot every link l knows the queue-lengths of all links in its interference set $\mathcal{E}_l$ and also the queue-lengths of all links in the interference set $\mathcal{E}_k$ for every $k \in \mathcal{E}_l$. Each time slot is divided into two parts: a scheduling slot and a data transmission slot. Links that are chosen in the scheduling slot will transmit their packets in the data transmission slot. The scheduling slot is further divided into M mini-slots. At the beginning of each time-slot n, each link l first computes:

$$P_l(n) = \alpha \frac{\frac{q_l(n)}{c_l}}{\max_{i \in \mathcal{E}_l} \sum_{k \in \mathcal{E}_i} \frac{q_k(n)}{c_k}},$$

where $\alpha = \log(M)$. Then, each link l picks a backoff time $Y_l(n)$ from $\{1, 2, ..., M+1\}$ according to the following probabilities:

$$\mathbf{P}(Y_l(n) = M+1) = e^{-P_l(n)};$$

$$\mathbf{P}(Y_l(n) = m) = e^{-P_l(n)\frac{m-1}{M}} - e^{-P_l(n)\frac{m}{M}}, m = 1, 2, ..., M.$$

A link that chooses backoff time $Y_l(n) = k \le M$ will start transmission at the k-th mini-slot unless it has already heard a transmission from one of its interfering links. If a link chooses a backoff time equals to $M+1$ it will not attempt to transmit in this time slot. If two or more links that interfere with each other begin to transmit simultaneously, collision will occur and all of these transmissions will fail. Finally, any link that hears the collision will not attempt to transmit in this time slot.

We now present an important lemma proved in [2] for Q-SCHED, which will be used in our derivation. Define

$$V(n) = \max_{i \in E} \sum_{l \in \mathcal{E}_i} \frac{q_l(n)}{c_l},$$

which denotes the largest sum of backlog in any interference neighborhood.

Lemma 1. *Q-SCHED scheduling policy guarantees that for any $\epsilon_0 \ge 0$ and constants $C_1, C_2 \ge 0$, there exists a constant R such that if $V(n) \ge R$, then for any $\eta \in [0,1]$ and for any link i such that*

$$\sum_{l \in \mathcal{E}_i} \frac{q_l(n)}{c_l} \ge \eta \left(V(n) - C_1 - C_2\epsilon_0\right),$$

the following holds,

$$\sum_{l \in \mathcal{E}_i} Pr\{\textit{Link } l \textit{ is scheduled}\} \ge \eta \left(1 - \frac{\log(M)+1}{M} - \epsilon_0\right).$$

Note that although the original statement of Lemma 1 in [2] requires that $\epsilon_0, C_1, C_2 > 0$, the proof there also trivially holds for the case when $\epsilon_0, C_1, C_2 \ge 0$. Letting $\eta = 1$, this then implies that, when $V(n)$ is large, with high probability at least one link will be scheduled in those interference neighborhood with sum of backlog close to $V(n)$. In [2], this lemma has been used to establish the negative drift of the Lyapunov function $V(n)$ whenever the offered load satisfies that, for some $\epsilon_0 > 0$,

$$\sum_{l \in \mathcal{E}_i} \frac{\lambda_l}{c_l} \le 1 - \frac{\log(M)+1}{M} - \epsilon_0, \text{ for all links } i. \quad (1)$$

For the rest of the paper, we assume that (1) holds because otherwise we do not know the stability of the system.

In this paper, we are interested in queue-overflow probabilities. For example, we may want to know the probability that the maximum queue length exceeds a given threshold B. On the other hand, with the techniques developed in this paper, it is more convenient to work with the probability

$$\mathbf{P}\left(\max_{i \in E} \sum_{l \in \mathcal{E}_i} \frac{q_l}{c_l} \ge B\right). \quad (2)$$

However, even calculating this probability is mathematically intractable. Hence, we will use large-deviations theory to estimate it. We are interested in the following limits:

$$I_0(\vec{\lambda}) \triangleq -\limsup_{B \to \infty} \frac{1}{B} \log \mathbf{P}\left(\max_{i \in E} \sum_{l \in \mathcal{E}_i} \frac{q_l}{c_l} \ge B\right),$$

$$J_0(\vec{\lambda}) \triangleq -\liminf_{B \to \infty} \frac{1}{B} \log \mathbf{P}\left(\max_{i \in E} \sum_{l \in \mathcal{E}_i} \frac{q_l}{c_l} \ge B\right).$$

Clearly, $I_0(\vec{\lambda})$ provides a lower bound on the decay rate of (2) and $J_0(\vec{\lambda})$ provides an upper bound.

III. The lower bound

We first develop a lower bound for $I_0(\vec{\lambda})$. For any link i in E, define the scaled queue length:

$$q_i^B(t) = \frac{1}{B} q_i\left(\lfloor Bt \rfloor\right).$$

Note that this expression represents the standard large-deviations scaling that shrinks both time and magnitude. We also define the scaled version of the Lyapunov function:

$$v^B(t) = V(\vec{q}^B(t)).$$

The queue overflow criterion is $\left\{v^B(t) \geq 1\right\}$. For ease of exposition, we consider a system that starts at $t=0$. For a given $T>0$, we are interested in the following probability:

$$I_0^T(\vec{\lambda}) \triangleq -\limsup_{B\to\infty} \frac{1}{B}\log \mathbf{P}\left(v^B(T) \geq 1 \,\middle|\, v^B(0)=0\right).$$

Intuitively, as $T \to \infty$, one would expect that $I_0^T(\vec{\lambda})$ approaches $I_0(\vec{\lambda})$, the lower bound on the decay rate of the stationary overflow probability [8], [9]. We will use Lemma 1 to derive a lower bound for $I_0^T(\vec{\lambda})$. Note that Lemma 1 provides a lower bound on the service rate of those interference sets whose backlogs are almost the largest. However, these interference sets with largest backlog can change from time to time, which makes it difficult for us to track the system dynamics directly by Lemma 1. To address the problem, in the following derivation we divide the entire scaled time into many small intervals. In each small interval, the interference sets that have almost the largest backlog do not change and therefore we are able to use Lemma 1 to estimate $I_0^T(\vec{\lambda})$.

A. Local Rate Function

For a fixed t, let $\delta > 0$ be a small number. Let $\Delta v^B(\delta,t) = v^B(t+\delta) - v^B(t)$ denote the drift of the scaled Lyapunov function. Let $\mathcal{Q}$ be a closed and bounded set such that $V(\vec{q}) \geq v > 0$ for all $\vec{q} \in \mathcal{Q}$. Our first goal is to find the following limit given $\vec{q} \neq \vec{0}$ and $W > 0$,

$$\lim_{B\to\infty} \frac{1}{B} \log \sup_{\vec{q}\in\mathcal{Q}} \mathbf{P}\left(\Delta v^B(\delta,t) \geq \delta W \,\middle|\, \vec{q}^B(t) = \vec{q}\right). \quad (3)$$

We call (3) the local rate function, which is the maximum asymptotic decay rate of the probability that the growth rate of v^B is no smaller than W over all $\vec{q}\in\mathcal{Q}$, conditioned on $\vec{q}^B(t)=\vec{q}$. Since the arrival and departure are both bounded, for any $i \in E$ there must exist C_i such that

$$\left|\sum_{l\in\mathcal{E}_i} \frac{q_l(n+1)}{c_l} - \sum_{l\in\mathcal{E}_i}\frac{q_l(n)}{c_l}\right| \leq C_i$$

for all n. We next define the set $\mathcal{I}(\vec{q},\delta)$ as

$$\mathcal{I}(\vec{q},\delta) = \left\{ i \in E \,\middle|\, \sum_{l\in\mathcal{E}_i}\frac{q_l}{c_l} \geq V(\vec{q}) - \delta C_i \right\}. \quad (4)$$

Intuitively, $\mathcal{I}(\vec{q}^B(t),\delta)$ is the set of links that have the close-to-largest sum of backlog $\sum_{l\in\mathcal{E}_i}\frac{q_l^B(t)}{c_l}$ in their respective interference range. Given that $W > 0$, if the event $\{\Delta v^B(\delta,t) \geq \delta W\}$ happens, we will have $v^B(t) < v^B(\delta+t)$. For any B, if $i \in E$ and $i \notin \mathcal{I}(\vec{q}^B(t),\delta)$, then $\sum_{l\in\mathcal{E}_i}\frac{q_l^B(t)}{c_l} < v^B(t) - \delta C_i$, and hence

$$\sum_{l\in\mathcal{E}_i}\frac{q_l^B(t+\delta)}{c_l} < v^B(t) - \delta C_i + \delta C_i = v^B(t) < v^B(\delta+t).$$

Therefore, for any large B, only $i \in \mathcal{I}(\vec{q}^B(t),\delta)$ could potentially maximize $\sum_{l\in\mathcal{E}_i}\frac{q_l^B}{c_l}$ at time $t+\delta$. So the change between $v^B(t)$ and $v^B(t+\delta)$ can be bounded by the maximum increment of $\sum_{l\in\mathcal{E}_i}\frac{q_l^B}{c_l}$ among those $i\in\mathcal{I}(\vec{q}^B(t),\delta)$. More specifically, we have

$$\Delta v^B(\delta,t) \leq \max_{i\in\mathcal{I}(\vec{q}^B(t),\delta)} \sum_{n=\lfloor Bt\rfloor+1}^{\lfloor B(t+\delta)\rfloor} \frac{1}{B}\sum_{l\in\mathcal{E}_i}\frac{a_l(n)-d_l(n)}{c_l}.$$

Define:

$$A_i(n) \triangleq \sum_{l\in\mathcal{E}_i}\frac{a_l(n)}{c_l}, \qquad D_i(n) \triangleq \sum_{l\in\mathcal{E}_i}\frac{d_l(n)}{c_l}. \quad (5)$$

Let $\bar{A}_i$ be the mean of $A_i(n)$. Note that by our assumption, there exists $\epsilon_0 > 0$ such that $\bar{A}_i < 1 - \frac{\log(M)+1}{M} - \epsilon_0$, for all links i.

Now consider Equation (3), since $q(t)$ is Markovian, so is $q^B(t)$. We thus have

$$\begin{aligned}&\lim_{B\to\infty}\frac{1}{B}\log\sup_{\vec{q}\in\mathcal{Q}}\mathbf{P}\left(\Delta v^B(\delta,t)\geq\delta W \,\middle|\, \vec{q}^B(t)=\vec{q}\right)\\ &=\lim_{B\to\infty}\frac{1}{B}\log\sup_{\vec{q}\in\mathcal{Q}}\mathbf{P}\left(\Delta v^B(\delta,0)\geq\delta W \,\middle|\, \vec{q}^B(0)=\vec{q}\right).\end{aligned}$$

Hence, for the following derivation, we will take $t=0$, and drop the variable t when there is no source of confusion. Moreover, for ease of exposition, let $\mathbf{P}_{\vec{q}}(\cdot)$ denote the probability distribution conditioned on $\vec{q}(0)=\vec{q}$.

Lemma 2. *Assume that $\vec{q}^B(0)=\vec{q}$ and $V(\vec{q})\geq v>0$. For any $i\in\mathcal{I}(\vec{q},\delta)$, $D_i(n)$ is defined in (5). For any $v,\xi>0$, there exists δ_0 and B_0 such that for all $\delta\leq\delta_0$ and for all $B\geq B_0$, the following holds for $1\leq n\leq\lfloor B\delta\rfloor$,*

$$\begin{aligned}&\mathbf{P}_{\vec{q}}(D_i(n)\geq 1\,|\,D_i(n-1),A_i(n-1),...,D(1),A(1))\\ &\qquad\geq 1-\epsilon,\\ &\mathbf{P}_{\vec{q}}(D_i(n)=0\,|\,D_i(n-1),A_i(n-1),...,D(1),A(1))\leq\epsilon.\end{aligned}$$

where $\epsilon = \frac{\log M+1}{M}+\xi$.

Lemma 2 implies that, when δ is small, the service rate at each neighborhood of link $i\in\mathcal{I}(\vec{q},\delta)$ is no smaller than 1 with probability no smaller than $1-\epsilon$.

Let $\hat{D}(n)$, $n=1,2,...$ be i.i.d random variables with distribution

$$\hat{D}(n) = \begin{cases} 1, & \text{with prob. } 1-\epsilon\\ 0, & \text{with prob. } \epsilon\end{cases}$$

Let $A_i^B(\delta) \triangleq \frac{1}{B}\sum_{n=1}^{[B\delta]} A_i(n)$, $D_i^B(\delta) \triangleq \frac{1}{B}\sum_{n=1}^{[B\delta]} D_i(n)$, $\hat{D}^B(\delta) \triangleq \frac{1}{B}\sum_{n=1}^{[B\delta]} \hat{D}(n)$, $Z_i(n) \triangleq A_i(n) - D_i(n)$, $Z_i^B(\delta) \triangleq A_i^B(\delta) - D_i^B(\delta)$, $\hat{Z}(n) \triangleq A_i(n) - \hat{D}_i(n)$, and $\hat{Z}_i^B(\delta) \triangleq A_i^B(\delta) - \hat{D}_i^B(\delta)$. We then have

$$\Delta v^B(\delta) \le \max_{i\in\mathcal{I}(\vec{q}^B(0),\delta)} Z_i^B(\delta), \tag{6}$$

For each $i \in E$, let

$$H_i^{\hat{Z}} \triangleq \limsup_{B\to\infty} \frac{1}{B} \log \mathbf{P}\left(\hat{Z}_i^B(\delta) \ge \delta W\right),$$
$$H_{\max}^{Z} \triangleq \limsup_{B\to\infty} \frac{1}{B} \log \sup_{\vec{q}\in\mathcal{Q}} \mathbf{P}_{\vec{q}}\left(\max_{i\in\mathcal{I}(\vec{q},\delta)} Z_i^B(\delta) \ge \delta W\right).$$

From (6), we have

$$\limsup_{B\to\infty} \frac{1}{B} \log \sup_{\vec{q}\in\mathcal{Q}} \mathbf{P}\left(\Delta v^B(\delta) \ge \delta W \,\middle|\, \vec{q}^B(0) = \vec{q}\right) \le H_{\max}^{Z}.$$

Unfortunately, $H_{\max}^{Z}$ is difficult to compute directly. On the other hand, $H_i^{\hat{Z}}$'s are fairly easy to compute. Next, we will establish a relationship between $H_{\max}^{Z}$ and $H_i^{\hat{Z}}$'s so that we can estimate $H_{\max}^{Z}$ by $H_i^{\hat{Z}}$'s.

Lemma 3. *Assume that $\vec{q}^B(0) = \vec{q}$ and $V(\vec{q}) \ge v > 0$. For any $\xi > 0$, define ϵ as in Lemma 2. For any $v > 0$, there exists $\delta_0 > 0$ and $B_0 > 0$, such that for any $W > 0$, any $i \in \mathcal{I}(\vec{q},\delta)$ and for all $0 < \delta \le \delta_0$ and $B \ge B_0$, we will have*

$$\mathbf{P}_{\vec{q}}(Z_i^B(\delta) \ge W) \le \mathbf{P}(\hat{Z}_i^B(\delta) \ge W).$$

Lemma 4. *Let $\mathcal{Q}$ be a closed and bounded set such that $V(\vec{q}) \ge v > 0$ for all $\vec{q} \in \mathcal{Q}$. For any $\xi > 0$, define ϵ as in Lemma 2. There exists $\delta_0 > 0$, such that for any $W > 0$ and for all $0 < \delta \le \delta_0$, we have*

$$H_{\max}^{Z} \le \max_{i\in E} H_i^{\hat{Z}}. \tag{7}$$

Due to page limitations, we omit the proof of these lemmas, which is available in [10]. Lemma 4 actually says that $H_{\max}^{Z}$ can be bounded by the maximum of all $H_i^{\hat{Z}}$'s. Each individual $H_i^{\hat{Z}}$ is much easier to obtain. We therefore has the following theorem. We omit its proof here. Interested reader could refer to [10] for more details.

Theorem 5. *Assume that for some $\epsilon_0 > 0$, inequality* (1) *holds. For any small and positive ξ such that $0 < \xi < \epsilon_0$, define ϵ as in Lemma 2. Given t, also assume that $\vec{q}^B(t) = \vec{q} \in \mathcal{Q}$, where $\mathcal{Q}$ is a closed and bounded set such that $V(\vec{q}) \ge v > 0$ for all $\vec{q} \in \mathcal{Q}$. There exists $\delta_0 > 0$ such that for all $0 < \delta \le \delta_0$ and $W > 0$,*

$$\begin{aligned}\limsup_{B\to\infty} \frac{1}{B} \log \sup_{\vec{q}\in\mathcal{Q}} \mathbf{P}\left(\Delta v^B(\delta, t) \ge \delta W \,\middle|\, \vec{q}^B(t) = \vec{q}\right)\\ \le -\delta \min_{i\in E} \inf_{0\le d\le 1-\epsilon} (I_i^A(d+W) + I^{\hat{D}}(d)).\end{aligned} \tag{8}$$

where

$$\begin{aligned}I^{\hat{D}}(d) &= \sup_{\theta\in\mathbb{R}} \left\{\theta d - \log \mathbf{E}\left(e^{\theta \hat{D}(1)}\right)\right\}\\ I_i^A(a) &= \sup_{\theta\in\mathbb{R}} \left\{\theta a - \log \mathbf{E}\left(e^{\theta A_i(1)}\right)\right\}.\end{aligned} \tag{9}$$

B. The Lower Bound

For fixed $T > 0$, we now derive a lower bound on $I_0^T(\vec{\lambda})$ in this subsection. Fix a small $v \in (0,1)$, choose δ_0 as in Theorem 5. Let $0 < \delta < \delta_0$ such that $\delta = T/n$ for some integer n. Since $v(0) = 0$ and the arrivals are bounded, there exists $v_{\max} > 0$, such that $v^B(t) \le v_{\max}$ for all $t \in [0,T]$. Fix $\zeta < v$. Given $v_0 = 0, v_n = 1, 0 \le v_1, ..., v_{n-1} \le v_{\max}$, define $\Gamma_k(v_k) = \left\{v_k - \zeta \le v^B(k\delta) \le v_k + \zeta\right\}$ for $k = 1, 2, ..., n-1$, $\Gamma_0(v_0) = \left\{v^B(0) = v_0\right\}$ and $\Gamma_n(v_n) = \left\{v^B(n\delta) \ge v_n\right\}$. Let m_v, $0 \le m_v < n$, be the largest integer m such that $v_{m-1} < 2v$. We first fix $v_0, ..., v_n$. For ease of exposition, we will use Γ_k to denote $\Gamma_k(v_k)$ when there is no source of confusion. Consider the follow probability:

$$\mathbf{P}\left(\Gamma_1 \cap ... \cap \Gamma_n \,|\Gamma_0\right) \tag{10}$$

Roughly speaking, this is the probability that the trajectory $v^B(t)$ follows $v_0, v_1, ..., v_n$ given that it starts at v_0. Clearly,

$$\begin{aligned}(10) \le &\mathbf{P}\left(\Gamma_n \,|\Gamma_{n-1}, ..., \Gamma_{m_v}, \Gamma_0\right)\\ &\times \mathbf{P}\left(\Gamma_{n-1} \,|\Gamma_{n-2}, ..., \Gamma_{m_v}, \Gamma_0\right) \times ... \times \mathbf{P}\left(\Gamma_{m_v} \,|\Gamma_0\right).\end{aligned} \tag{11}$$

Define $\Psi_k^B(\delta,\zeta) \triangleq \left\{\Delta v^B(\delta, (k-1)\delta) \ge v_k - v_{k-1} - 2\zeta\right\}$. Let

$$\mathcal{Q}_k = \begin{cases} \left\{\vec{q}\,|V(\vec{q}) \in [v_{k-1} - \zeta, v_{k-1} + \zeta]\right\}, k = 2, ..., n \\ \left\{\vec{q}\,|V(\vec{q}) = v_0\right\}, k = 1 \end{cases}.$$

Also define

$$\phi_k^B(\delta,\zeta) = \sup_{\vec{q}\in\mathcal{Q}_k} \mathbf{P}\left(\Psi_k^B(\delta,\zeta) \,\middle|\, \vec{q}^B((k-1)\delta) = \vec{q}\right).$$

We then have the following lemma.

Lemma 6. *For any k, $m_v < k \le n$, the following holds*

$$\mathbf{P}\left(\Gamma_k \,|\Gamma_{k-1}, ..., \Gamma_{m_v}, \Gamma_0\right) \le \phi_k^B(\delta,\zeta). \tag{12}$$

The proof of Lemma is omitted due to page limits, which is available in [10].

Theorem 7. *Assume that for some $\epsilon_0 > 0$, inequality* (1) *holds. For any small and positive ξ such that $0 < \xi < \epsilon_0$, define ϵ as in Lemma 2. For any $T > 0$, the lower bound on the decay rate function satisfies*

$$I_0^T(\vec{\lambda}) \ge \inf_{W>0} \min_{i\in E} \inf_{d\le(1-\epsilon)} \frac{(I_i^A(d+W) + I^{\hat{D}}(d))}{W} \triangleq L. \tag{13}$$

Proof: For any $\zeta > 0$, there exists a finite set $\mathcal{V}$ of vectors $(v_n, ..., v_0)$, $v_0 = 0, v_n = 1, 0 \le v_1, ..., v_{n-1} \le v_{\max}$, such that

$$\begin{aligned}&\bigcup_{(v_n,...,v_0)\in\mathcal{V}} \Gamma_{n-1}(v_{n-1}) \times ... \times \Gamma_1(v_1)\\ &\supseteq \Big\{ \left(v^B((n-1)\delta), ..., v^B(\delta)\right) \big| 0 \le v^B((n-1)\delta)\\ &\qquad \le v_{\max}, ..., 0 \le v^B(\delta) \le v_{\max}\Big\},\end{aligned}$$

where $\times$ denotes the Cartesian product. Then taking the advantage of Lemma 6, one can show that

$$\mathbf{P}\left(v^B(T) \ge 1 \,\middle|\, v^B(0) = v_0\right) \le \sum_{(v_n,...,v_0)\in\mathcal{V}} \prod_{k=m_v+1}^{n} \phi_k^B(\delta,\zeta).$$

If we take the log and let B go to infinity, we will have

$$\limsup_{B\to\infty} \frac{1}{B} \log \mathbf{P}\left(v^B(T) \geq 1 \,\middle|\, v^B(0) = v_0\right)$$
$$\leq \max_{(v_n,\ldots,v_0)\in\mathcal{V}} \limsup_{B\to\infty} \frac{1}{B} \sum_{k=m_v+1}^{n} \log \phi_k^B(\delta,\zeta).$$

To estimate the limit in the above inequality, we will use the local rate function derived in Section III.A. From the definition of $\phi_k^B(\delta,\zeta)$ and Theorem 5, if $\zeta < v$, since $v_k \geq 2v, m_v < k \leq n$, we will have

$$\limsup_{B\to\infty} \frac{1}{B} \log \phi_k^B(\delta,\zeta) \leq -(v_k - v_{k-1} - 2\zeta)L$$

if $v_k - v_{k-1} - 2\zeta \geq 0$. On the other hand, if $v_k - v_{k-1} - 2\zeta < 0$, the above inequality still holds because the left hand side is always less than 0. Therefore, taking the sum from $k = m_v + 1$ to n, one will get

$$\max_{(v_n,\ldots,v_0)\in\mathcal{V}} \limsup_{B\to\infty} \frac{1}{B} \sum_{k=m_v+1}^{n} \log \phi_k^B(\delta,\zeta)$$
$$\leq -\min_{(v_n,\ldots,v_0)\in\mathcal{V}} \left(1 - 2v - 2(n-m_v)\zeta\right) L$$
$$\leq -\left(1 - 2v - 2n\zeta\right) L$$

Let $v \to 0$ and $\zeta \to 0$, we have

$$I_0^T(\vec{\lambda}) = -\limsup_{B\to\infty} \frac{1}{B} \log \mathbf{P}\left(v^B(T) \geq 1 \,\middle|\, v^B(0) = 0\right) \geq L.$$

Note that the bound in Theorem 7 is independent from T. As $T \to \infty$ we could infer that $I_0^T(\vec{\lambda}) \to I_0(\vec{\lambda})$. Hence we could expect that $I_0(\vec{\lambda}) \geq L$. Such a limiting argument can be rigorously made using the Freidlin-Wentzell construction as in [8], [9].

IV. An Upper Bound

In this section, we will develop an upper bound for the decay-rate function $J_0(\vec{\lambda})$ under the node-exclusive interference model. In the node-exclusive model, each interference set may have at most two links scheduled in the same slot. Consider an fictitious algorithm such that

$$\mathbf{P}\left(\sum_{l\in\mathcal{E}_i} \frac{d_l(n)}{c_l} = 2\right) = 1, \text{for all } n.$$

Clearly, this fictitious algorithm will provide a lower bound on the overflow probability over all possible algorithms. We denote such algorithm as OPTIMAL. Note that OPTIMAL may not exist, and it is only used to derive an upper bound on $J_0(\vec{\lambda})$. We now consider (2) under algorithm OPTIMAL. Let $Z_i^B(t), A_i^B(t), D_i^B(t)$ have the same meaning as before. Now, the derivation are much easier since $Z_i^B(t)$ are i.i.d. across t. According to Theorem 6.6 in [11], the overflow rate function of OPTIMAL for each link i is given by

$$\lim_{B\to\infty} \frac{1}{B} \log \mathbf{P}\left(\sum_{l\in\mathcal{E}_i} \frac{q_l^B(0)}{c_l} > 1\right) = -\inf_{x>0} \frac{I_{i,\text{opt}}^Z(x)}{x}$$

where $I_{i,\text{opt}}^Z(x)$ is the rate function of Z_i^B under OPTIMAL algorithm. It is trivial to show that $I_{i,\text{opt}}^Z(x) = I_i^A(x+2)$. Hence, the decay-rate of the queue-overflow probability of this fictitious system is given by

$$\lim_{B\to\infty} \frac{1}{B} \log \mathbf{P}\left(v^B(0) > 1\right) = -\min_{i\in E} \inf_{a>0} \frac{I_i^A(a+2)}{a}. \quad (14)$$

Then, we have the following upper bound:

$$J_0(\vec{\lambda}) \leq I_{\text{opt}} \triangleq \min_{i\in E} \inf_{a>0} \frac{I_i^A(a+2)}{a}.$$

We now pose a constraint on this decay rate function. Suppose that we want to guarantee that $I_{\text{opt}} \geq \theta_0$. Let Λ_i^A be the cumulant generating function of the arrival process, $\Lambda_i^A(\theta) = \log \mathbf{E}\left(e^{\theta A_i(1)}\right)$. Using the fact that I_i^A is the Legendre transform of Λ_i^A and Λ_i^A is also the Legendre transform of I_i^A, one can show that

$$I_{opt} \geq \theta_0 \Leftrightarrow \max_{i\in E} \frac{\Lambda_i^A(\theta_0)}{\theta_0} \leq 2. \quad (15)$$

The quantity $\frac{\Lambda_i^A(\theta_0)}{\theta_0}$ is often called the effective bandwidth of the arrival process. The inequality (15) implies that the maximum possible effective capacity region of the system under any algorithm is such that the effective bandwidth in every interference range must be no greater than 2.

V. Comparisons

We now compare our lower bound in Section III with the upper bound in Section IV in three cases.

A. Deterministic Arrivals

We first consider the case when the arrivals are deterministic, i.e., $a_i(n) = \lambda_i$ for all $i \in E$ and $n = 1, 2, \ldots$. Recall that $\bar{A}_i = \sum_{l\in\mathcal{E}_i} \frac{\lambda_l}{c_l}$ is the mean of $A_i(n)$, and the rate function of A_i will be

$$I_i^A(a) = \begin{cases} 0, a = \bar{A}_i \\ \infty, \text{otherwise} \end{cases}.$$

Then, the lower bound of the decay rate of the queue-overflow probability is

$$L = \inf_{W\geq 0} \min_{i\in E} \frac{I^{\hat{D}}(\bar{A}_i - W)}{W}.$$

We then put a constraint on the decay rate. Using similar technique as we did when deriving (15), we could get

$$L \geq \theta_0 \Leftrightarrow \Lambda^{\hat{D}}(-\theta_0)/(-\theta_0) \geq \bar{A}_i, \forall i \in E.$$

Note that under the assumption of deterministic arrival, we have the effective bandwidth of arrival process equals to its mean, i.e., $\Lambda_i^A(\theta_0)/\theta_0 = \bar{A}_i$. Therefore, the effective capacity region of Q-SCHED is such that the sum of effective bandwidth in each interference range is no greater than $\Lambda^{\hat{D}}(-\theta_0)/(-\theta_0)$. Note that $\Lambda^{\hat{D}}(-\theta_0) = \log\left(\epsilon + (1-\epsilon)e^{-\theta_0}\right)$. Thus, under the assumption of deterministic arrivals the effective capacity region of Q-SCHED at a given constraint is at least $\frac{\log\left(\epsilon+(1-\epsilon)e^{-\theta_0}\right)}{-2\theta_0}$ of that of any other algorithms.

B. Infinite Number of Mini-slots

In this case, we assume that the number of mini-slots M is infinite. We also assume that the variable ξ in Lemma 2 is 0, which implies that ϵ in Lemma 2 equals to 0. It follows that the rate function of $\hat{D}$ will be:

$$I^{\hat{D}}(d) = \begin{cases} 0, d = 1 \\ \infty, \text{otherwise} \end{cases}.$$

Consequently, the right-hand-side of (13) can be simplified

$$L = \inf_{W \geq 0} \min_{i \in E} \frac{I_i^A(W+1)}{W}.$$

Using the same method as in the section IV, if we pose a constraint on the decay rate, e.g. if we want to guarantee that $L \geq \theta_0$, then we could get the "effective bandwidth" in this case:

$$\max_{i \in E} \frac{\Lambda_i^A(\theta_0)}{\theta_0} \leq 1.$$

This result implies that, at a given θ_0, the effective capacity region of the Q-SCHED algorithm is such that the sum of the effective bandwidth in each interference range is no greater than 1. Comparing with (15), we note that the effective capacity region of Q-SCHED is at least 1/2 of that of any other algorithms.

C. General Case

Now we consider the general case. Note that when the value of ϵ_0 in (1) satisfies $0 < \xi < \epsilon_0$ and $W > 0$, we have $W + 1 - \epsilon > \bar{A}_i$ for all links i by our assumption. If $d \geq 1 - \epsilon$, we have $W + d \geq W + 1 - \epsilon > \bar{A}_i$, which means that $I_i^A(d+W)$ and $I^{\hat{D}}(d)$ are increasing with respect to d. Therefore,

$$\inf_{d \leq 1-\epsilon} I_i^A(d+W) + I^{\hat{D}}(d) = \inf_d I_i^A(d+W) + I^{\hat{D}}(d).$$

Then, we have

$$\begin{aligned} & L \geq \theta_0 \\ \Leftrightarrow & \inf_{W>0} \min_{i \in E} \inf_{d \leq (1-\epsilon)} \{I_i^A(d+W) + I^{\hat{D}}(d) - \theta W\} \geq 0 \\ \Leftrightarrow & \inf_{W>0,d} \{I_i^A(d+W) + I^{\hat{D}}(d) - \theta W\} \geq 0, \forall i \in E. \end{aligned}$$

Further note that $I_i^A(d+W) + I^{\hat{D}}(d) - \theta W \geq 0$ if $W \leq 0$. Hence, we have, for all $i \in E$

$$\begin{aligned} & \inf_{W>0,d} \{I_i^A(d+W) + I^{\hat{D}}(d) - \theta W\} \geq 0 \\ \Leftrightarrow & \inf_{W,d} \{I_i^A(d+W) + I^{\hat{D}}(d) - \theta W\} \geq 0 \\ \Leftrightarrow & \sup_{W+d} \{(d+W)\theta - I_i^A(d+W)\} \\ & \qquad + \sup_d \{-d\theta - I^{\hat{D}}(d)\} \leq 0 \\ \Leftrightarrow & \Lambda_i^A(\theta) + \Lambda^{\hat{D}}(-\theta) \leq 0. \end{aligned}$$

Hence

$$\begin{aligned} L \geq \theta_0 \Leftrightarrow & \Lambda_i^A(\theta) + \Lambda^{\hat{D}}(-\theta) \leq 0, \forall i \in E \\ \Leftrightarrow & \max_{i \in E} \frac{\Lambda_i^A(\theta_0)}{\theta_0} \leq \frac{\Lambda^{\hat{D}}(-\theta_0)}{-\theta_0}. \end{aligned}$$

Similarly to the case in Section V.A, this means that the effective capacity region of Q-SCHED is at least $\frac{\log\left(\epsilon + (1-\epsilon)e^{-\theta_0}\right)}{-2\theta_0}$ of that of any other algorithms.

VI. Conclusion

In this paper, we developed a lower bound on the decay rate of the overflow probability for scheduling algorithm for ad-hoc wireless networks called Q-SCHED. We also show that the effective capacity that Q-SCHED could support is a provable fraction of the maximum possible effective capacity over all other algorithms, subject to a given constraint on the decay-rate of the queue-overflow probability. For future work, we will extend the approach of this paper to other wireless scheduling algorithms and other types of performance guarantees.

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