

B7-7 The solution to such a problem is not unique. We shall present 2 solutions to the problem in what follows. Note that from the requirement stated in the problem, the dominant closed-loop poles must have $\zeta = 0.5$ and $\omega_n = 3$, or

$$s = -1.5 \pm j 2.5981$$

Notice that the angle deficiency is

$$\phi = 180^\circ - (-120^\circ - 100.894^\circ) = 40.894^\circ$$

Method 1: If we choose the zero of the lead compensator at $s = -1$ so that it will cancel the plant pole at $s = -1$, then the compensator pole must be located at $s = -3$, or

$$G_c(s) = K \frac{T_1 s + 1}{T_2 s + 1} = \frac{K T_1}{T_2} \left(\frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}} \right) = \frac{K T_1}{T_2} \frac{s+1}{s+3}$$

$$\text{or } G_c(s) = 3K \cdot \frac{s+1}{s+3} \quad \text{i.e. } \alpha = \frac{T_1}{T_2} = \frac{\frac{1}{T_1}}{\frac{1}{T_2}} = \frac{3}{1} = 3$$

The value of K can be determined by use of the magnitude condition.

$$\left| 3K \cdot \frac{s+1}{s+3} \frac{1}{s(s+1)} \right|_{s=-1.5+j2.5981} = 1$$

$$\text{or } K = \left| \frac{s(s+3)}{30} \right|_{s=-1.5+j2.5981} = 0.3$$

Hence

$$G_c(s) = 0.9 \frac{s+1}{s+3}$$

The open-loop transfer function is

$$G_c(s) G(s) = \frac{9}{s(s+3)}$$

The closed-loop transfer function $C(s)/R(s)$ becomes as follows:

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 3s + 9}$$

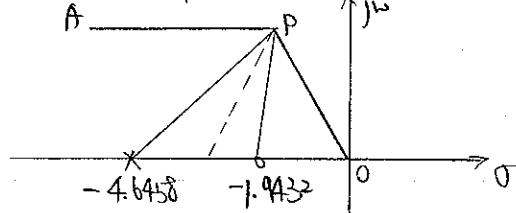
Method 2:* Referring to the figure shown below, if we bisect angle $\angle OPA$ and take 20.44° on each side, then the locations of the zero and pole are found as follows:

zero at $s = -1.9432$

pole at $s = -4.6458$

Thus, $G_c(s)$ can be given as

$$G_c(s) = K \frac{T_1 s + 1}{T_2 s + 1} = K \frac{T_1}{T_2} \frac{s + 1.9432}{s + 4.6458} = 2.391 K \frac{s + 1.9432}{s + 4.6458}$$



Method 2 follows the example in book and is not covered in class.

The value of K can be determined by use of the magnitude condition

$$|2.391K \frac{s+1.9432}{s+4.6458} \frac{1}{s(s+1)}|_{s=-1.5+2.598i} = 1$$

or $K = \left| \frac{(s+4.6458)s(s+1)}{23.91(s+1.9432)} \right|_{s=-1.5+2.598i} = 0.5138$

Hence, the compensator $G_C(s)$ is given by

$$G_C(s) = 1.2285 \frac{s+1.9432}{s+4.6458} = 0.5138 \frac{0.5146s+1}{0.2152s+1}$$

Then, the open-loop transfer function becomes as

$$G_C(s) G(s) = 0.5138 \left(\frac{0.5146s+1}{0.2152s+1} \right) \frac{1}{s(s+1)}$$

The closed-loop transfer function is

$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{5.138 (0.5146s+1)}{s(s+1)(0.2152s+1) + 5.138 (0.5146s+1)} \\ &= \frac{2.6448s + 5.138}{0.2152s^3 + 1.2152s^2 + 3.6448s + 5.138} \end{aligned}$$

It is interesting to compare the static velocity error constants for the two systems designed above

For the system designed by Method 1:

$$K_V = \frac{2}{s \rightarrow 0} s \frac{1}{s(s+3)} = 3$$

For the system designed by Method 2:

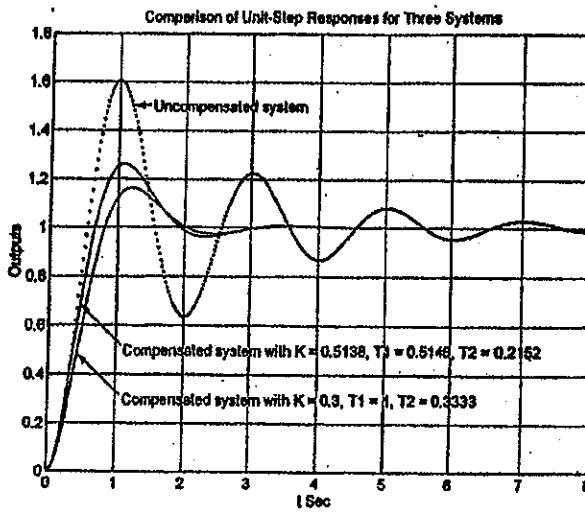
$$K_V = \frac{2}{s \rightarrow 0} s (-0.5138) \frac{0.5146s+1}{0.2152s+1} \frac{1}{s(s+1)} = 5.138$$

The system designed by Method 2 gives a larger value of the static velocity error constant. This means that the systems designed by Method 2 will give smaller steady-state errors in following ramp inputs than the system designed by method 1.

In what follows, we compare the unit-step responses of the 3 systems: the original uncompensated system, the system designed by Method 1, and the system designed by Method 2. The MATLAB program used to obtain the unit-step response curves and the resulting unit-step response curves are shown on the next page.

% ***** Comparison of unit-step responses for three systems *****

```
num = [0 0 10];
den = [1 1 10];
num1 = [0 0 9];
den1 = [1 3 9];
num2 = [0 0 2.644 5.138];
den2 = [0.2152 1.2152 3.644 5.138];
t = 0:0.02:8;
c = step(num,den,t);
c1 = step(num1,den1,t);
c2 = step(num2,den2,t);
plot(t,c,'.',t,c1,'-',t,c2,'-.')
grid
title('Comparison of Unit-Step Responses for Three Systems')
xlabel('t Sec')
ylabel('Outputs')
text(1.5,1.5,'Uncompensated system')
text(1.1,0.5,'Compensated system with K = 0.5138, T1 = 0.5146, T2 = 0.2152')
text(1.1,0.3,'Compensated system with K = 0.3, T1 = 1, T2 = 0.3333')
```



B-7-8 Method 1:

The closed-loop transfer function $(C(s)/R(s))$ is given by

$$\frac{C(s)}{R(s)} = \frac{K(Ts+1)}{s(s+2)+K(Ts+1)}$$

Since the closed-loop poles are specified to be

$$s = -2 \pm j2$$

we obtain

$$s(s+2)+K(Ts+1) = (s+2+j2)(s+2-j2)$$

$$\text{or } s^2 + (2+KT)s + K = s^2 + 4s + 8 \quad \text{true for any } s$$

Hence, we require

$$2+KT=4, \quad K=8$$

which results in

$$T=0.25, \quad K=8$$

Method 2: Since the desired closed-loop pole is at $-2+j2$, the angle deficiency is

$$\phi = 180^\circ - [-135^\circ - 45^\circ] = 45^\circ$$

The additional zero of the compensation must provide an angle that compensates ϕ .

Hence, the zero should be at -4 , i.e.,

$$\frac{1}{T} = 4, \quad T = \frac{1}{4}$$

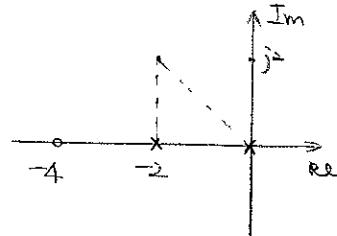
Further, according to the standard form

$$1+KT \frac{(s+\frac{1}{T})}{s(s+2)} = 0$$

the gain KT can be determined from the magnitude condition, i.e.,

$$KT = \frac{\sqrt{2} \times 2}{2\sqrt{2}} = 2$$

$$\therefore K = \frac{2}{T} = 8$$



B7-9 The angle deficiency at the closed-loop pole $s = -2 + j2\sqrt{3}$ is

$$180^\circ - (-120^\circ - 40^\circ) = 30^\circ$$

The lead compensator must contribute 30° .

Let us choose the zero of the lead compensator at $s = -2$. Then, the pole of the compensator must be located at $s = -4$. Thus,

$$G_c(s) = K \cdot \frac{s+2}{s+4}$$

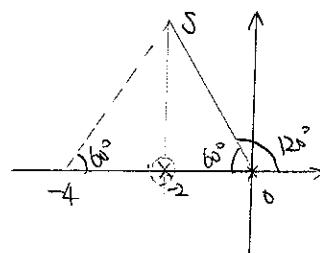
The gain K is determined from the magnitude condition.

$$\left| K \frac{s+2}{s+4} \right| \Big|_{s=-2+j2\sqrt{3}} = 1$$

or $K = \left| \frac{s+4}{10} \right| \Big|_{s=-2+j2\sqrt{3}} = 1.6$

Hence,

$$G_c(s) = 1.6 \cdot \frac{s+2}{s+4}$$



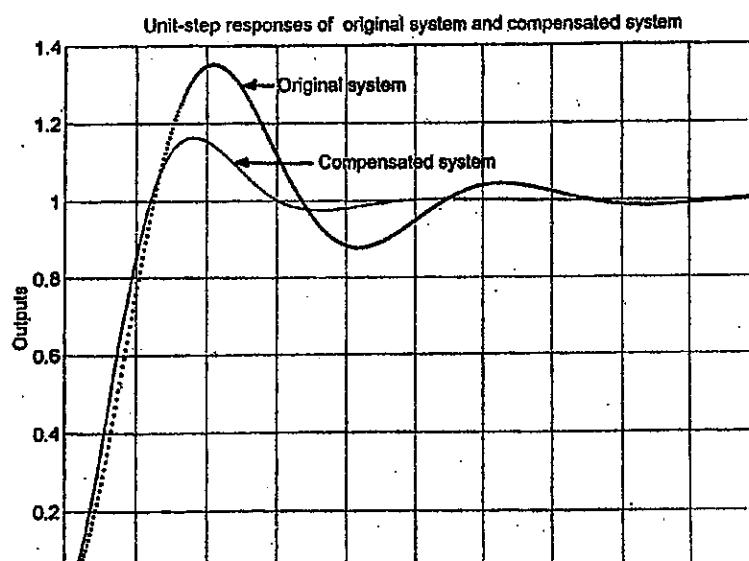
Next, we shall obtain unit-step responses of the original system and the compensated system. The original system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 2s + 10}$$

The compensated system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 4s + 16}$$

The unit-step response curves of the original system and compensated system are shown below:



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The original uncompensated system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 4s + 16}$$

The two closed-loop poles are located at $s = -2 \pm j2\sqrt{3}$. Choose a lag compensator of the following form:

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}, \quad (\beta > 1)$$

Then, the static velocity error constant K_v can be given by

$$K_v = \lim_{s \rightarrow 0} s G_c(s) G(s) = \lim_{s \rightarrow 0} s K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} \frac{16}{s(s+4)} = 4\beta K_c = 20$$

Let us choose $K_c = 1$. Then

$$\beta = 5$$

The pole and zero of the lag compensator must be located close to the origin.

Let us choose $T = 20$. Then, the lag compensator becomes

$$G_c(s) = \frac{s + \frac{1}{20}}{s + \frac{1}{100}} = \frac{s + 0.05}{s + 0.01}$$

Notice that

$$\left| \frac{s + 0.05}{s + 0.01} \right|_{s = -2 + j2\sqrt{3}} = 0.9950$$

$$\angle \left(\frac{s + 0.05}{s + 0.01} \right)_{s = -2 + j2\sqrt{3}} = \angle(-1.95 + j2\sqrt{3}) - \angle(-1.99 + j2\sqrt{3}) \\ = -60.6241^\circ + 60.1242^\circ = -0.4999^\circ$$

The angle contribution of this lag network is very small (-0.4999°) and the magnitude of $G_c(s)$ is approximately unity at the desired closed-loop pole. Hence, the designed lag compensator is satisfactory. Thus

$$G_c(s) = \frac{s + 0.05}{s + 0.01}$$

Let us compare the unit-step response curves of the uncompensated and compensated systems. The closed-loop transfer function of the uncompensated system is

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + 4s + 16}$$

For the compensated system the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{16(s+0.95)}{(s+0.01)s(s+4)+16(s+0.05)}$$

$$= \frac{16s + 0.8}{s^3 + 4.01s^2 + 16.04s + 0.8}$$

The closed loop poles can be found by entering the following matlab program into the computer.

```
p = [1 4.01 16.04 0.8];
roots(p)
```

The dominant closed-loop poles are located at $s = -1.9797 \pm j3.4526$. These locations are very close to the original closed-loop poles.

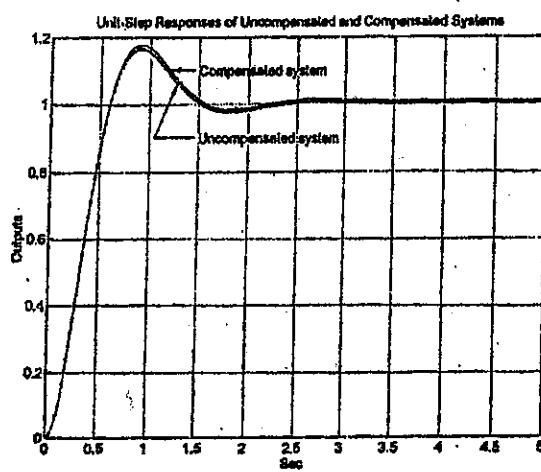
The following MATLAB program produces a plot of unit-step response curves.

```
p = [1 4.01 16.04 0.8];
roots(p)
ans =
-1.9797 + 3.4526i
-1.9797 - 3.4526i
-0.0505
```

The unit-step response curves obtained are shown below.

```
% ***** Comparison of Unit-Step Responses for Two Systems *****
num = [0 0 16];
den = [1 4 16];
numc = [0 0 16 0.8];
denc = [1 4.01 16.04 0.8];
t = 0:0.02:5;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,c1,'.',t,c2,'-')
grid
title('Unit-Step Responses of Uncompensated and Compensated Systems')
xlabel('Sec')
ylabel('Outputs')
text(1.5,1.1,'Compensated system')
text(1.5,0.9,'Uncompensated system')
```

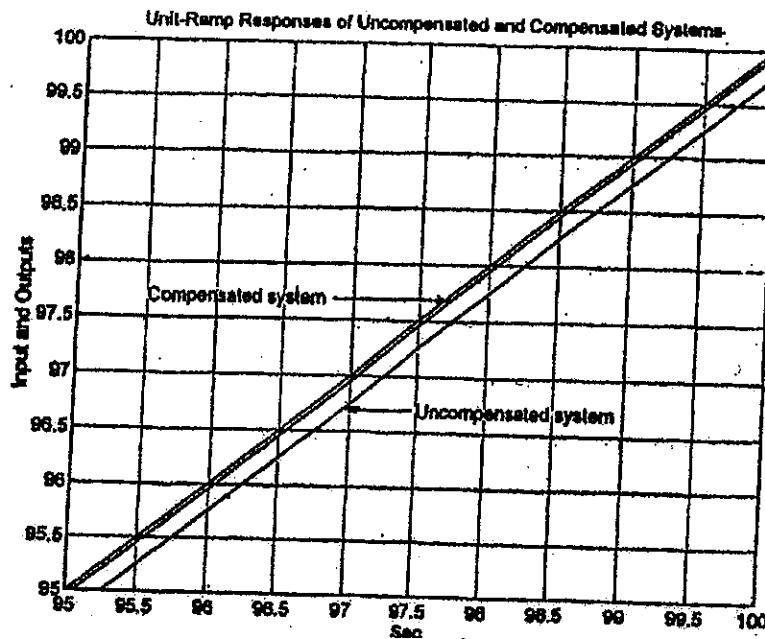
The unit-step response curves obtained are shown below.



Clearly, the unit-step response curves for the two systems are approximately the same.

For the unit-ramp response, the response curves for the two systems differ, because the original uncompensated system gives the steady-state error of 0.25, while the compensated system exhibits the steady-state error of 0.05. The following MATLAB program gives the unit-ramp response curves in the time range 95 sec \leq t \leq 100 sec. The resulting unit-ramp response curves are shown below.

```
% ***** Comparison of Unit-Ramp Responses for Two Systems *****
num = [0 0 0 16];
den = [1 4 16 0];
numc = [0 0 0 16 0.8];
denc = [1 4.01 16.04 0.8 0];
t = 0:0.1:100;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,t,'-',t,c1,'-.',t,c2,'-')
v = [95 100 95 100]; axis(v)
grid
title('Unit-Ramp Responses of Uncompensated and Compensated Systems')
xlabel('Sec')
ylabel('Input and Outputs')
text(95.5,97.7,'Compensated system')
text(97.5,96.7,'Uncompensated system')
```



B-7-13 The uncompensated system has open-loop poles at $0, -2, -5$.

Hence, from the desired closed-loop pole of $S = -2 + 2\sqrt{3}j$,

$$\angle G(s) = -49.10^\circ - 90^\circ - 120^\circ = 100.90^\circ$$

The angle deficiency is $\phi = 180^\circ - \angle G(s) = 79.10^\circ$

Design a lead compensator of the following form

$$K = \frac{s+2}{s+\alpha}$$

$$\therefore \text{Thus, } -\alpha = -2 - \frac{2\sqrt{3}}{\tan 10.90^\circ} \\ = -2\sqrt{3}$$

Thus, the lead compensator is

$$K = \frac{s+2}{s+2\sqrt{3}}$$

The gain K combined with the gain 10 in the uncompensated system, can be determined from the magnitude condition, i.e.,

$$10K = \frac{4 \times 2\sqrt{3} \times \sqrt{(2\sqrt{3})^2 + 3^2} \times \frac{2\sqrt{3}}{\sin 10.90^\circ}}{2\sqrt{3}} = 33.6$$

$$\therefore K = 33.6$$

The steady-state velocity error constant is

$$K_v = \frac{2}{5\sqrt{3}} \times 33.6 \times \frac{s+2}{s+2\sqrt{3}} \cdot \frac{1}{s(s+2)(s+5)} = 3.36$$

In order to improve K_v to 5 sec^{-1} , we need an additional lag-compensator with

DC gain

$$\frac{5}{3.36} = 14.88$$

We can design the lag-compensator as

$$\frac{s+0.1488}{s+0.01}$$

The angle from the desired closed-loop pole $-2 + 2\sqrt{3}j$ is less than 2° .

Hence, the overall lag-lead compensator is

$$33.6 \frac{s+2}{s+2\sqrt{3}} \frac{s+0.1488}{s+0.01}$$

