

B-6-1

Step 1: the standard form

$$1 + k \cdot \frac{s+1}{s^2} = 0$$

Step 2: Locate the poles & zeros

one zero at -1

two poles at 0

Step 3: Find real segments of the root-locus

on the left of -1, and at the poles at 0.

Step 4: Find asymptotes

At large s

$$G(s) = \frac{s+1}{s^2} \approx \frac{1}{s}$$

$$\text{we need } \angle G(s) \approx -\angle s = 180^\circ + k \cdot 360^\circ$$

$$\therefore \angle s = 180^\circ$$

Hence, one branch goes to the negative real axis. No intersections of asymptotes.

Step 5: Find imaginary close-loop poles

$$s^2 + ks + k = 0$$

| | | | |
|--------------|-------|-----|-----|
| Routh array: | s^2 | 1 | k |
| | s^1 | k | |
| | s^0 | 1 | |

When $k=0$, the s^1 row is entirely zero. The auxiliary polynomial is $s^2=0 \Rightarrow s=0$

Only imaginary closed-loop poles are at the origin.

Step b: Find break-in / breakaway points

$$k = -\frac{s^2}{s+1}$$

$$\frac{dk}{ds} = -\frac{s^2 + 2s}{(s+1)^2} = 0$$

$$\therefore s=0, \text{ and } s=-2$$

Both are on the root locus.

Step 7. Find angle of arrival / departure

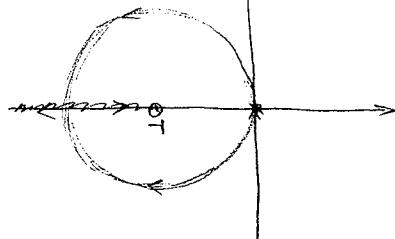
To find the angle of departure from the poles at 0, consider S very close to 0.

$$\angle(0 - (-1)) + 2\angle(S-0) = 180^\circ + l \cdot 360^\circ$$

$$\Rightarrow 0^\circ + 2\angle(S-0) = 180^\circ + l \cdot 360^\circ$$

$$\therefore \angle(S-0) = 90^\circ + l \cdot 180^\circ$$

$$= \pm 90^\circ$$



(In fact, we can show that

B-6-3 open loop transfer function $G(s)H(s) = \frac{K}{s(s+1)(s^2+4s+5)}$

no zeros, m=0; poles: $p_1=0, p_2=-1, p_3, 4 = -2 \pm j$, n=4

\therefore then asymptotes have angles $\pm \frac{1}{4}(180^\circ + 360^\circ k) = \begin{cases} \pm 45^\circ & k=0, -1 \\ \pm 135^\circ & \end{cases}$

and they meet on the negative real axis at

$$\sigma_a = \frac{1}{4}(\sum p_i - \sum z_i) = \frac{1}{4}(-5 - 0) = -1.25$$

To find the intersection of two branches at the imaginary axis,

consider $1+G(s)H(s)=0$, we get

$$s^4 + 5s^3 + 9s^2 + 5s + K = 0$$

$$s^4 \quad 1 \quad 9 \quad K$$

$$K > 0, 5 - \frac{5}{8}K = 0 \quad \therefore K = 8$$

$$s^3 \quad 5 \quad 5 \quad 0$$

\therefore From s^2 item

$$s^2 \quad 8 \quad K \quad 0$$

$$8s^2 + 8 = 0 \quad s = \pm j$$

$$s^1 \quad 5 - \frac{5}{8}K \quad 0 \quad 0$$

are two intersections

$$s^0 \quad K \quad 0 \quad 0$$

To find the break away points, again we consider $1+G(s)H(s)=0$

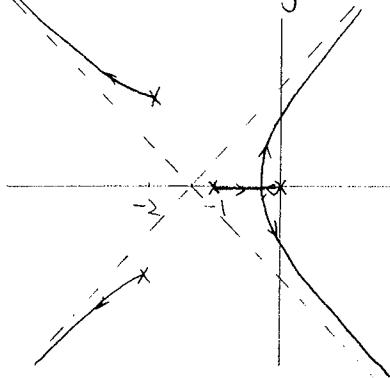
$$K + s^4 + 5s^3 + 9s^2 + 5s = 0$$

$$\therefore \frac{dK}{ds} = -(4s^3 + 15s^2 + 18s + 5) = 0$$

we pick $s = -0.393$ as the break away point

The angle of departure from $p_3 = -2 + j$ is 162°

The root locus is given in the figure below



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$$1 + G(s)H(s) = \frac{(1+k)s^2 + (2+6k)s + 10 + 10k}{s^2 + 2s + 10}$$

\therefore The characteristic equation

$$(1+k)s^2 + (2+6k)s + 10 + 10k = 0$$

Write s as $s = \sigma + j\omega$ and plug into the characteristic equation

$$(1+k)(\sigma + j\omega)^2 + (2+6k)(\sigma + j\omega) + 10(1+k) = 0$$

$$(1+k)(\sigma^2 - \omega^2 + 2\sigma\omega j) + (2+6k)(\sigma + j\omega) + 10(1+k) = 0 \quad (1)$$

make the real and imaginary parts of the equation apart

$$(1+k)(\sigma^2 - \omega^2) + (2+6k)\sigma + 10(1+k) = 0 \quad (2)$$

$$(1+k)2\sigma\omega + (2+6k)\omega = 0 \Rightarrow 2+6k = -2\sigma(1+k) \quad (3)$$

(3) \rightarrow (2)

$$(1+k)(\sigma^2 - \omega^2) - 2\sigma^2(1+k) + 10(1+k) = 0$$

$$\sigma^2 - \omega^2 - 2\sigma^2 + 10 = 0 \Rightarrow \sigma^2 + \omega^2 = 10$$

\therefore The root loci are on a circle about the origin of radius $\sqrt{10}$.

B-6-9

$$G(s)H(s) = \frac{k(s+q)}{s(s^2 + 4s + 11)}$$

zeros: $z_1 = -q$, $m=1$; poles: $p_1 = 0$, $p_{2,3} = -2 \pm \sqrt{7}j$ $n=3$

The asymptotes have angles $\frac{1}{3-1}(180^\circ + 360^\circ k) = \pm 90^\circ$ ($k=0, -1$)

They meet at $\sigma_a = \frac{1}{3-1}(\sum p_i - \sum z_i) = \frac{1}{2}(-4+q) = 2.5$

To find the intersections with the imaginary axis,

(3)

Consider $1 + G(s)H(s) = 0$ and we get

$$s^3 + 4s^2 + (11+k)s + 9k = 0$$

$$\begin{array}{r|cc} s^3 & 1 & 11+k \\ s^2 & 4 & 9k \\ \hline s^1 & \frac{44-5k}{4} & 0 \\ s^0 & 9k \end{array}$$

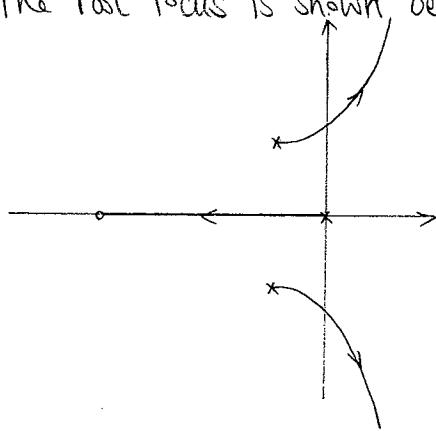
$$\text{we take } \frac{44-5k}{4} = 0 \quad \therefore k = \frac{44}{5}$$

and $4s^2 + 9k = 0$ we have

$$s = \pm 4.45$$

The angle of departure from $\rho_2 = -2 + \sqrt{7}j$ is -16.5°

The root locus is shown below.



$$\text{Problem 2: } \frac{Y(s)}{W(s)} = \frac{G_2(s)}{1 - C(s)G_1(s)}$$

$$\therefore G_1(s) = \frac{(M+m)s^2 + b_2 s + k_2}{\Delta}$$

$$G_2(s) = \frac{-M b_2 s^3 - M k_2 s^2}{\Delta}$$

$$\text{where } \Delta = (Ms^2 + b_1 s + k_1)[ms^2 + (b_1 + b_2)s + (k_1 + k_2)] - (b_1 s + k_1)^2$$

$$\text{and } C(s) = k_p + \frac{k_i}{s} + k_d s$$

$$\therefore \frac{Y(s)}{W(s)} = \frac{-M b_2 s^3 - M k_2 s^2}{\Delta - (M+m)s^2 + b_2 s + k_2 (k_p + \frac{k_i}{s} + k_d s)}$$

Steps of designing a PID controller:

- ① Obtain an open-loop response and determine what needs to be improved
- ② Add controls:
 - Add a proportional control to improve the rise time
 - Add a derivative control to improve the overshoot
 - Add an integral control to eliminate the steady-state error
- ③ Trial and Error

Adjust each of K_p , K_i and K_d until you obtain the desired overall response.

| CL response | τ_r | M_p | t_s | e_{ss} |
|-------------|----------------|-------|----------------|--------------|
| K_p | ↓ | ↑ | small change ↓ | |
| K_i | ↓ | ↑ | ↑ | eliminate |
| K_d | small change ↓ | | ↓ | small change |

```
M=2500.0;
m=320.0;
k1=80000.0;
k2=500000.0;
b1=350.0;
b2=15020.0;

KD=208025.0*5;
KP=832100.0*2;
KI=624075.0*1;

num=[b2*M k2*M 0 0 0];
den=[M*m M*b2+m*b1+M*b1-(M+m)*KD k2*M+b1*b2+k1*m+k1*M-(M+m)*KP-b2*KD k2*b1+k1*b2-(M+m)*KI-b2*KP-k2*KD k1*k2-b2*KI-k2*KP -k2*KI];
sys=tf(num,den);
step(sys);
```

The parameters used seem to work well.

