

B-5-3 In order to have $M_p = 0.05$, we must have

$$e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} = 0.05$$

or equivalently,

$$-\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\ln 0.05}{\pi}$$

Taking the square of both sides, we have a quadratic equation. Its only positive root gives us the desired ζ :

$$\zeta = 0.69$$

Then, in order to have $t_s = \frac{4}{\zeta \omega_n} = 2$, we have

$$\omega_n = \frac{2}{\zeta} = \frac{2}{0.69} = 2.90 \text{ rad/s}$$

B-5-4 One can verify that the transfer function from $R(s)$ to $C(s)$ is

$$\frac{C(s)}{R(s)} = \frac{K(Ts+1)}{s^2+KTs+K} = \frac{K}{J} \frac{(Ts+1)}{s^2 + \frac{K}{J}Ts + \frac{K}{J}}$$

Since $T=3$ and $K/J = 2/9$, we have

$$\frac{C(s)}{R(s)} = \frac{\frac{2}{9}(3s+1)}{s^2 + \frac{2}{3}s + \frac{2}{9}}$$

Comparing the polynomials in the denominator with that of the standard form of 2nd order system (and ignoring the extra first order term in the numerator), we have $\omega_n^2 = \frac{2}{9}$ and $2\zeta\omega_n = \frac{2}{3}$. Therefore,

$$\omega_n = \frac{\sqrt{2}}{3}, \quad \zeta = \frac{1}{3\omega_n} = 0.707$$

B-5-8 Using the envelope of the transient response,

we know $x_1 = e^{-\zeta\omega_n t_1}$

$$x_n = e^{-\zeta\omega_n t_n}$$

Hence $\frac{x_1}{x_n} = e^{+\zeta\omega_n(t_n - t_1)}$ and $\ln \frac{x_1}{x_n} = \zeta\omega_n(t_n - t_1)$

Note that $\omega_d(t_n - t_1) = 2(n-1)\pi$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Hence $\ln \frac{x_1}{x_n} = \frac{2\zeta(n-1)\pi}{\sqrt{1-\zeta^2}}$

Solving the equation we get $\zeta = \frac{\frac{1}{n-1} \ln \left(\frac{x_1}{x_n} \right)}{\sqrt{4\pi^2 + \left[\frac{1}{n-1} \ln \left(\frac{x_1}{x_n} \right) \right]^2}} > 0$

B-5-11'

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16k)s + 16}$$

From the characteristic polynomial, we find

$$\omega_n = 4, \quad 2\zeta\omega_n = 2 \times 0.5 \times 4 = 0.8 + 16k$$

Hence

$$k = 0.2$$

The rise time t_r is obtained from

$$t_r = \frac{\pi - \beta}{\omega_d}$$

Since $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4 \sqrt{1 - 0.25} = 3.46$

$$\beta = \sin^{-1} \frac{\omega_d}{\omega_n} = \sin^{-1} 0.866 = \frac{\pi}{3}$$

We have $t_r = \frac{\pi - \frac{1}{3}\pi}{3.46} = 0.605 \text{ sec}$

The peak time t_p is obtained as

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{3.46} = 0.907 \text{ sec}$$

The maximum overshoot M_p is

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{0.5 \times 3.14}{\sqrt{1-0.25}}} = e^{-1.814} = 0.163$$

The settling time t_s is (2% criterion)

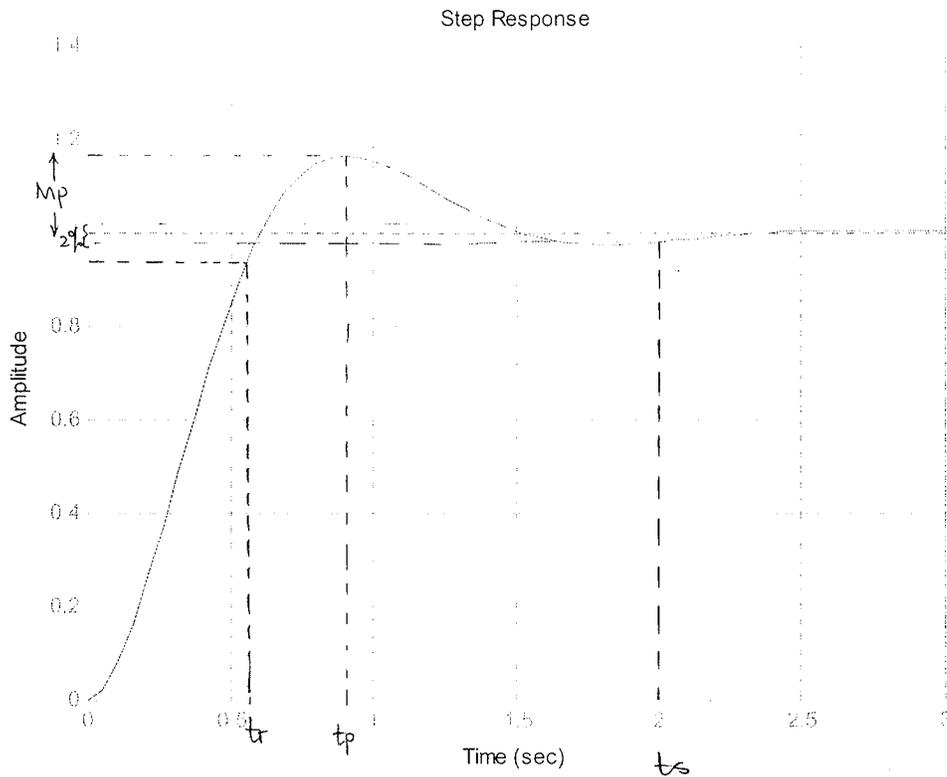
$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5 \times 4} = 2 \text{ sec}$$

The matlab code and the step response are shown on the next page.

Matlab Code:

```
num=16;  
den=[1 4 16];  
step(num,den);  
grid;
```

Step Response:



Read from the plot of the step response, we find

$$t_r \approx 0.6 \text{ sec}$$

$$t_p \approx 0.9 \text{ Sec}$$

$$M_p \approx 0.17$$

$$t_s \approx 2 \text{ sec}$$

which all fit quite well with the performance spec we derived.

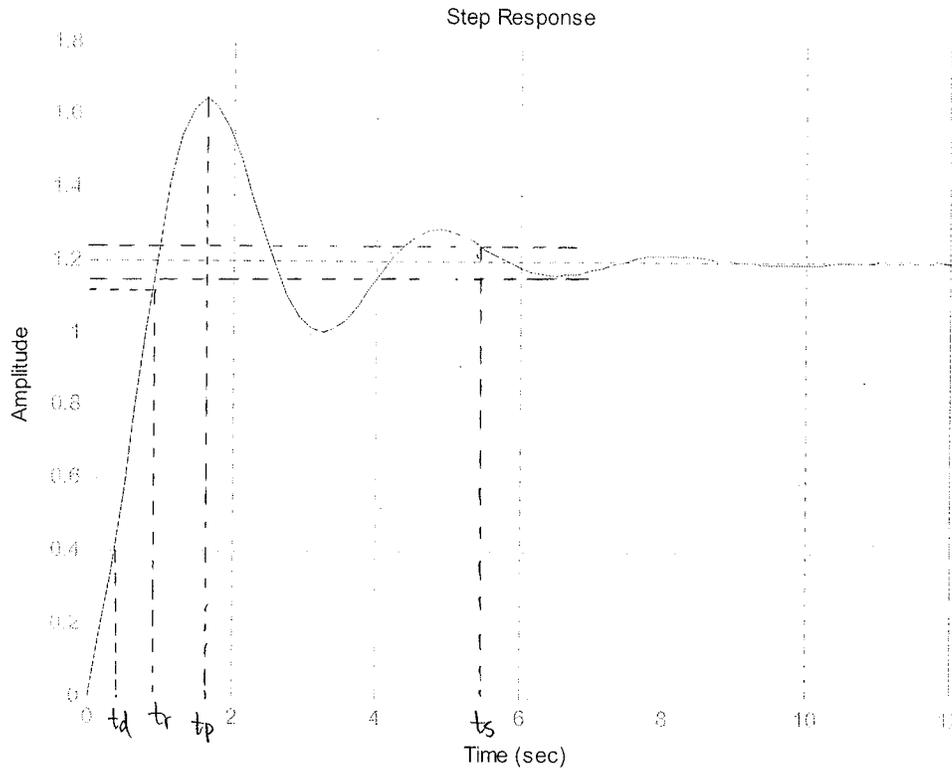
Problem 3

a. Write $H(s) = \frac{s^2 + 5s + 24}{s^3 + 6s^2 + 9s + 20}$

Matlab Code:

```
num=[0 1 5 24];
den=[1 6 9 20];
step(num,den);
grid;
```

Step Response:



Use final value theorem $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

∴ Since $H(0) = 1.2$, the final value of the step response $s(t)$ is 1.2.

Thus when evaluating the quantities t_d , t_r , m_p , etc., we need to normalize the scale. For example, the delay time t_d is the time for $s(t)$ to rise from 0 to 50% of the final value, namely, to 0.6. By inspection of the figure, we have approximately $t_d = 0.514$, $t_r = 0.94$

$t_p = 1.64$, $m_p = \frac{1.63 - 1.2}{1.2} = 35.8\%$, and $t_s = 5.27$

b. The real form partial sum expansion of $H(s)$ is

$$H(s) = \frac{1}{s+5} + \frac{4}{s^2+s+4}$$

Since the poles $0.5 \pm 1.93j$ of $H(s)$ are much closer to the imaginary axis and have a much larger residue than the pole -3 , they are the dominant poles. Thus $H(s)$ can be approximated by

$$H_{\text{appr}}(s) = \frac{4}{s^2 + s + 4}$$

which is a standard second order system with $\omega_n = 2$ and $\zeta = 0.25$

c. As mentioned in the class notes, there is no analytical expression for t_d except that it is the solution to the equation $S_{\text{appr}}(t) = 0.5$, where $S_{\text{appr}}(t)$ is the step response of $H_{\text{appr}}(s)$ with final value $H_{\text{appr}}(0) = 1$. From numerical results, t_d is approximately 0.575. We can apply the formulae in the notes to derive the other quantities:

$$t_r = \frac{\pi - \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n \sqrt{1-\zeta^2}} = 0.942$$

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 1.622$$

$$M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \pi} = 44.4\%$$

$$t_s = \frac{3}{\zeta \omega_n} = 6$$

Comparing them with the corresponding quantities for $H(s)$, we can see that the step response specifications of $H_{\text{appr}}(s)$ approximate those of $H(s)$ very well, indicating that the complex conjugate poles $0.5 \pm 1.93j$ are indeed the dominant forces in determining the step response of $H(s)$.