

ECE 382 HW2

$$B-2-9 \quad \mathcal{L}[f(t)] = \frac{1}{s} \mathcal{L}[f(s)] = \frac{1}{s} \frac{10s}{s(s+1)} = 10$$

To verify this result,

$$\mathcal{L}^{-1}\left[\frac{10}{s(s+1)}\right] = 10(1 - e^{-t}) \mathbf{1}(t)$$

$$\mathcal{L}^{-1}[10(1 - e^{-t}) \mathbf{1}(t)] = 10.$$

B-2-11 In the function $F(s)$ involves a pair of complex-conjugate poles, it is convenient not to expand $F(s)$ into the usual partial fractions, but to expand it into the sum of a damped sine and a damped cosine function. For detailed explanation of this technique, see Example 2-5 in P34 of textbook.

We first write $F(s)$ as

$$F(s) = \frac{s+1}{s(s^2+s+1)} = \frac{\alpha}{s} + \frac{\beta s + \gamma}{s^2+s+1}$$

$$\therefore \frac{\alpha}{s} + \frac{\beta s + \gamma}{s^2+s+1} = \frac{(\alpha+\beta)s^2 + (\alpha+\gamma)s + \alpha}{s(s^2+s+1)} = \frac{s+1}{s(s^2+s+1)}$$

$$\begin{cases} \alpha = -1 \\ \beta = -1 \\ \gamma = 1 \end{cases}$$

$$\therefore F(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1}$$

To continue, we want to write the second term as the sum of $\frac{s+0}{(s+0)^2+w^2}$ and $\frac{w}{(s+0)^2+w^2}$, which are the Laplace Transform of cos and sin respectively.

$$\begin{aligned} s^2+s+1 &= \left(s+\frac{1}{2}\right)^2 + \frac{3}{4} \\ \frac{s+1}{s^2+s+1} &= \frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} \frac{\sqrt{\frac{3}{4}}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} \\ \therefore f(t) &= \mathbf{1}(t) - e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t \end{aligned}$$

B-2-12 Note that for a translated function $g(t-\alpha)\mathbf{1}(t-\alpha)$, we have

$$\mathcal{L}[g(t-\alpha)\mathbf{1}(t-\alpha)] = e^{-\alpha s} g(s) \quad (\alpha \geq 0)$$

So define $G(s) = \frac{5}{s+1}$

Then $f(t) = 5e^{-t}$

$$\text{So we have } \mathcal{L}[5e^{-(t-1)} \mathbb{1}(t-1)] = e^{-s} \frac{5}{s+1} = F(s)$$

$$\therefore f(t) = \mathcal{L}^{-1}[F(s)] = 5e^{-(t-1)} \mathbb{1}(t-1)$$

$$\text{B-2-13 (a)} \quad F_1(s) = \frac{bs+3}{s^2} = \frac{b}{s} + \frac{3}{s^2}$$

$$f_1(t) = b + 3t$$

$$(b) \quad F_2(s) = \frac{5s+2}{(s+1)(s+2)^2} = \frac{\alpha}{s+1} + \frac{\beta}{s+2} + \frac{\gamma}{(s+2)^2}$$

$$\text{Then } \alpha = \lim_{s \rightarrow -1} (s+1)F_2(s) = \frac{5s+2}{(s+2)^2} \Big|_{s=-1} = -3$$

To determine β & γ , multiply $F_2(s)$ by $(s+2)^2$, we have

$$\frac{5s+2}{s+1} = \beta(s+2) + \gamma + \frac{\alpha}{(s+1)}(s+2)^2 \quad (*)$$

$$\text{Hence, } \gamma = \frac{5s+2}{s+1} \Big|_{s=-2} = 8$$

Take the derivative of $(*)$,

$$\frac{3}{(s+1)^2} = \beta + \left[\frac{\alpha}{(s+1)} (s+2)^2 \right]'$$

$$\therefore \beta = \frac{3}{(s+1)^2} \Big|_{s=-2} = 3$$

$$\therefore F_2(s) = -\frac{3}{s+1} + \frac{3}{s+2} + \frac{8}{(s+2)^2}$$

$$f_2(t) = -3e^{-t} + 3e^{-2t} + 8te^{-2t}$$

B-2-20 Using the differential rule,

$$\mathcal{L}[\ddot{x}(t)] = sX(s) - x(0) = sX(s) - a$$

$$\mathcal{L}[\ddot{x}(t)] = s\mathcal{L}'(\dot{x}) - \dot{x}(0) = s^2 X(s) - as - b$$

Hence, taking the I.T. of both sides of the equation, we have

$$[s^2 + 2\zeta\omega_n s + \omega_n^2] X(s) = as + (2a\zeta\omega_n + b)$$

$$X(s) = \frac{as + (2a\zeta\omega_n + b)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

There are three cases, depending on whether the poles are real or complex

Case I, if $\zeta > 1$,

$$\Delta = (2\zeta\omega_n)^2 - 4\omega_n^2 = 4\omega_n^2(\zeta^2 - 1) > 0$$

The poles are real

$$r_1 = \frac{-2\zeta w_n + \sqrt{\Delta}}{2} = -\zeta w_n + w_n \sqrt{\zeta^2 - 1}$$

$$r_2 = -\zeta w_n - w_n \sqrt{\zeta^2 - 1}$$

Let us write

$$X(s) = \frac{\alpha_1}{s - (-\zeta w_n + w_n \sqrt{\zeta^2 - 1})} + \frac{\alpha_2}{s - (-\zeta w_n - w_n \sqrt{\zeta^2 - 1})}$$

To determine α_1, α_2 , we have

$$\begin{aligned}\alpha_1 &= \underset{s \rightarrow r_1}{\lim} (s - r_1) X(s) \\ &= \frac{as + (2a\zeta w_n + b)}{s - (-\zeta w_n - w_n \sqrt{\zeta^2 - 1})} \Big|_{s = -\zeta w_n + w_n \sqrt{\zeta^2 - 1}} \\ &= \frac{a w_n (\zeta + \sqrt{\zeta^2 - 1}) + b}{2 w_n \sqrt{\zeta^2 - 1}}\end{aligned}$$

$$\begin{aligned}\alpha_2 &= \underset{s \rightarrow r_2}{\lim} (s - r_2) X(s) \\ &= \frac{as + (2a\zeta w_n + b)}{s - (-\zeta w_n + w_n \sqrt{\zeta^2 - 1})} \Big|_{s = -\zeta w_n - w_n \sqrt{\zeta^2 - 1}} \\ &= \frac{a w_n (\zeta - \sqrt{\zeta^2 - 1}) + b}{-2 w_n \sqrt{\zeta^2 - 1}}\end{aligned}$$

Hence,

$$\begin{aligned}X(t) &= \alpha_1 e^{rt} + \alpha_2 e^{rst} \\ &= \left\{ \frac{a}{2} \left(\frac{\sqrt{\zeta^2 - 1} + \zeta}{\sqrt{\zeta^2 - 1}} \right) + \frac{b}{2 w_n \sqrt{\zeta^2 - 1}} \right\} e^{-(\zeta - \sqrt{\zeta^2 - 1}) w_n t} \\ &\quad + \left\{ \frac{a}{2} \left(\frac{\sqrt{\zeta^2 - 1} - \zeta}{\sqrt{\zeta^2 - 1}} \right) - \frac{b}{2 w_n \sqrt{\zeta^2 - 1}} \right\} e^{-(\zeta + \sqrt{\zeta^2 - 1}) w_n t}\end{aligned}$$

Case 2: $r = 1$

$$\Delta = (2w_n\zeta)^2 - 4w_n^2 = 0$$

The pole has multiplicity of 2, i.e.,

$$r = -\zeta w_n = -w_n$$

Let us write:

$$X(s) = \frac{\alpha}{s + w_n} + \frac{\beta}{(s + w_n)^2}$$

Take common denominator and arrange terms, we have

$$X(s) = \frac{as + (2aw_n + b)}{(s + w_n)^2} = \frac{as + (aw_n + \beta)}{(s + w_n)^2}$$

Comparing terms, we have

$$\begin{aligned} \alpha &= a \\ 2aw_n t + b &= \alpha w_n t + \beta \end{aligned} \Rightarrow \left\{ \begin{array}{l} \alpha = a \\ \beta = aw_n t + b \end{array} \right.$$

$$\therefore X(t) = a e^{-w_n t} + \beta t e^{-w_n t}$$

$$= a e^{-w_n t} + t(a w_n + b) e^{-w_n t}$$

Case 3: $\zeta < 1$

$$\Delta = (2w_n \zeta)^2 - 4w_n^2 = 4w_n^2(\zeta^2 - 1) < 0$$

The poles are complex-conjugate pair.

Note that

$$s^2 + 2\zeta w_n s + w_n^2 = (s - \zeta w_n)^2 + w_n^2(1 - \zeta^2)$$

Let us write

$$\begin{aligned} X(s) &= \frac{as + (2a\zeta w_n + b)}{(s + \zeta w_n)^2 + w_n^2(1 - \zeta^2)} \\ &= \frac{a(s + \zeta w_n) + (a\zeta w_n + b)}{(s + \zeta w_n)^2 + w_n^2(1 - \zeta^2)} \\ &= \frac{a(s + \zeta w_n)}{(s + \zeta w_n)^2 + w_n^2(1 - \zeta^2)} + \frac{a\zeta w_n + b}{w_n \sqrt{1 - \zeta^2}} \cdot \frac{w_n \sqrt{1 - \zeta^2}}{(s + \zeta w_n)^2 + w_n^2(1 - \zeta^2)} \end{aligned}$$

Hence

$$X(t) = a e^{-\zeta w_n t} \cos w_n \sqrt{1 - \zeta^2} t + \frac{a\zeta w_n + b}{w_n \sqrt{1 - \zeta^2}} e^{-\zeta w_n t} \sin w_n \sqrt{1 - \zeta^2} t$$

To conclude, the final answers are:

$$X(t) = a e^{-5w_n t} (\cos w_n \sqrt{1 - \zeta^2} t + \frac{b + a\zeta w_n}{a w_n \sqrt{1 - \zeta^2}} \sin w_n \sqrt{1 - \zeta^2} t) \quad (0 \leq \zeta < 1)$$

$$X(t) = a e^{-w_n t} + \frac{(b + w_n a) e^{-w_n t}}{w_n} t \quad (\zeta = 1)$$

$$X(t) = \left\{ -\frac{a}{2} \left(\frac{-\sqrt{\zeta^2 - 1} + \zeta}{\sqrt{\zeta^2 - 1}} \right) - \frac{b}{2w_n \sqrt{\zeta^2 - 1}} \right\} e^{-(\zeta + \sqrt{\zeta^2 - 1}) w_n t}$$

$$+ \left\{ \frac{a}{2} \left(\frac{\sqrt{\zeta^2 - 1} + \zeta}{\sqrt{\zeta^2 - 1}} \right) + \frac{b}{2w_n \sqrt{\zeta^2 - 1}} \right\} e^{-(\zeta - \sqrt{\zeta^2 - 1}) w_n t} \quad (\zeta > 1)$$