

Higher-order Systems p239

The order of the denominator is greater than 2.
To study the step response, we use partial fraction expansion of $\frac{1}{s} H(s)$.

Note that the poles of $\frac{1}{s} H(s)$ are

0,

$-P_1, -P_2, \dots -P_q$ real poles

$-(\zeta_k \pm j \sqrt{1-\zeta_k^2})\omega_k, k=1, \dots, r$ complex poles

Thus

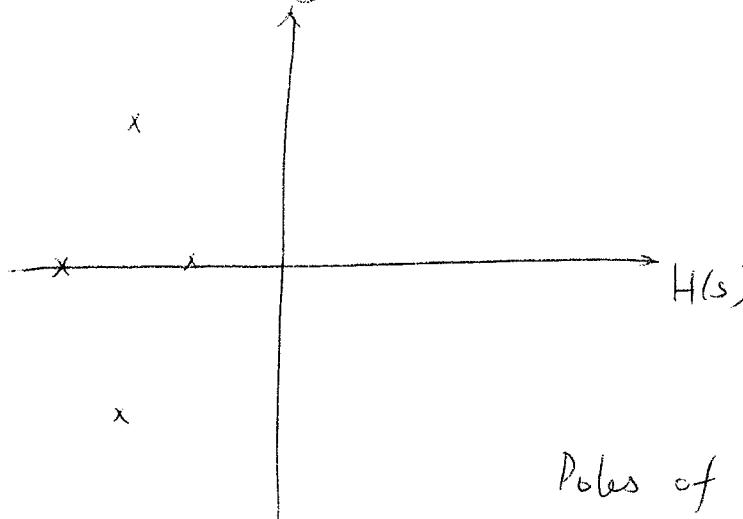
$$\frac{1}{s} H(s) = \frac{a}{s} + \sum_{i=1}^q \frac{a_i}{s+P_i} + \sum_{k=1}^r \frac{b_k(s+\zeta_k \omega_k) + c_k \omega_k \sqrt{1-\zeta_k^2}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

$$x(t) = a + \sum_{i=1}^q a_i e^{-P_i t} + \sum_{k=1}^r b_k e^{-\zeta_k \omega_k t} \cos(\omega_k \sqrt{1-\zeta_k^2} t) + \sum_{k=1}^r c_k e^{-\zeta_k \omega_k t} \sin(\omega_k \sqrt{1-\zeta_k^2} t)$$

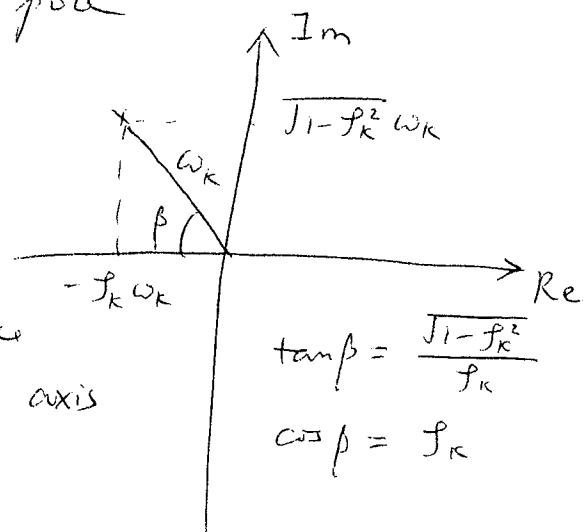
In other words, the response of a higher-order system is the combination of that of the first-order systems and second-order systems.

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The Art of Reading Poles



- ① Poles are either real or complex conjugate pairs.
Each pole has a residue value.
- ② In order for the step response to settle down (stability), all poles must have negative real parts.
- ③ Each real pole $-p_i \Rightarrow$ exponential decay term
Each complex pole $(-f_k \pm j\omega_k) e^{j\omega_k t}$
 \Rightarrow decayed oscillation.
Note that the value of f_k & ω_k can be derived from the position of the pole
- ④ The real parts determine how fast the term decays
The residue determines the magnitude of the term
- ⑤ Poles with a larger residue and closer to the imaginary axis
 \rightarrow "dominant poles"

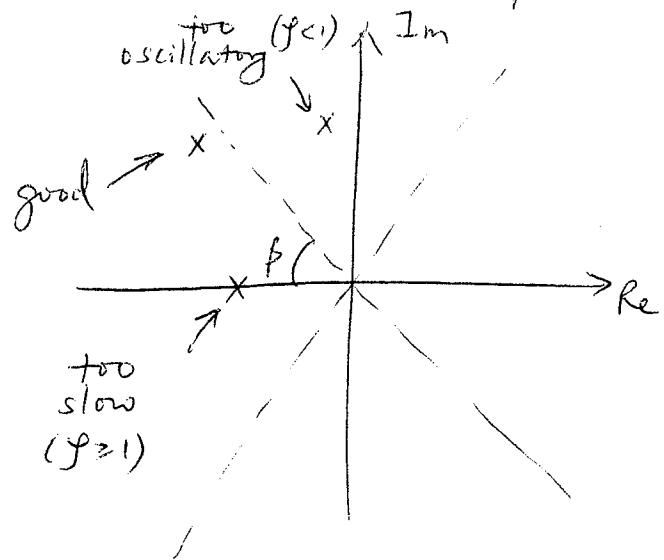


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For design purpose, we want the dominant poles be along the direction that corresponds to "good" β

$$\cos \beta = \frac{1}{\kappa}$$

\Rightarrow Basis for
"root-locus"
design method.



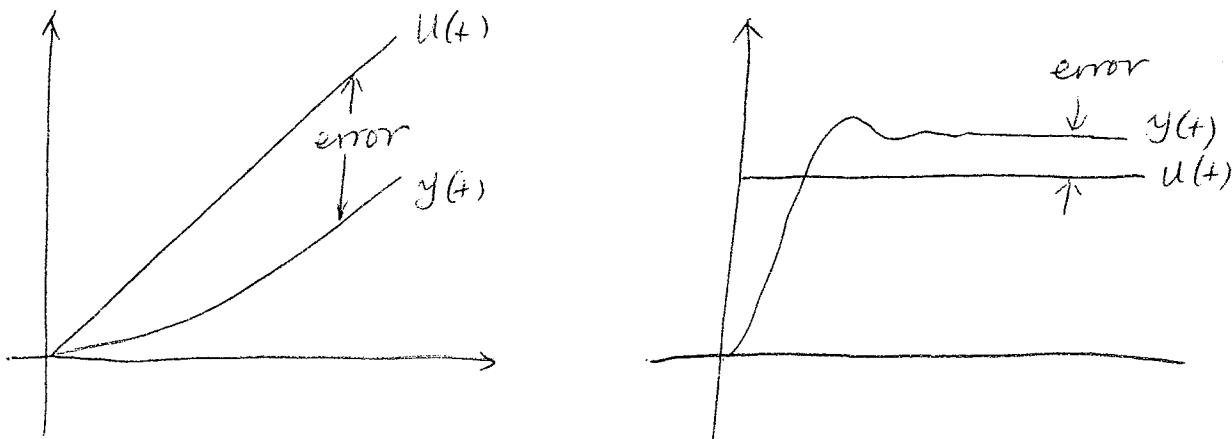
Steady-state Analysis

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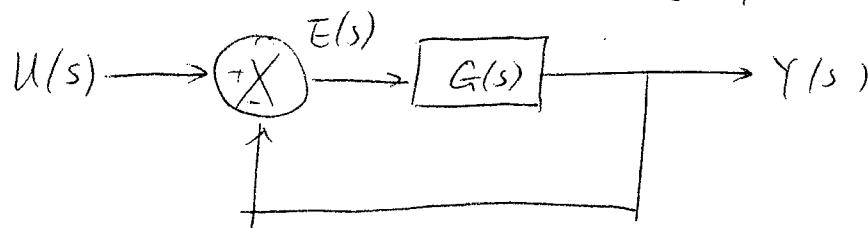
Most convenient way is to use the Final Value Theorem

$$f(+\infty) = \lim_{s \rightarrow 0} s F(s)$$

However, for feedback control systems, we often more interested in the difference between output and input, rather than the output itself.



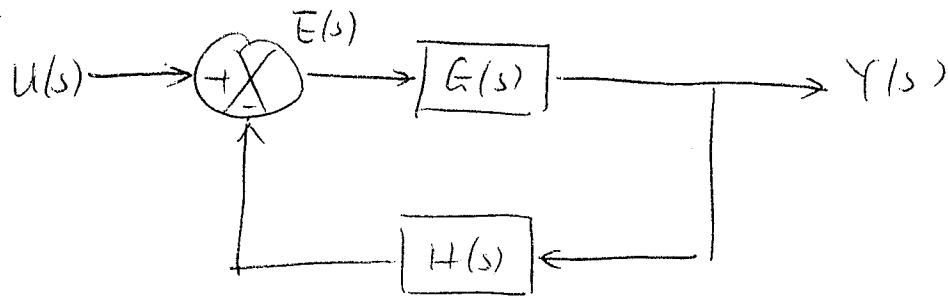
For a unity-feedback system, the error is the signal right after the summing point



$$E(s) = U(s) - Y(s).$$

In general, the input & output can be of different metric. In that case we take the difference of input and the feedback as the error signal.

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$$E(s) = U(s) - H(s)Y(s)$$

We are interested in the following question: For an input signal U , what is the steady-state error?

Overview of results

The steady-state error depends on the number of poles of $G(s)H(s)$ at $s=0$. If

$$G(s)H(s) = \frac{K(T_1 s + 1)(T_2 s + 1) \dots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \dots (T_p s + 1)}$$

then the number of poles of $G(s)H(s)$ at $s=0$ is N . The system is then called "type N ".

A type- N system can track input $1/s^N$ with no error, track input $1/s^{N+1}$ with constant error, and will not be able to track input $1/s^{N+2}$.

Recall that :	step input	$1/s$	$1(+)$
	ramp input	$1/s^2$	$+1(+)$
	acceleration input	$1/s^3$	$\frac{t^2}{2} 1(+)$

As $N \uparrow$, the input $1/s^N$ increases faster.
 \Rightarrow harder to track

	Type 0	Type 1	Type 2
step $1/s$	constant error c_0	0	0
ramp $1/s^2$	$+\infty$	constant error c_1	0
acceleration $1/s^3$	$+\infty$	$+\infty$	constant error c_2

We will also be able to compute $c_0, c_1, c_2 \dots$.

Derivation of the steady-state error

Note $\bar{E}(s) = \frac{U(s)}{1 + G(s)H(s)}$

Using the final-value theorem

$$\lim_{t \rightarrow +\infty} e(t) = \lim_{s \rightarrow 0} \frac{sU(s)}{1 + G(s)H(s)}$$

Take input as $u(s) = 1/s^{k+1}$ ($k=0$ for step input,
 $k=1$ for ramp input, etc.)

Then

$$\lim_{s \rightarrow 0} \frac{s \cdot 1/s^{k+1}}{1 + G(s)H(s)} = \lim_{s \rightarrow 0} \frac{1}{s^k + s^k G(s)H(s)}$$

If $K=0$

$$e(+\infty) = \lim_{s \rightarrow 0} \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + G(0)H(0)} \triangleq C_0$$

C_0 is called the steady-state error for step input

If $K=1$

$$e(+\infty) = \lim_{s \rightarrow 0} \frac{1}{s + sG(s)H(s)} = \lim_{s \rightarrow 0} \frac{1}{sG(s)H(s)} \triangleq C_1$$

C_1 is called the steady-state error for ramp input

In general, if input is $1/s^{K+1}$

$$e(+\infty) = \lim_{s \rightarrow 0} \frac{1}{s^k + s^k G(s)H(s)} = \lim_{s \rightarrow 0} \frac{1}{s^k G(s)H(s)} \triangleq C_k.$$

Take $G(s)H(s)$ of the form

$$G(s)H(s) = \frac{\bar{K}(T_1 s + 1)(T_2 s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)}.$$

↑ type-N.

$$\lim_{s \rightarrow 0} s^K G(s)H(s) = \begin{cases} 0 & \text{if } K > N \\ \frac{\bar{K}}{K} & \text{if } K = N \\ +\infty & \text{if } K < N \end{cases}$$

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Define

Steady-state position error constant

$$C_0 = \frac{1}{1+K_p}$$

$$K_p = \lim_{s \rightarrow 0} s G(s) H(s)$$

Steady-state velocity error constant

$$C_1 = \frac{1}{K_V}$$

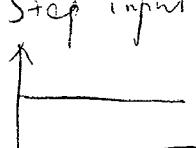
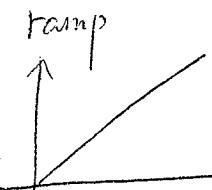
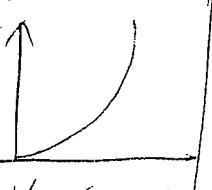
$$K_V = \lim_{s \rightarrow 0} s G(s) H(s)$$

Steady-state acceleration constant

$$C_2 = \frac{1}{K_a}$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

We then have

	Type 0	Type 1	Type 2
 $\gamma_s (K=0)$	$K_p = \bar{K}$ $C_0 = \frac{1}{1+\bar{K}}$	$K_p = +\infty$ $C_0 = 0$	$K_p = +\infty$ $C_0 = 0$
 $\gamma_s (K=1)$	$K_V = 0$ $C_1 = +\infty$	$K_V = \bar{K}$ $C_1 = \frac{1}{\bar{K}}$	$K_V = +\infty$ $C_1 = 0$
 $\gamma_s (K=2)$	$K_a = 0$ $C_2 = +\infty$	$K_a = 0$ $C_2 = +\infty$	$K_a = \bar{K}$ $C_2 = \frac{1}{\bar{K}}$

Conclusion: A system with Type-N can
 track $\frac{1}{s^N}$ input with zero error
 track $\frac{1}{s^{N+1}}$ input with constant error
 can not track $\frac{1}{s^{N+2}}$ input

Higher Type = better steady-state
 tracking capability
 = smaller steady-state error.

$$\text{Ex) } (1) \quad G(s) = \frac{1}{1+sT}, \quad H(s) = 1$$

$$K_p = G(0)H(0) = 1$$

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s) = 0$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s) = 0$$

$$\therefore C_0 = \frac{1}{1+K_p} = \frac{1}{2} \quad \text{step}$$

$$C_1 = \frac{1}{K_v} = +\infty \quad \text{ramp}$$

$$C_2 = \frac{1}{K_a} = +\infty \quad \text{acceleration}$$

$$(2) \quad G(s) = \frac{1}{s(1+sT)}, \quad H(s) = 1$$

$$K_p = \cancel{\lim_{s \rightarrow 0}} sG(s)H(s) = +\infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s) = 1$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s) = 0$$

$$\therefore C_0 = \frac{1}{1+K_p} = 0 \quad \text{step}$$

$$C_1 = \frac{1}{K_v} = 1 \quad \text{ramp}$$

$$C_2 = \frac{1}{K_a} = +\infty \quad \text{acceleration}$$

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$$\textcircled{3} \quad G(s) = \frac{1}{s^2} - \frac{1}{1+sT} \quad H(s) = 1$$

$K_p = +\infty$
 $K_r = +\infty$
 $K_a = 1$

$\therefore C_0 = 0$ step
 $C_1 = 0$ ramp
 $C_2 = 1$ acceleration.