Higher-order Systems

The order of the denominator is greater than 2. To study the step response, we use partial fraction expansion of $\frac{1}{s}H(s)$.

Note that the poles of $\frac{1}{s}H(s)$ are:
- $0$,
- $-p_1, -p_2, \ldots, -p_q$ (real poles)
- $-(f_k \pm j\sqrt{1-f_k^2})\omega_k$, $k = 1, \ldots, r$ (complex poles)

Thus,

$$\frac{1}{s}H(s) = \frac{a}{s} + \sum_{i=1}^{q} \frac{a_i}{s+p_i} + \sum_{k=1}^{r} \frac{b_k}{s^2 + 2f_k \omega_k s + \omega_k^2}$$

$$s(t) = a + \sum_{i=1}^{q} a_i e^{-p_i t} + \sum_{k=1}^{r} \frac{b_k e^{-f_k \omega_k t}}{(\omega_k \sqrt{1-f_k^2})} \left(c_3/(\omega_k \sqrt{1-f_k^2} t)\right)
+ \sum_{k=1}^{r} c_k e^{-f_k \omega_k t} \sinh(\omega_k \sqrt{1-f_k^2} t)$$

In other words, the response of a higher-order system is the combination of that of the first-order systems and second-order systems.
The Art of Reading Poles

- Poles are either real or complex conjugate pairs. Each pole has a residue value.
- In order for the step response to settle down (stability), all poles must have negative real parts.
- Each real pole - $p_i$ \(\Rightarrow\) exponential decay term
  Each complex pole \((-f_k \pm j\omega_k)\) \(\Rightarrow\) decayed oscillation
  Note that the value of $f_k$ & $\omega_k$ can be derived from the position of the pole.
- The real parts determine how fast the term decays. The residue determines the magnitude of the term.
- Poles with a larger residue and closer to the imaginary axis \(\Rightarrow\) "dominant poles"
For design purpose, we want the dominant poles be along the direction that corresponds to "good" if 
\[ \cos \beta = x_k \]

⇒ Basis for "root-locus" design method.
Steady-state Analysis

Most convenient way is to use the Final Value Theorem

\[ f(+\infty) = \frac{1}{s} \lim_{s \to 0} s F(s) \]

However, for feedback control systems, we often more interested in the difference between output and input, rather than the output itself.

For a unity-feedback system, the error is the signal right after the summing point

\[ E(s) = U(s) - Y(s) \]

In general, the input & output can be of different metric. In that case we take the difference of input and the feedback as the error signal.
We are interested in the following question: For an input signal $U(s)$, what is the steady-state error?

**Overview of results**

The steady-state error depends on the number of poles of $G(s)H(s)$ at $s = 0$. If

$$G(s)H(s) = \frac{K}{s^N (T_{s+1})(T_{s+2}) \cdots (T_{s+N})}$$

then the number of poles of $G(s)H(s)$ at $s = 0$ is $N$. The system is then called “type N.”

A type-N system can track input $1/s^N$ with no error, track input $1/s^{N+1}$ with constant error, and will not be able to track input $1/s^{N+2}$.

Recall that:
- step input $1/s$, $1(+)$
- ramp input $1/s^2$, $1(+)$
- acceleration input $1/s^3$, $\frac{t^2}{2}$ $1(+)$.  

As $N$ increases, the input $1/s^N$ increases faster.

$\Rightarrow$ harder to track.
\[
\begin{array}{|c|c|c|c|}
\hline
\text{Type 0} & \text{Type 1} & \text{Type 2} \\
\hline
\text{step } \frac{1}{s} & \text{constant error} & 0 & 0 \\
\text{ramp } \frac{1}{s^2} & \infty & \text{constant error} & 0 \\
\text{acceleration } \frac{1}{s^3} & \infty & \infty & \text{constant error} \\
\hline
\end{array}
\]

We will also be able to compute \( C_0, C_1, C_2 \ldots \).

\underline{Derivation of the steady-state error}

\[ E(s) = \frac{U(s)}{1 + G(s)H(s)} \]

Note

Using the final value theorem

\[ \lim_{t \to \infty} E(t) = \lim_{s \to 0} \frac{sU(s)}{1 + G(s)H(s)} \]

Take input as \( U(s) = \frac{1}{s^{k+1}} \) (\( k = 0 \) for step input, \( k = 1 \) for ramp input, etc.)

Then

\[ \lim_{s \to 0} \frac{s \cdot \frac{1}{s^{k+1}}}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{1}{s^{k+1} + s^k G(s)H(s)} \]
2. If \( K = 0 \)
\[
\lim_{s \to 0} \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + G_H(s)} \triangleq C_0
\]

\( C_0 \) is called the steady-state error for step input.

3. If \( K = 1 \)
\[
\lim_{s \to 0} \frac{1}{s + s G(s)H(s)} = \frac{1}{s + s G_H(s)} \triangleq C_1
\]

\( C_1 \) is called the steady-state error for ramp input.

In general, if input is \( \frac{1}{s} \)
\[
\lim_{s \to 0} \frac{1}{s^{k+1}} \triangleq C_k
\]

Take \( G(s)H(s) \) of the form
\[
G(s)H(s) = \frac{K(T_0 + 1)(T_1 + 1)\cdots(T_{n+1})}{s^N(T_0 + 1)(T_1 + 1)\cdots(T_{N+1})}
\]

\( \triangleq \) type \( -N \).

\[
\lim_{s \to 0} s^k G(s)H(s) = \begin{cases} 0 & \text{if } K > N \\ \frac{K}{K} & \text{if } K = N \\ +\infty & \text{if } K < N \end{cases}
\]
Define:

**Steady-state position error constant**
\[ K_p = \lim_{s \to 0} \frac{G(s)H(s)}{1 + G(s)H(s)} \]

**Steady-state velocity error constant**
\[ K_v = \lim_{s \to 0} s \cdot \frac{G(s)H(s)}{1 + G(s)H(s)} \]

**Steady-state acceleration error constant**
\[ K_a = \lim_{s \to 0} s^2 \frac{G(s)H(s)}{1 + G(s)H(s)} \]

We then have:

<table>
<thead>
<tr>
<th>Step input</th>
<th>Type 0</th>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{s}) ((k=0))</td>
<td>(K_p = \bar{K})</td>
<td>(K_p = +\infty)</td>
<td>(K_p = +\infty)</td>
</tr>
<tr>
<td>(C_0 = \frac{1}{1 + \bar{K}})</td>
<td>(C_0 = 0)</td>
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</tr>
<tr>
<td>Ramp</td>
<td>(K_v = 0)</td>
<td>(K_v = \bar{K})</td>
<td>(K_v = +\infty)</td>
</tr>
<tr>
<td>(\frac{1}{s^2}) ((k=1))</td>
<td>(C_1 = +\infty)</td>
<td>(C_1 = \frac{1}{\bar{K}})</td>
<td>(C_1 = 0)</td>
</tr>
<tr>
<td>Acceleration</td>
<td>(K_a = 0)</td>
<td>(K_a = 0)</td>
<td>(K_a = \bar{K})</td>
</tr>
<tr>
<td>(\frac{1}{s^3}) ((k=2))</td>
<td>(C_2 = +\infty)</td>
<td>(C_2 = +\infty)</td>
<td>(C_2 = \frac{1}{\bar{K}})</td>
</tr>
</tbody>
</table>
Conclusion: A system with Type-N can
track $\frac{1}{s^N}$ input with zero error
track $\frac{1}{s^{N+1}}$ input with constant error
cannot track $\frac{1}{s^{N+2}}$ input

Higher Type = better steady-state
tracking capability

= smaller steady-state error.

\[ G(s) = \frac{1}{1+sT}, \quad H(s) = 1 \]

<table>
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<tr>
<th>$K_p = \frac{G(0)H(0)}{1}$</th>
<th>$K_v = \frac{\lim_{s \to 0} sG(s)H(s)}{s}$</th>
<th>$K_a = \frac{\lim_{s \to 0} s^2G(s)H(s)}{s^2}$</th>
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\[ G(s) = \frac{1}{s(1+sT)}, \quad H(s) = 1 \]

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<td>$K_a = \frac{G(s)}{s^2}$</td>
</tr>
</tbody>
</table>

\[ c_0 = \frac{1}{1+K_p} = \frac{1}{2} \quad \text{step} \]
\[ c_1 = \frac{1}{K_v} = +\infty \quad \text{ramp} \]
\[ c_2 = \frac{1}{K_a} = +\infty \quad \text{acceleration} \]
(3) \[ G(s) = \frac{1}{s^2} \frac{1}{1 + sT} \quad H(s) = 1 \]

\[ K_p = +\infty \]
\[ K_v = +\infty \]
\[ K_a = 1 \]

\[- C_0 = 0 \quad \text{step} \]
\[- C_1 = 0 \quad \text{ramp} \]
\[- C_2 = 1 \quad \text{acceleration}. \]