

Second-order Systems P224-229

Standard form:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n : undamped natural frequency. $\omega_n > 0$

ζ : damping ratio $\zeta > 0$

$\sigma = \zeta\omega_n$: attenuation.

$$\text{Ex)} \quad H(s) = \frac{8}{2s^2 + as + 8}$$

To convert to the standard form

$$H(s) = \frac{4}{s^2 + \frac{a}{2}s + 4}$$

$$\Rightarrow \begin{cases} \omega_n^2 = 4 \Rightarrow \omega_n = 2 \\ \frac{a}{2} = 2\zeta\omega_n \Rightarrow \zeta = \frac{a}{8} \end{cases}$$

Overview of results

Note that the roots for the denominator is
 $-\zeta\omega_n \pm \sqrt{\zeta^2 - 1} \cdot \omega_n$

$0 < \zeta < 1$, under-damped case

Two complex conjugate poles at $(-\zeta \pm j\sqrt{1-\zeta^2})\omega_n$
Step response is a decayed oscillation.

As $\zeta \rightarrow 0$, no damping. Poles at $\pm j\omega_n$

Step response is oscillation with frequency ω_n .

② $\zeta = 1$. Critically damped case.

Two real poles, both at $-w_n$

No oscillation in step response

③ $\zeta > 1$. Over-damped case

Two real poles at $(-\zeta \pm \sqrt{\zeta^2 - 1})w_n$

Step response almost looks like exponential decay.

When ζ is between 0.5 and 0.8, the step response gets close to the final value fastest.

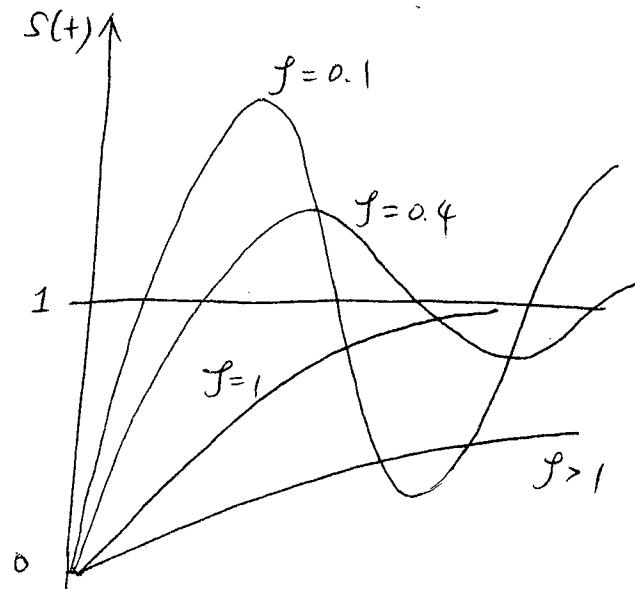
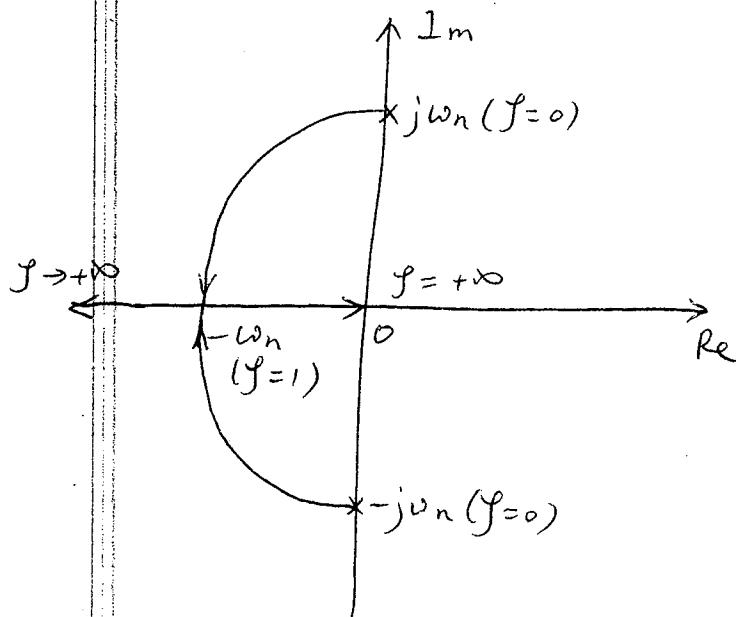


Fig 5-7, p 229 of text.

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Let us now derive the step-response for second-order systems

$$S(t) = \mathcal{L}^{-1} \left[\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \right]$$

$$= \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right)$$

① The under-damped case $0 < \zeta < 1$

$$S(t) = \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \right) \quad \text{|| } \omega_d^2$$

Let $\omega_d = \omega_n \sqrt{1-\zeta^2}$ damped natural frequency

$$S(t) = \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \right)$$

$$= 1 - e^{-\zeta\omega_n t} \cos \omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t} \quad s = \omega_d t \\ \text{|| } \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$= 1 - e^{-\frac{\zeta\omega_n t}{\sqrt{1-\zeta^2}}} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \quad s = \omega_d t \right) \\ \text{attenuation}$$

decayed oscillation

As $\zeta \rightarrow 0$ (no damping), $\omega_d = \omega_n \leftarrow$ undamped natural frequency

$$S(t) = 1 - \cos \omega_n t$$

(can also be obtained directly from $\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right)$)

As $\zeta \rightarrow 1$, the decay of the envelop $e^{-\zeta \omega_n t}$ becomes faster.

② Critically damped case $\zeta = 1$

$$\begin{aligned}s(t) &= \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{s + 2\omega_n}{(s + \omega_n)^2} \right) \\ &= \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2} \right) \\ &= 1 - e^{-\omega_n t} (1 + \omega_n t)\end{aligned}$$

No oscillation

③ Over-damped case $\zeta > 1$

$$s(t) = \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{s + 2\zeta \omega_n}{(s + \zeta \omega_n + \tilde{\omega}_d)(s + \zeta \omega_n - \tilde{\omega}_d)} \right)$$

$$\text{where } \tilde{\omega}_d = \omega_n \sqrt{\zeta^2 - 1}$$

Use partial-fraction expansion again, we can show

$$s(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right)$$

$$\text{where } s_1 = (\zeta + \sqrt{\zeta^2 - 1}) \omega_n$$

$$s_2 = (\zeta - \sqrt{\zeta^2 - 1}) \omega_n$$

When $\zeta \gg 1$, $e^{-s_1 t} \ll e^{-s_2 t}$, we then have

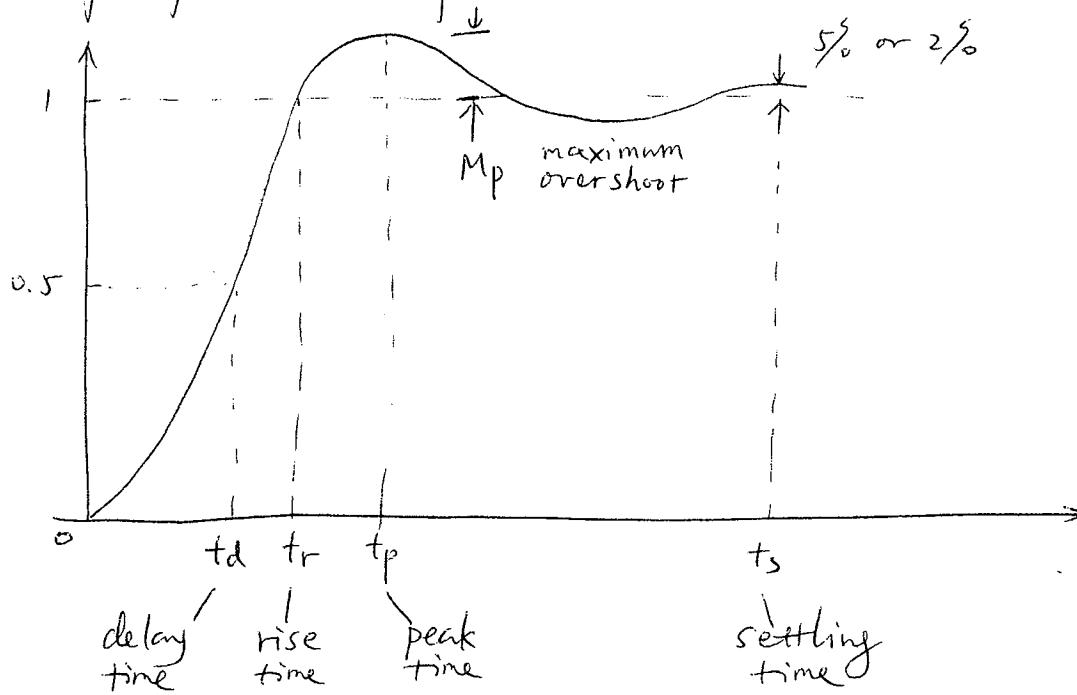
$$s(t) \approx 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1}) \omega_n t}$$

Almost looks like exponential decay.

As $\zeta \rightarrow +\infty$, the decay becomes slower.

Transient-response Specifications P229-235-

Given a step response, we can state a number of specifications of interest.



Delay time t_d : time required for the response to reach half of the steady-state value.

Rise time t_r : time required for the response to reach from 10% to 90%, or 5% to 95%, or 0% to 100% of the steady-state value.

Peak time t_p : time to reach the first peak of the overshoot

Maximum overshoot M_p : percentage with respect to the final value

Settling time t_s : time for the response to reach & stay within a range (2% or 5%) about the steady-state value.

Note: the step response shown have been scaled so that $s(+\infty) = 1$.

Some of the definitions do not apply (e.g. M_p , t_p) for critically damped or over-damped systems.

For second-order systems, we have explicit expression for relating these specs to ζ , ω_n .

Case 1: Underdamped ($\zeta < 1$)

Recall that

$$s(t) = 1 - e^{-\zeta \omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right)$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

① Delay time t_d : solve for $s(t) = 0.5$

$$\frac{1}{2} = 1 - e^{-\zeta \omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right)$$

$$\Rightarrow \frac{1}{2} e^{\zeta \omega_n t} = \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t$$

Solved by numerical tools

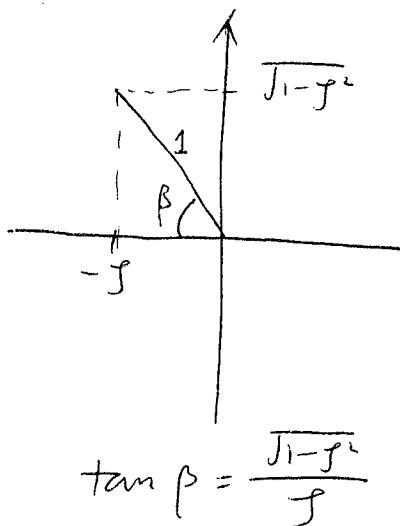
② Rise time t_r : solve for $s(t) = 1$

$$1 = 1 - e^{-\zeta \omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right)$$

$$\Rightarrow \tan \omega_d t = -\frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$= -\frac{\omega_d}{\sigma} \quad \text{damped natural frequency}$$

$$\quad \quad \quad \sigma = \zeta \omega_n \quad \text{attenuation}$$



$$\therefore \omega_d t = \pi - \beta$$

$$\text{where } \tan \beta = \frac{J_1 - f^2}{f}$$

$$\Rightarrow t_r = \frac{\pi - \beta}{\omega_d}$$

As f changes from 0 to 1,

β changes from $\frac{\pi}{2}$ to 0,

t_r changes from $\frac{\pi}{2\omega_d}$ to $\frac{\pi}{\omega_d}$ (between a quarter and a half of the cycle of damped oscillation)

Note: To make t_r small, need f small and ω_n large. But we will see that this leads to other potentially undesirable behavior.

③ Peak time t_p : Solve for $s'(t) = 0$

$$f\omega_n e^{-f\omega_n t} \left(\cos \omega_d t + \frac{f}{J_1 - f^2} \sin \omega_d t \right)$$

$$-e^{-f\omega_n t} \left(-\omega_d \sin \omega_d t + \frac{f\omega_d}{J_1 - f^2} \cos \omega_d t \right) = 0$$

$$\Rightarrow s = \omega_d t = 0$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

half of the cycle of damped oscillation

Note: To make t_p small, need ζ small and ω_n large.

④ Maximum overshoot M_p : $s(t_p) - 1$

$$\begin{aligned} M_p &= s(t_p) - 1 \\ &= e^{-\zeta \omega_n t_p} \\ &= e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \pi} \end{aligned}$$

Thus, as ζ changes from 0 to 1
 M_p changes from 1 to 0

Note: making ζ small leads to large overshoot!

⑤ Settling time t_s

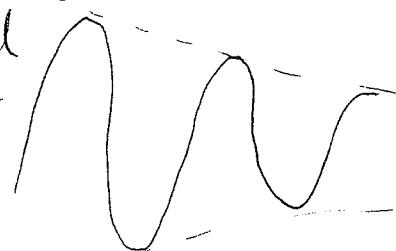
Rewrite $s(t)$ as

$$s(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta})$$

Thus, $s(t)$ is bounded by two envelopes

$$1 \pm \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}}$$

This envelop has exponential decay with time-constant $\frac{1}{\zeta \omega_n}$. For convinience, we commonly use this envelop to compare the responses of systems, in



which case we "re-define" the settling time

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta \omega_n} \quad (2\% \text{ criterion})$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta \omega_n} \quad (5\% \text{ criterion})$$

Note: making ζ small leads to longer settling time.

See text p234 for precise quantification of t_s .
 There the smallest settling time is achieved
 when $\zeta = 0.76$ for 2% criterion
 0.68 for 5% criterion.

Ex) p235

For $\zeta = 0.6$, $\omega_n = 5 \text{ rad/sec}$

Compute the various parameters

$$\text{Recall } \omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$\sigma = \zeta \omega_n$$

$$\tan \beta = \frac{\omega_d}{\sigma} = \frac{\sqrt{1 - \zeta^2}}{\zeta}$$

Hence

$$t_r = \frac{1}{\omega_d} \left(\lambda - \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right) = 0.55 \text{ sec}$$

$$t_p = \frac{\lambda}{\omega_d} = 0.785 \text{ sec}$$

$$M_p = e^{-\frac{\sqrt{1 - \zeta^2}}{\zeta} \lambda} = 0.095$$

$$t_s = \frac{4}{\zeta \omega_n} = 1.33 \text{ sec (2\%)}$$

Summary of Step response specs ($0 \leq \xi < 1$)

- ① Delay time t_d , solve for smallest t such that
- $$\frac{1}{2} \frac{1}{J_1 - \xi^2} e^{J_1 \omega_n t} = \sin(\omega_d t + \tan^{-1} \frac{J_1 - \xi^2}{\xi})$$

- ② Rise time t_r

$$t_r = \frac{\pi - \beta}{\omega_d}, \quad \beta = \tan^{-1} \frac{J_1 - \xi^2}{\xi}$$

- ③ Peak time t_p

$$t_p = \frac{\pi}{\omega_d}$$

- ④ Maximum overshoot

$$M_p = e^{-\frac{\pi}{J_1 - \xi^2} \xi}$$

- ⑤ Settling time, use approximations

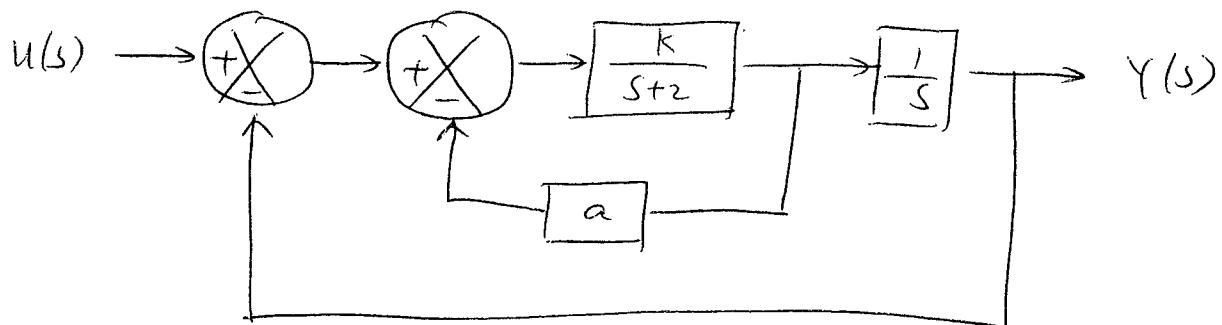
$$t_s = \frac{4}{\xi \omega_n}, \quad 2\%$$

$$t_s = \frac{3}{\xi \omega_n}, \quad 5\%$$

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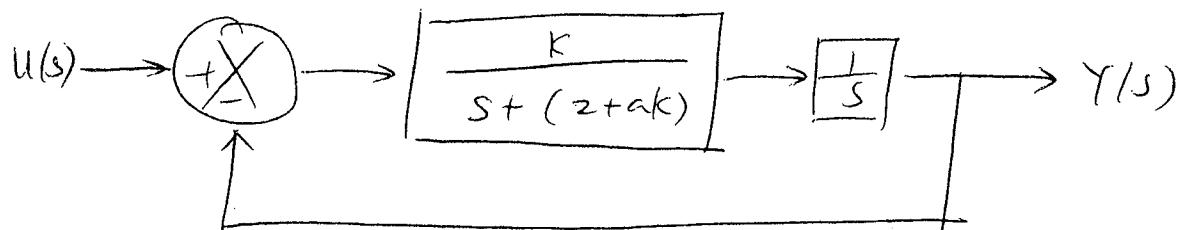
Conversely, if we are given some design specs, we can choose ζ & ω_n (or related parameters) using these relationships. Often, the value of ζ is determined from the maximum overshoot. Then, the value of ω_n is determined from rise time/delay time/settling time.

Ex) B-5-10 p332



Find the value of K, a such that the system has a damping ratio of $\zeta = 0.5$, and an undamped natural frequency of $\omega_n = 4 \text{ rad/sec}$. Find the transient-response specifications.

Solution: First, we block-diagram reduction to compute the end-to-end transfer function



$$\frac{Y(s)}{U(s)} = \frac{\frac{K}{s(s + (2 + ak))}}{1 + \frac{K}{s(s + (2 + ak))}} = \frac{K}{s^2 + s(2 + ak) + K}$$

$$\text{Hence: } \omega_n^2 = K \Rightarrow K = 16$$

$$2 + \alpha K = 2\zeta\omega_n \Rightarrow 2 + 16\alpha = 4 \Rightarrow \alpha = \frac{1}{8}$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{16}{s^2 + 4s + 16}$$

Compute the various performance specs.

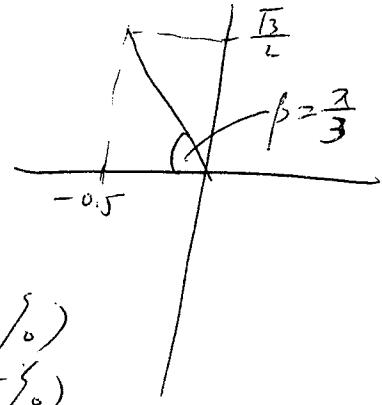
$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 2\sqrt{3}$$

$$\Rightarrow t_r = \frac{1}{\omega_d} (\pi - \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}) = \frac{\pi}{3\sqrt{3}}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2\sqrt{3}}$$

$$M_p = e^{-\frac{\pi}{\sqrt{1-\zeta^2}} \cdot \zeta} = e^{-\frac{\pi}{\sqrt{13}} \cdot \zeta} \\ = e^{-\frac{\pi}{\sqrt{13}}}$$

$$t_s = \frac{4}{\zeta \omega_n} = \frac{4}{0.5 \times 4} = 2s \quad (2\%) \\ = \frac{3}{\zeta \omega_n} = \frac{3}{2}s \quad (5\%)$$



The performance specs for critically-damped and over-damped case can be derived analogously.

Critically-damped

$$s(t) = 1 - e^{-\omega_n t} (1 + \omega_n t)$$

Over-damped

$$s(t) \approx 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1}) \omega_n t}$$

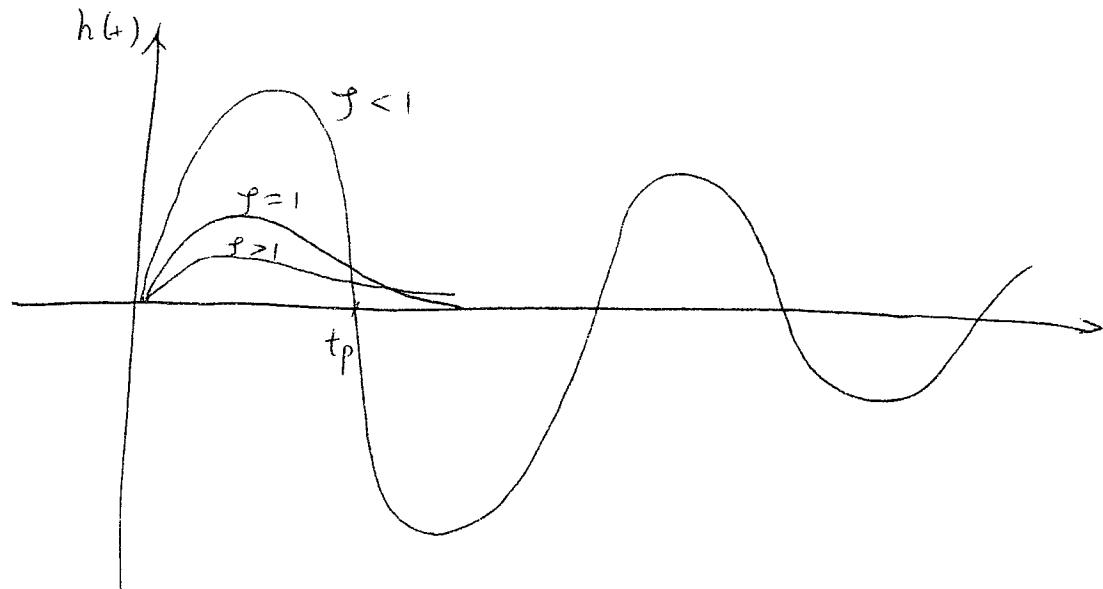
(details for you to work out)

Impulse response of 2nd-order Systems P238

$$h(t) = \mathcal{L}^{-1}\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

$$= \begin{cases} \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} s - \omega_n \sqrt{1-\zeta^2} t & \zeta < 1 \\ \omega_n^2 + e^{-\omega_n t} & \zeta = 1 \\ \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2-1}} e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t} & \zeta > 1 \end{cases}$$

or, take derivative of $s(t)$.



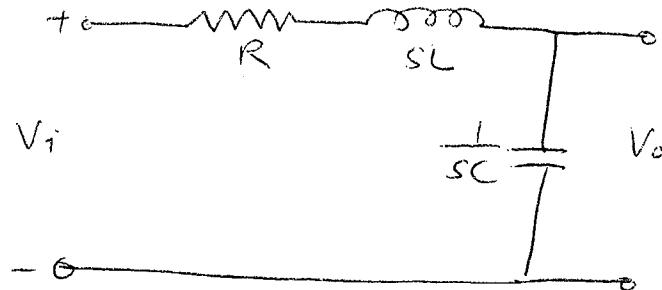
Impulse Response

Step Response

- ① $\zeta \geq 1 \Rightarrow$ no always positive $\rightarrow \zeta \geq 1 \Rightarrow$ no oscillation
- ② impulse response = 0 \rightarrow peak time
- ③ area under the impulse response \rightarrow Maximum overshoot

Examples of 2nd-order systems

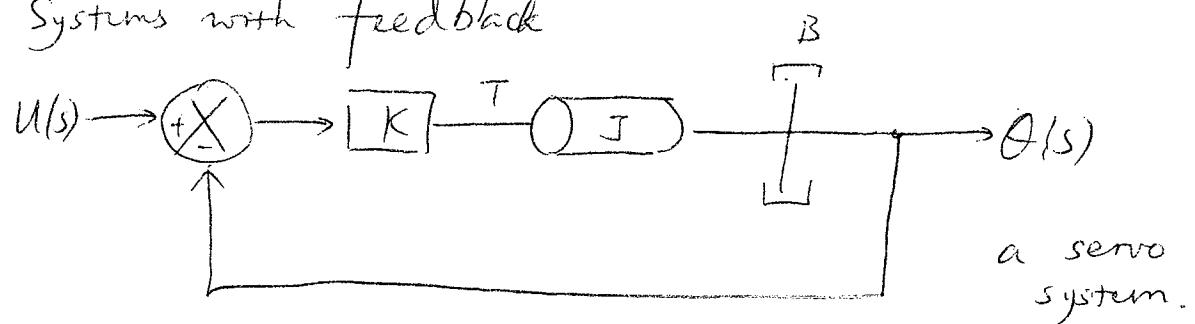
- ① Electrical system with both capacitor & inductor



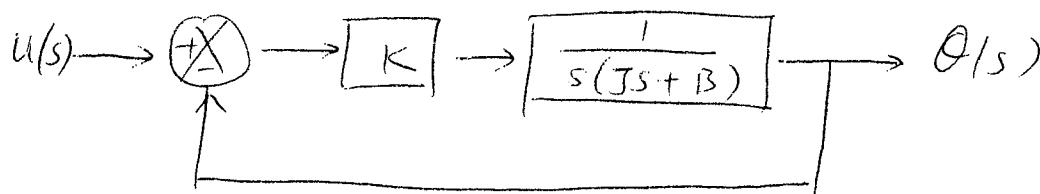
$$\begin{aligned} \frac{V_o(s)}{V_i(s)} &= \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1} \\ &= \frac{\frac{1}{LC}}{s^2 + \underbrace{\frac{R}{L}s}_{2f\omega_n} + \frac{1}{LC}} \end{aligned}$$

$$\gamma = \frac{1}{2} R \sqrt{\frac{1}{LC}}$$

- ② Systems with feedback



$$\begin{aligned} T - B \dot{\theta} &= J \ddot{\theta} \\ \frac{\theta(s)}{T(s)} &= \frac{1}{Bs + Js^2} \end{aligned}$$



$$\begin{aligned} \frac{\theta(s)}{U(s)} &= \frac{\frac{K}{s(Js+B)}}{1 + \frac{K}{s(Js+B)}} = \frac{K}{Js^2 + Bs + K} = \frac{\frac{K}{J}}{s^2 + \underbrace{\frac{B}{J}s}_{2f\omega_n^2} + \frac{K}{J\omega_n^2}} \end{aligned}$$

Summary: The performance of 2nd-order systems depends on the value of ξ — damping ratio.

(6) What if there is an extra zero?

$$H(s) = \frac{s + \alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s(t) = \mathcal{L}^{-1} \left[\frac{1}{s} \frac{s + \alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right]$$

$$= \mathcal{L}^{-1} \left[\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\alpha}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \right]$$

↗
impulse response
of the standard
form

↗
step response
of the standard
form

(7) What if $\xi < 0$

$$H(s) = \frac{\omega_n}{s^2 - 2|\xi|\omega_n s + \omega_n^2}$$

Poles at $(|\xi| \pm j \sqrt{1-\xi^2}) \omega_n$, on the right-half-plane!

The envelop becomes

$$1 \pm \frac{\epsilon}{\sqrt{1-\xi^2}}$$

System is unstable!