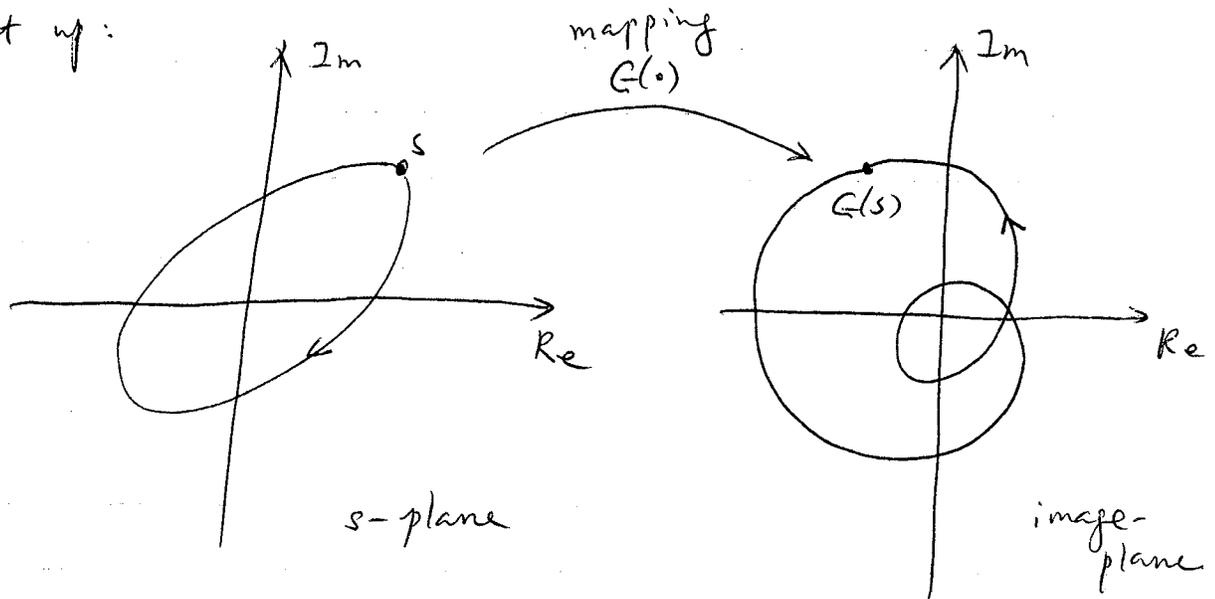


Mathematics behind Nyquist-Stability Criterion

Set up:



- There are two complex planes: the original complex plane for the variable s , and the image complex plane. There is also a mapping function $G(\cdot)$.
- For each point s on the s -plane, the mapping function $G(\cdot)$ maps it to a corresponding point $G(s)$ on the image-plane.
- $G(s)$ is called the "image" of s .
- A "contour" is a continuous closed curve. The image of the contour is also a curve on the image-plane.
- As s moves on the contour in the clockwise direction, $G(s)$ will follow the image-curve (either clockwise or counter clockwise).

The mapping theorem : When s follows a

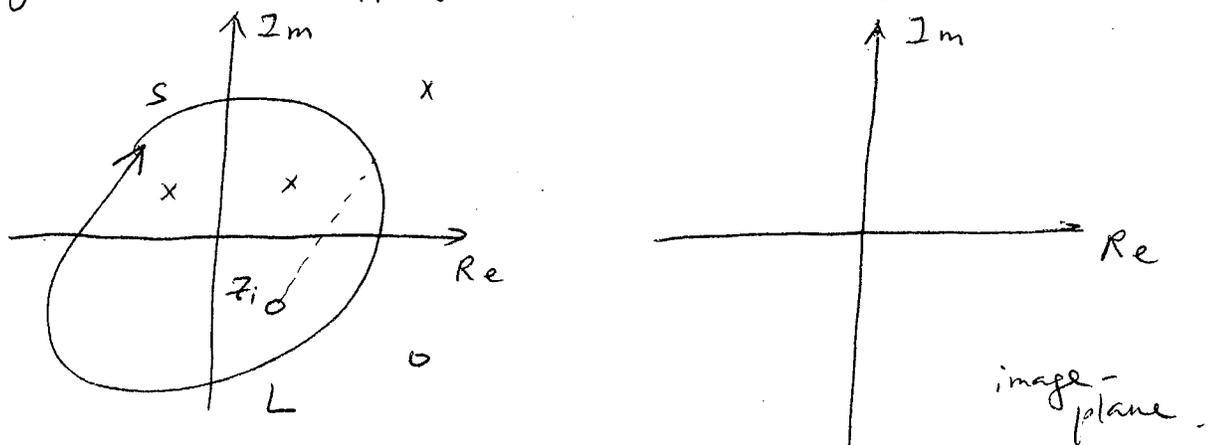
contour L on the complex plane in the clockwise direction, the number of times N its image $G(s)$ circles the origin in the clockwise direction is equal to

$$N = Z - P$$

of times $G(s)$ encircles the origin clockwise
of zeros of $G(s)$ enclosed by L
of poles of $G(s)$ enclosed by L .

See Fig 8-47 in text. (p544)

Why is the Mapping Theorem true?



$$\text{Let } G(s) = \frac{\prod_{i=1}^m (s - z_i)}{\prod_{j=1}^n (s - p_j)}$$

$$\Rightarrow \angle G(s) = \sum_{i=1}^m \angle (s - z_i) - \sum_{j=1}^n \angle (s - p_j)$$

(17)

When s moves along the contour L , the change of angle of $\angle G(s)$ is the sum of the change of each individual angles.

Each zero z_i inside $L \rightarrow \angle(s-z_i)$ changes -360°

Each zero z_i outside $L \rightarrow \angle(s-z_i)$ changes 0°

Each pole p_j inside $L \rightarrow \angle(s-p_j)$ changes -360°

Each pole p_j outside $L \rightarrow \angle(s-p_j)$ changes 0°

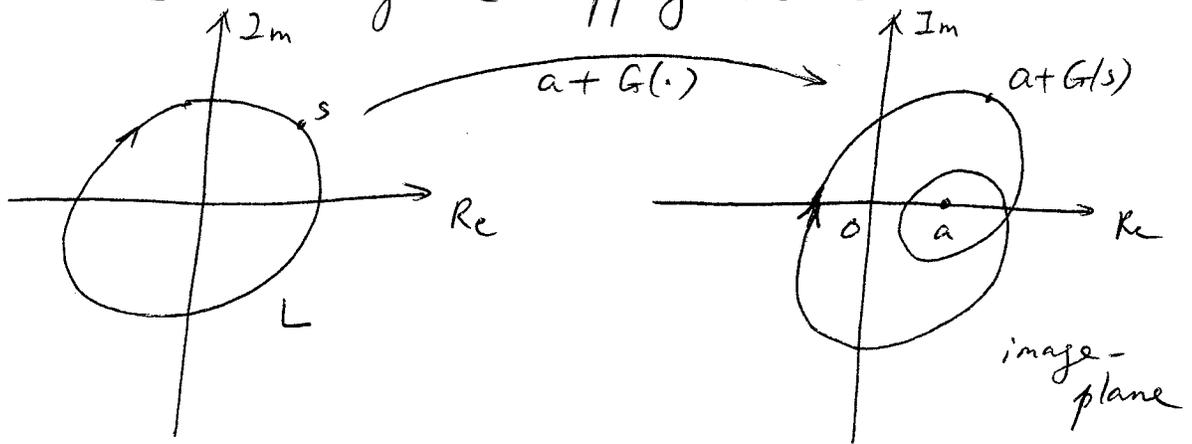
$$\begin{aligned} \text{Hence, the change of angle } \angle G(s) &= -360^\circ (\# \text{ of zeros in } L \\ &\quad - \# \text{ of poles in } L) \\ &= -360^\circ (z-p) \end{aligned}$$

Each time the angle $\angle G(s)$ changes -360° , the image point circles the origin on the image-plane once.

$$\Rightarrow N = z - p \quad (\text{the mapping theorem}).$$

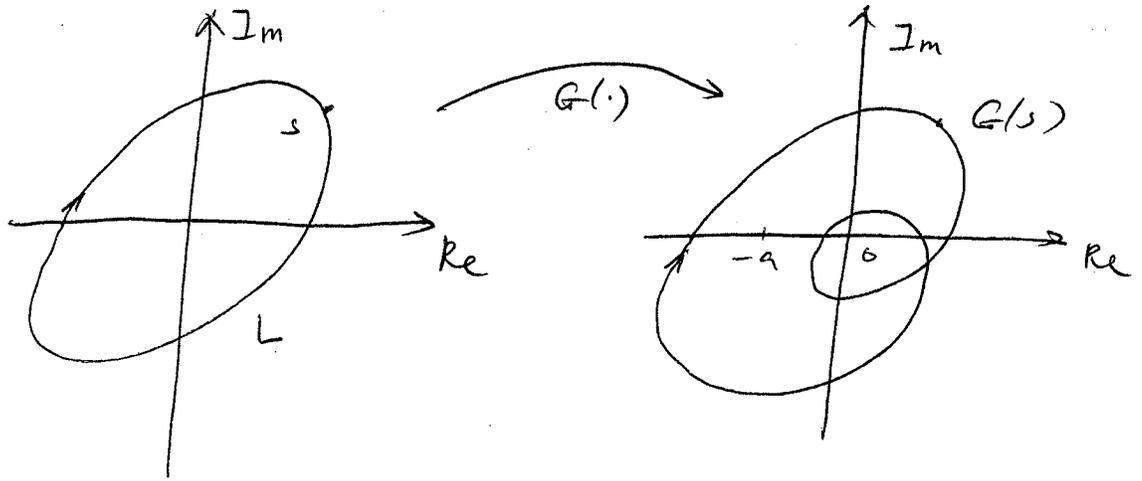
What if the mapping function is $a + G(s)$?

We can consider $a + G(s)$ as the image of s , and use the original mapping theorem.



$N = \#$ of encirclement of the origin
 by $a + G(s)$ in the clockwise direction
 $Z = \#$ of zeros of $a + G(s)$ (i.e.,
 $\#$ of roots of $a + G(s) = 0$) enclosed
 by L
 $P = \#$ of poles of $a + G(s)$. (the same
 as the $\#$ of poles of $G(s)$) enclosed
 by L .

Or, since the image $a + G(s)$ is simply the image of $G(s)$ moved to the right by a , we can still use $G(s)$ as the image of s , and count N as the number of times the image $G(s)$ circles the point $-a$ in the clockwise direction.



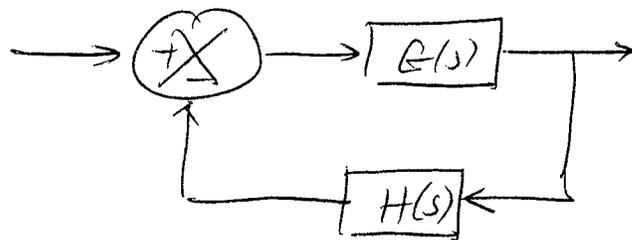
$N = \#$ of encirclement of $-a$ by $G(s)$ in the clockwise direction

$Z = \#$ of roots of $1 + G(s)H(s) = 0$ enclosed by L

$P = \#$ of poles of $G(s)H(s)$ enclosed by L

The Final Connection to Nyquist Stability

Consider a feedback system



For stability, we need to know the # of roots of

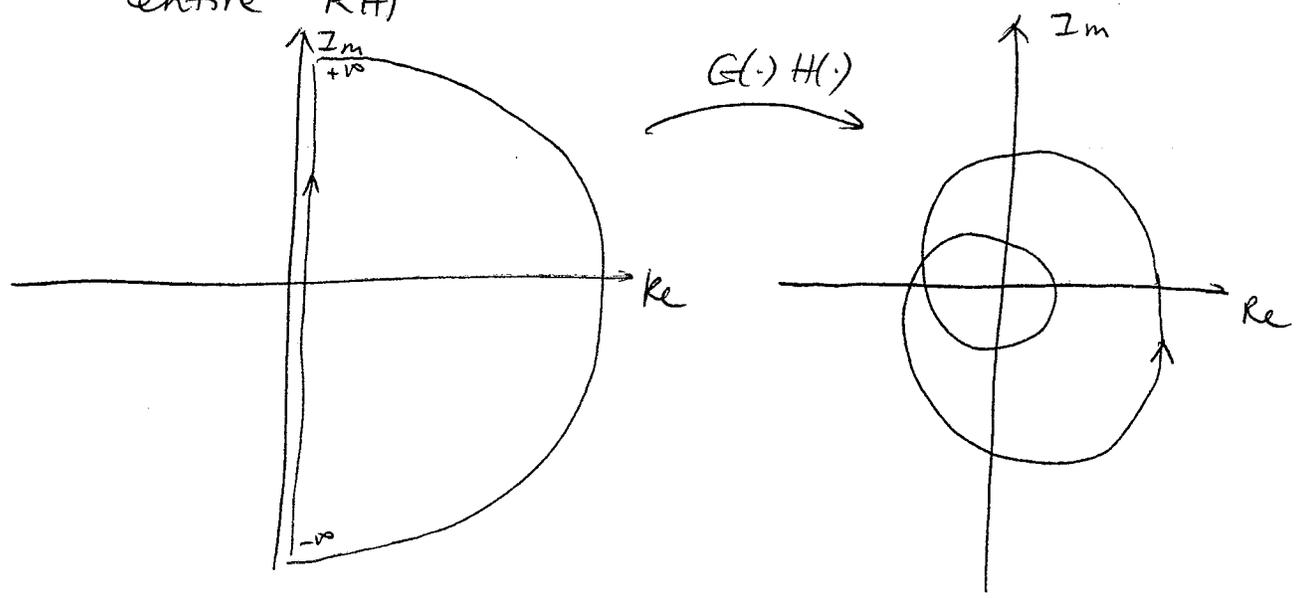
$$1 + G(s)H(s) = 0$$

on the right-half-plane. In general, let us

consider $a + G(s)H(s) = 0$

To use the mapping Theorem:

- ① Pick a special contour L that encloses the entire RHP



- ② When s moves from $-j\infty$ to $+j\infty$, the image is the frequency-response $G(j\omega)H(j\omega)$.

- ③ Assuming $\lim_{s \rightarrow \infty} G(s) \cdot H(s) = \text{constant}$.
When s moves along the big half-circle (with infinite radius), the image stays at the same position, which is also on the frequency-response $G(j\omega)H(j\omega)$

⇒ The image of L is the Nyquist-Plot.

By Mapping Theorem

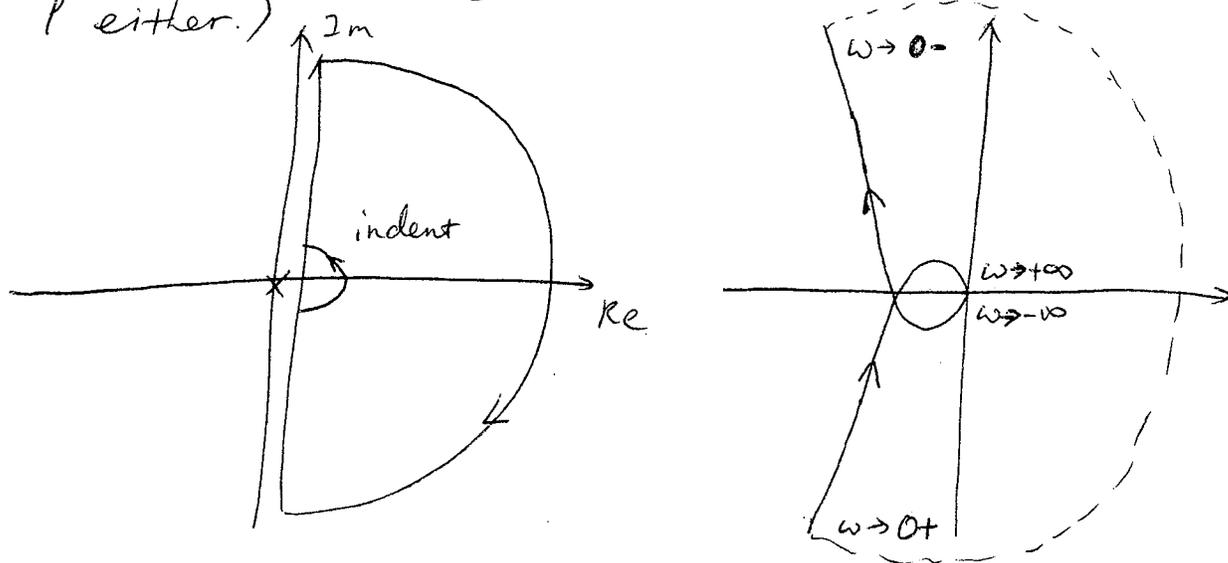
$$\begin{array}{ccc}
 N & = & Z - P \\
 \swarrow & & \downarrow \quad \quad \downarrow \\
 \text{\# of clockwise} & & \text{\# of zeros} \\
 \text{encirclement of} & & \text{of } a + G(s)H(s) \\
 -a \text{ by } G(j\omega)H(j\omega) & & \text{on RHP} \\
 & & \text{\# of poles} \\
 & & \text{of } G(s)H(s) \\
 & & \text{on RHP}
 \end{array}$$

We can then pick $a = 1$ (or $\frac{1}{K}$) to get the Nyquist Stability Criterion.

Special case when $G(s)H(s)$ involves poles on the $j\omega$ -axis p547

If we still use the contour that passes through the $j\omega$ -axis poles, the image will be a Nyquist-plot that is not closed (has discontinuous branches going to infinity).

Solution: we indent the contour into the RHP, so that the contour does not pass through the $j\omega$ -axis poles. (Hence, they will not be counted to get P either.)



What will the image of this contour look like?

Other parts still look the same as the Nyquist-plot. Each of the small arc is infinitesimally small, but it results in a large arc that swings 180° clockwise in the image.

To see this, assuming $G(s)$ has only one pole 0 on the $j\omega$ -axis

$$G(s) = \frac{1}{s} \cdot \frac{(s-z_1) \dots (s-z_m)}{(s-p_1) \dots (s-p_n)}$$



① As s approaches B, $|G(s)|$ approaches infinity this is part of the Nyquist plot. ($\omega \rightarrow 0^-$)

② As s moves from B to C, $|G(s)|$ is large.

$$\angle G(s) = -\angle s + \sum \angle (s-z_i) - \sum \angle (s-p_j)$$

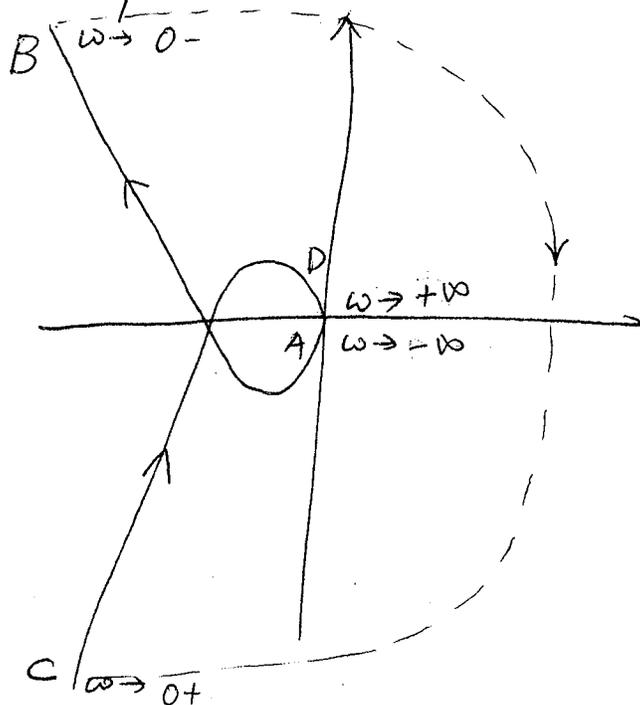
does not change much

$\angle s$ increases from -90° to 90°

$\Rightarrow \angle G(s)$ decreases by 180°

The result is a large arc that swings 180° clockwise

③ As s moves from C to D, it is another part of the Nyquist-plot



Now the image is closed, and we can then count the number of encirclements around -1 (or $-k$) and apply the Nyquist-Stability Criterion

Note: P does not include the $j\omega$ -axis poles.

In general, if there are K repeated poles on the $j\omega$ -axis (we just saw an example with $K=1$), the Nyquist-plot swings clockwise by $180^\circ \cdot K$

Example 1: P 548

$$G(s)H(s) = \frac{K}{s(Ts+1)}$$

Determine the range of K for stability of the closed-loop system.

Important: use our knowledge of asymptotes to figure out what point corresponds to what frequency.

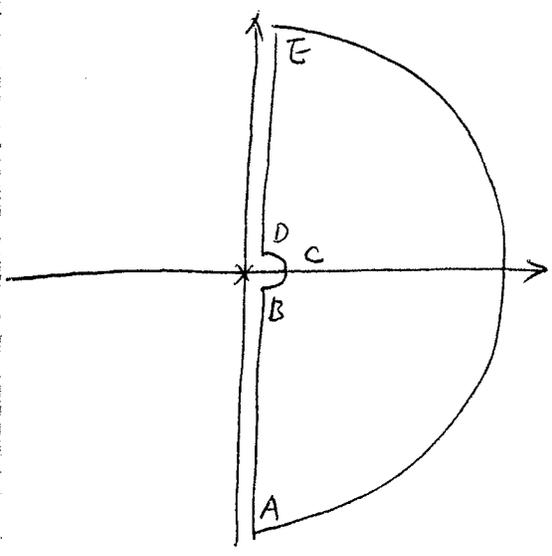
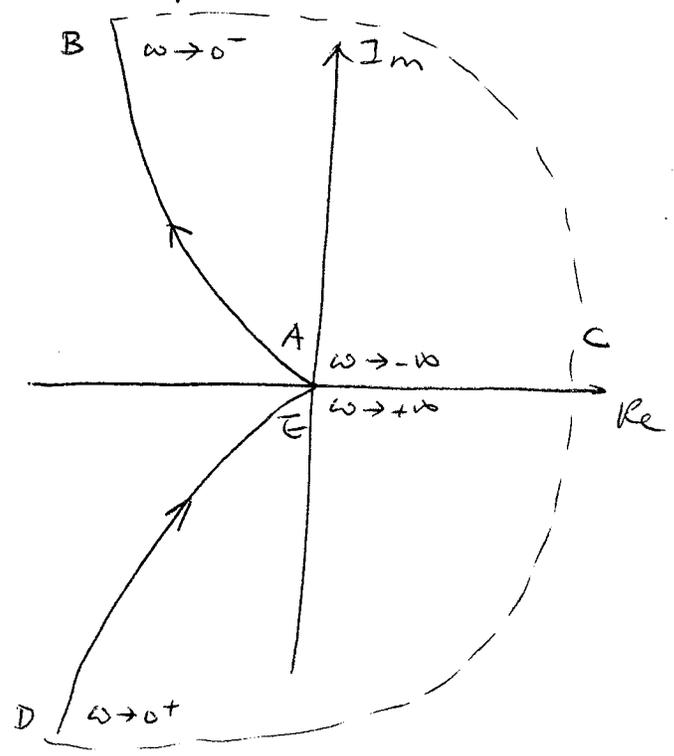
$$G(j\omega)H(j\omega) = \frac{K}{j\omega(1+j\omega T)}$$

$$\omega \rightarrow 0^+$$

$$G(j\omega)H(j\omega) \approx \frac{K}{j\omega}$$

$$\omega \rightarrow +\infty$$

$$G(j\omega)H(j\omega) \approx \frac{K}{(j\omega)^2 T}$$



Take the Nyquist path around the open-loop pole at the origin

Close the Nyquist-plot: $B \rightarrow C \rightarrow D$ swings 180° clockwise

Step 1: one open-loop pole at zero ✓

$$\lim_{s \rightarrow \infty} G(s)H(s) = 0$$

Step 2: $P = 0$ (not including the $j\omega$ -axis poles)

Step 3: no encirclement of $-1/K$
 \Rightarrow STABLE for all K .

Example 2: P549

$$G(s)H(s) = \frac{K}{s^2(Ts+1)}$$

Determine the range of K for stability.

$$G(j\omega)H(j\omega) = \frac{K}{(j\omega)^2(1+j\omega T)}$$

As $\omega \rightarrow 0^+$

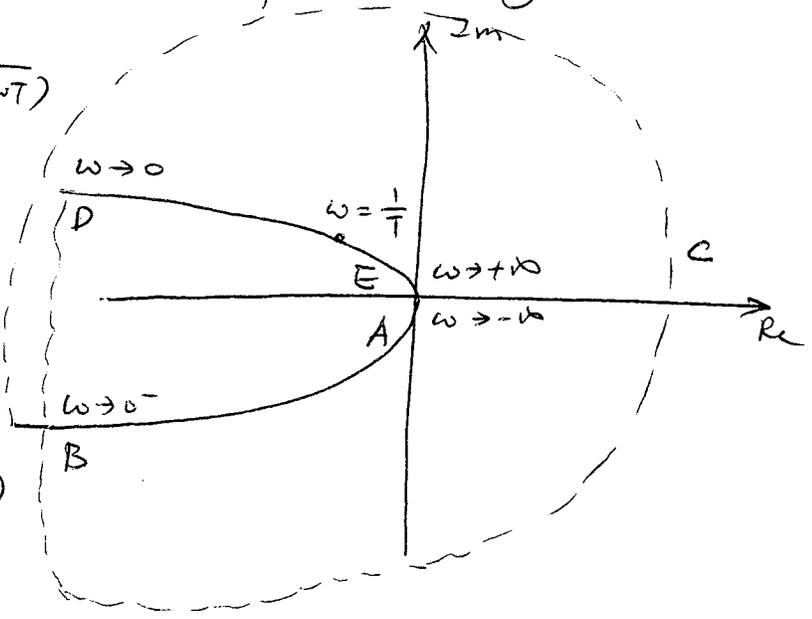
$$G(j\omega)H(j\omega) \approx \frac{K}{(j\omega)^2}$$

As $\omega \rightarrow +\infty$

$$G(j\omega)H(j\omega) \approx \frac{K}{(j\omega)^3 T}$$

As $\omega \rightarrow \frac{1}{T}$

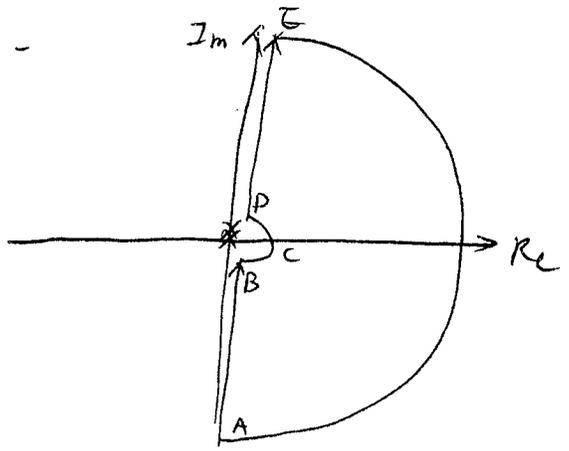
$$G(j\omega)H(j\omega) = \frac{K}{(j\frac{1}{T})^2(1+j)} = \frac{KT^2}{T^2} \angle -225^\circ$$



Sometimes the initial MATLAB Nyquist-plot can be quite confusing. Use the axis command and the asymptotes to make sense.

Take the Nyquist-path around the open-loop poles at origin

$B \rightarrow C \rightarrow D$ swings $180^\circ \times 2$ clockwise



Step 1: two open-loop poles at origin ✓

$$\lim_{s \rightarrow 0} G(s)H(s) = 0$$

Step 2: $P = 0$

Step 3: 2 clockwise encirclement of $-\frac{1}{K}$
 \Rightarrow UNSTABLE for all K .

Example 3: P552

$$G(s)H(s) = \frac{K(s+3)}{s(s-1)}$$

Determine the range of K for stability

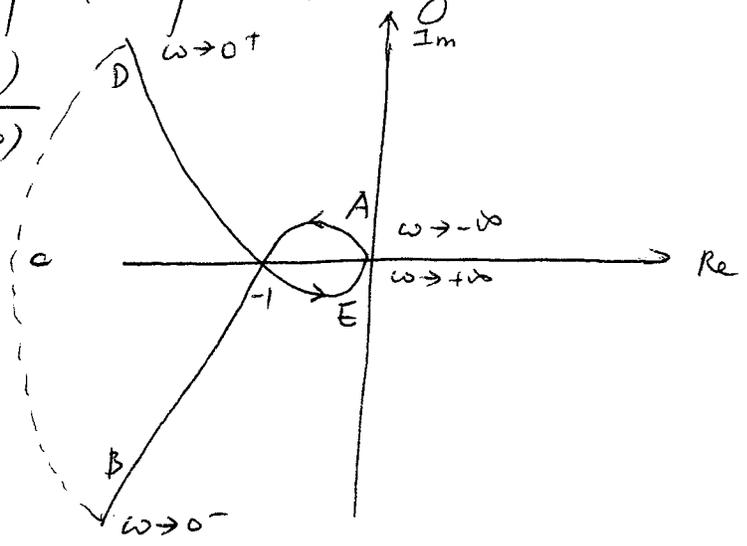
$$G(j\omega)H(j\omega) = \frac{3K(1 + \frac{j\omega}{3})}{j\omega(-1 + j\omega)}$$

$\omega \rightarrow 0^+$,

$$G(j\omega)H(j\omega) \approx -\frac{3K}{j\omega}$$

$\omega \rightarrow +\infty$

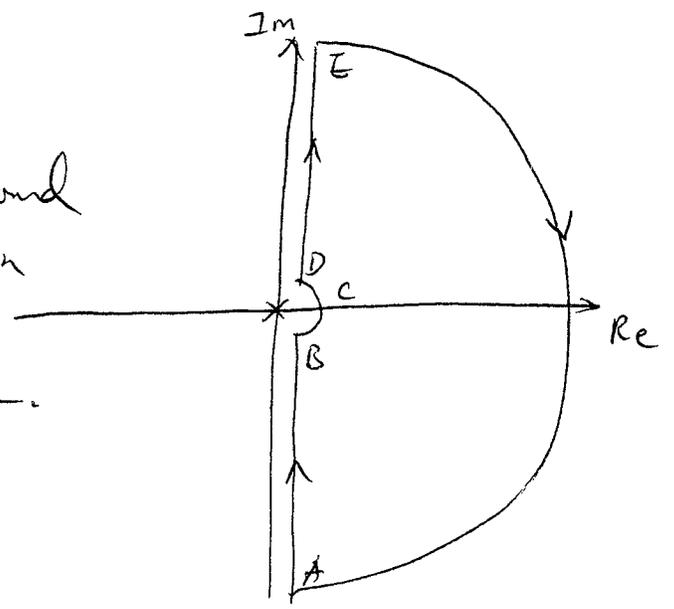
$$G(j\omega)H(j\omega) \approx \frac{Kj\omega}{j\omega \cdot j\omega}$$



Note: This is a non-minimum-phase system!

Take the Nyquist-path around the open-loop pole at origin

$B \rightarrow C \rightarrow D$ swings 180° clockwise.



Step 1: one open-loop pole at origin ✓

$$\lim_{s \rightarrow +\infty} G(s)H(s) = 0$$

Step 2: $P = 1$

Step 3: For stability, the Nyquist-plot must encircle $-\frac{1}{K}$ counter-clockwise once.

encirclement
counter-clockwise

$-1 < -\frac{1}{K} < 0$	1	STABLE	$K > 1$
$-\frac{1}{K} < -1$	-1	UNSTABLE	$0 < K < 1$

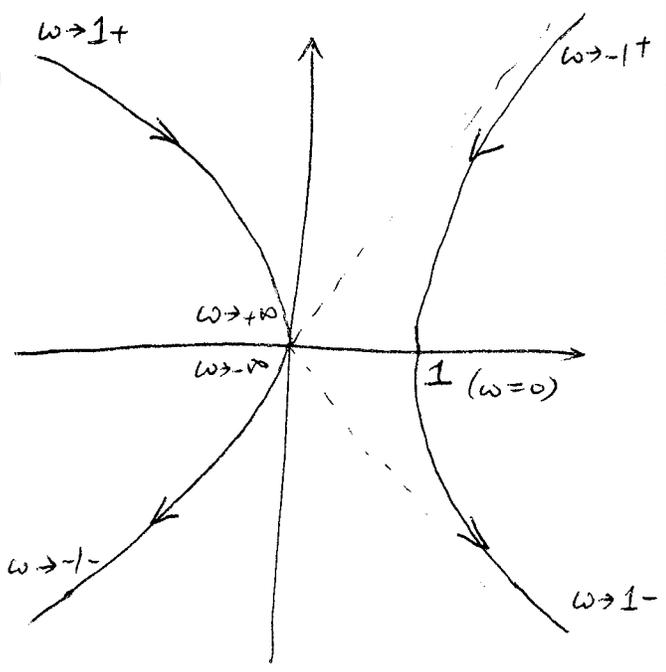
∴ For stability, $K > 1$. (Another "side-effect" of non-minimum-phase systems.)

Example 4:

$$G(s)H(s) = \frac{1}{(s+1)(s^2+1)}$$

Determine the range of K for stability

The Nyquist-plot is quite complicated. Need to use our knowledge of the asymptotes to determine the value of ω for each segment.



As $\omega \rightarrow 0$, $G(j\omega)H(j\omega) = 1$

The Nyquist-plot will approach ∞ as $j\omega \rightarrow \pm j1$

As $\omega \rightarrow j1^-$

$$G(j\omega)H(j\omega) \approx \frac{1}{(1+j)(1-\omega^2)} \approx +\infty \angle -45^\circ$$

As $\omega \rightarrow j1^+$

$$G(j\omega)H(j\omega) \approx \frac{1}{(1+j)(1-\omega^2)} \approx +\infty \angle 135^\circ$$

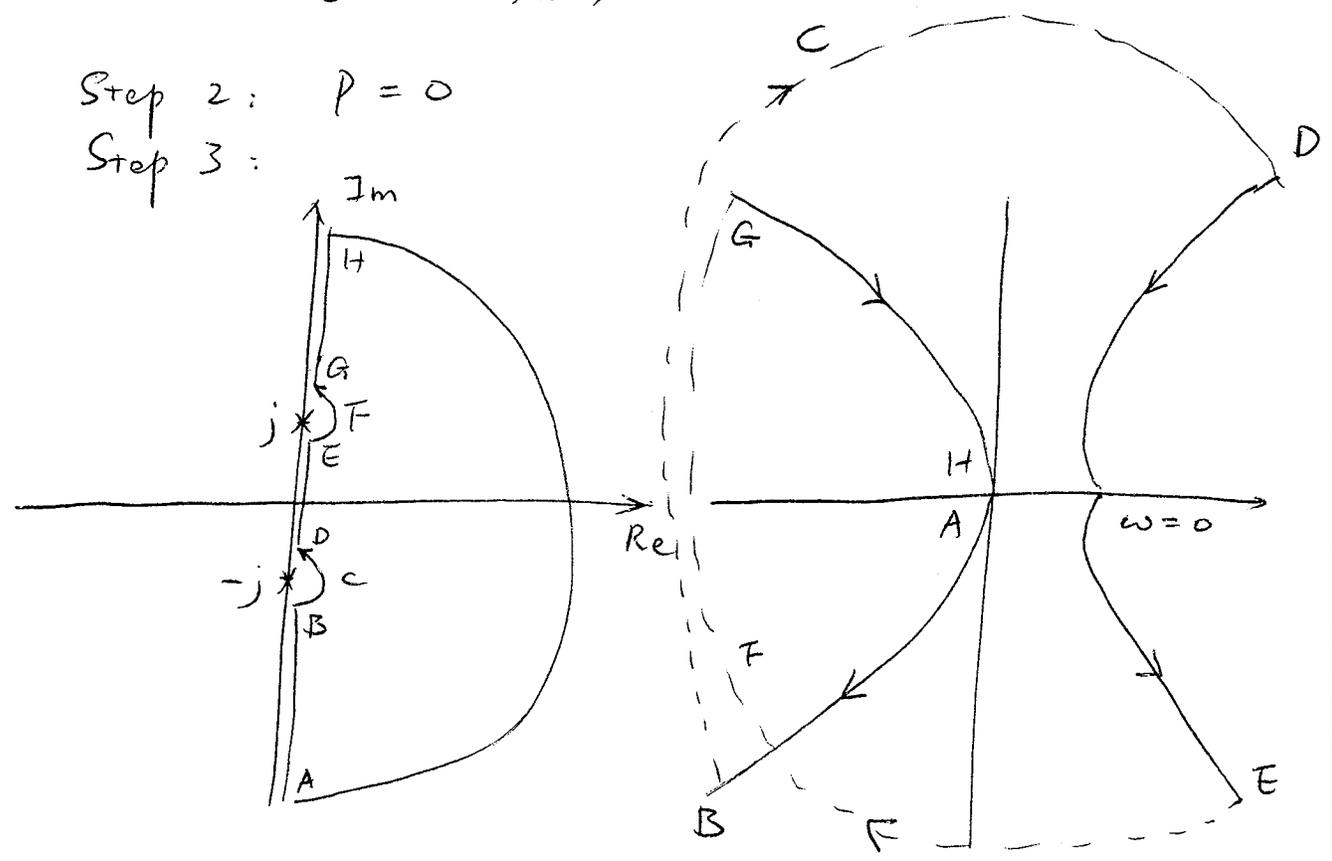
As $\omega \rightarrow +\infty$, $G(j\omega)H(j\omega) \approx \frac{1}{(j\omega)^3} \approx 0 \angle +90^\circ$

Step 1: two poles on $j\omega$ -axis

$$\lim_{s \rightarrow \infty} \frac{1}{(s+1)(s^2+1)} = 0$$

Step 2: $P = 0$

Step 3:



Use the Nyquist-path on the left. Both $D \rightarrow C \rightarrow D$ and $E \rightarrow F \rightarrow G$ swing clockwise 180° .
 Two clockwise encirclement of $-1/k$ for all $k > 0$
 \Rightarrow UNSTABLE for all $k > 0$.