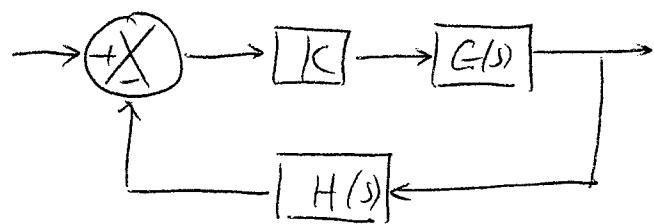


(1)

The Nyquist Stability Criterion P540-547

The closed-loop poles play an essential role in the transient-response of the system.



Closed-loop poles satisfy

$$1 + KG(s)H(s) = 0$$

For stability, the above equation should not have roots on the RHP (including the imaginary axis).

Methods:

- ① Routh Array } require knowledge of $G(s)H(s)$
- ② Root-locus
- ③ Nyquist-Stability Criterion

Nyquist-stability criterion is an alternate method to determine stability, which only requires the evaluation of the open-loop frequency-response

(2)

Main advantage:

- It does not require knowledge of the open-loop transfer function
- Can be used when only experimentally determined frequency-response is available.

(3)

Nyquist Stability Criterion

Basic idea: consider the equation

$$a + G(s) = 0$$

where a is a real number.

Let Z be the number of zeros of $a + G(s)$ on RHP
(i.e., the roots of $a + G(s) = 0$ on RHP)

P be the number of poles of $a + G(s)$ on RHP
(same as the # of poles of $G(s)$ on RHP)

N be the number of times the Nyquist plot
of $G(j\omega)$ (as ω varies from $-\infty$ to $+\infty$)
circles the point $-a$ in the clockwise
direction

The basic idea behind Nyquist-Stability is the
following relationship between N , Z & P

Simplest case: Assume that $G(s)$ has no poles on
the $j\omega$ -axis, and

$$\lim_{s \rightarrow +\infty} G(s) H(s) = \text{constant}$$

(The latter is true when the denominator has higher or
equal order as the numerator). Then

(4)

$$N = Z - P$$

/ \ \

of clockwise encirclement of
 $-a$ by $G(j\omega)$ # of zeros of $a + G(s)$
 on RHP # of poles of $G(s)$
 on RHP

Two immediate cases

① $a=0$: $Z = \#$ of zeros of $G(s)$ on RHP
 $N = \#$ of clockwise encirclement of the origin by $G(j\omega)$

② $a=1$: $Z = \#$ of closed-loop poles (i.e., zeros of $1 + G(s)$) on RHP
 $N = \#$ of clockwise encirclement of -1 by $G(j\omega)$.

Study the two examples in the next two pages.

The Nyquist plots of three open-loop transfer functions $G(s)$ are shown in Figure 1. In each case, the Nyquist plot is shown for $-\infty < \omega < +\infty$, where arrows indicating the direction of increasing ω .

Fill in the following table.

$G(s)$	$\frac{s+4}{11s+6}$	$\frac{(s-1)(s+2)}{(s+1)^2}$	$\frac{-(s+2)(s+3)}{(s-1)^2}$
Number of RHP zeros	0	1	0
Number of RHP poles	0	0	2
Number of clockwise encirclement of 0	0	1	-2

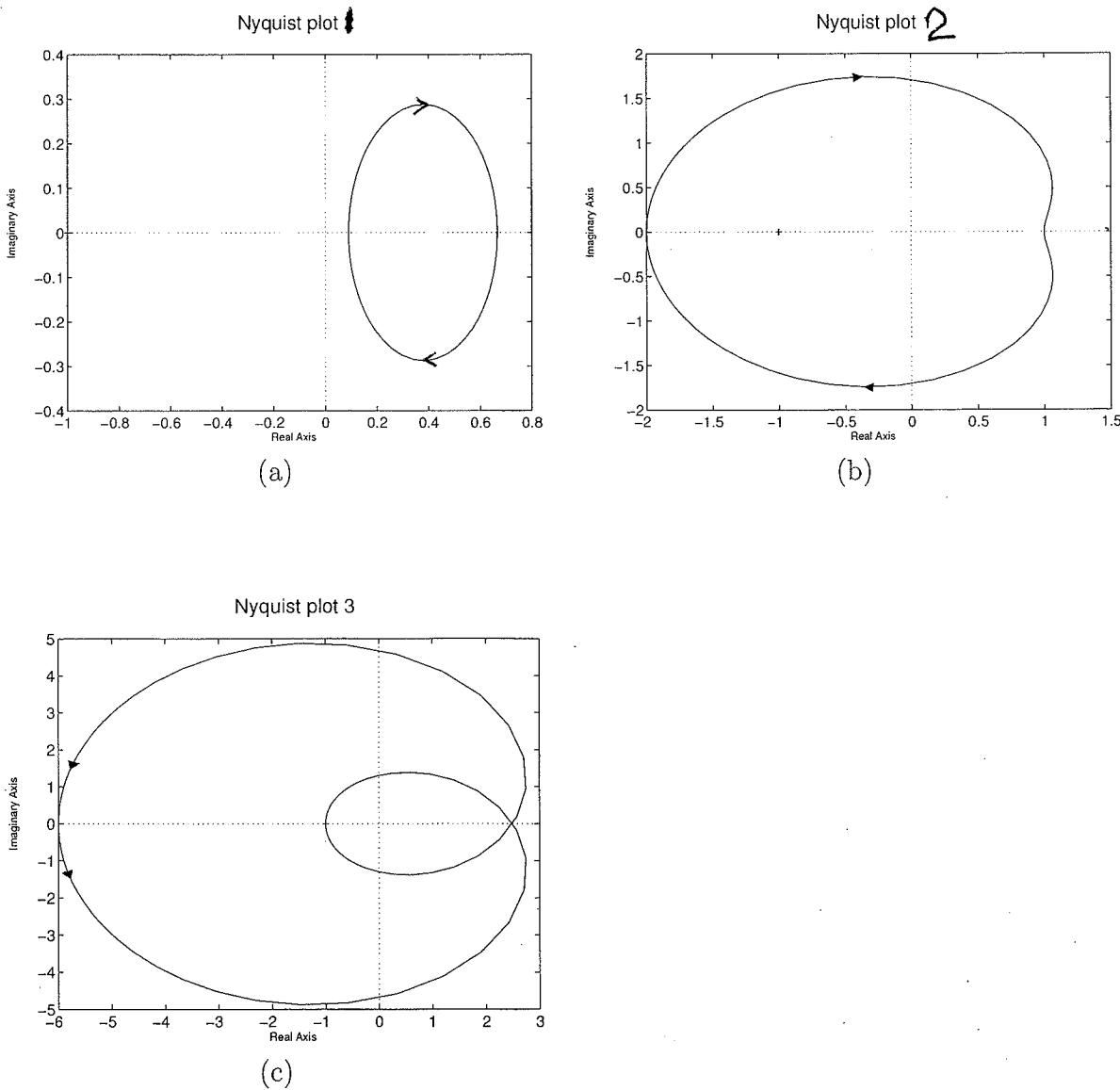


Figure 1: Nyquist plots

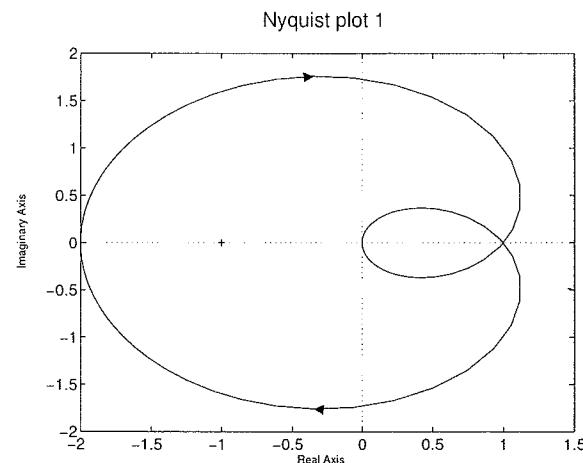
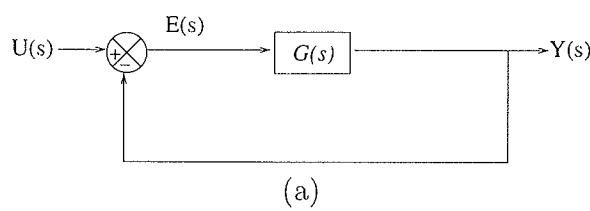
Problem 3 (15 points)

Consider the closed-loop system shown in Figure 3(a). The Nyquist plots of three open-loop transfer functions $G(s)$ are also shown in Figure 3(b)-(d). In each case, the Nyquist plot is shown for $-\infty < \omega < +\infty$, where arrows indicating the direction of increasing ω .

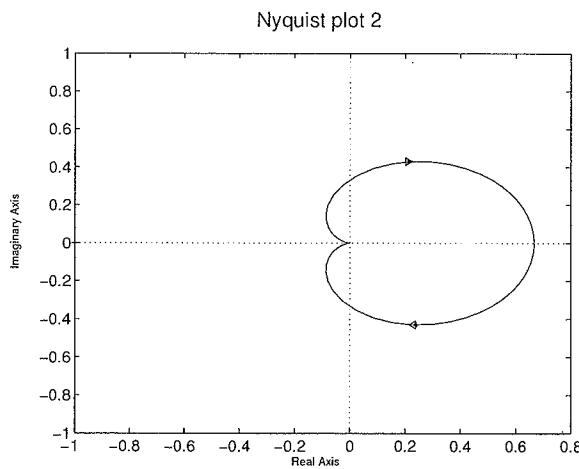
Assume that all *open-loop* transfer functions have **one** pole with positive real part. Find the number of *closed-loop* poles with positive real parts. Do not justify your answer.

P = 1

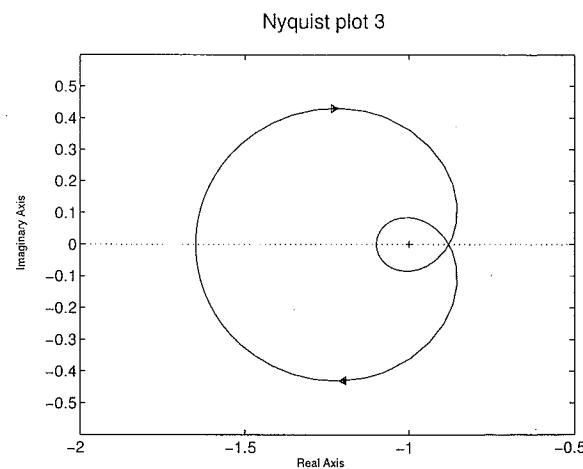
Nyquist plot Number	1	2	3
Number of RHP closed-loop poles	2	1	3



$$N=1 \Rightarrow Z=N+P \\ = 2$$



$$N=0 \\ \Rightarrow N=1$$



$$N=2 \\ \Rightarrow Z=N+P \\ = 3$$

Figure 3: Nyquist plots

Nyquist-Stability Criterion: Assume that the open-loop transfer function $G(s)H(s)$ has no poles on the $j\omega$ axis, and $\lim_{s \rightarrow +\infty} G(s)H(s) = \text{constant}$. If $G(s)H(s)$ has p open-loop poles on RHP, then for stability - then Nyquist-plot of $G(j\omega)H(j\omega)$, as ω varies from $-\infty$ to $+\infty$, must encircle the point -1 p times in the counter-clockwise direction.

In other words, in order for $Z=0$, we must have

$$N = -P$$

of clockwise encirclement of -1 by $G(j\omega)H(j\omega)$

of poles of $G(s)H(s)$ on RHP

Corollary: If $P=0$, i.e., there is no open-loop poles on RHP, then the Nyquist plot should not encircle the -1 point.

What if the system has a proportional gain K ?

The closed-loop poles are roots of

$$1 + KG(s)H(s) = 0$$

(8)

To use Nyquist-Stability, we can either

- ① Draw the Nyquist plot of $KG(j\omega)H(j\omega)$, and count the encirclement of -1
- ② Still use the Nyquist plot of $G(j\omega)H(j\omega)$, and count the encirclement of $-\frac{1}{K}$

$$1 + KG(s)H(s) = 0 \Leftrightarrow -\frac{1}{K} + G(s)H(s) = 0$$

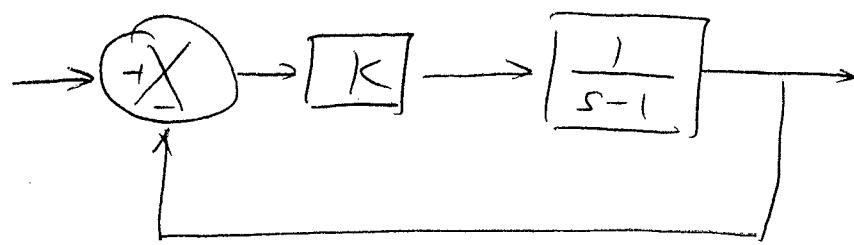
everything else should remain the same.

Steps to use the Nyquist Stability Criterion:

- ① Check:
 - a $G(s)H(s)$ has no poles on the $j\omega$ -axis
 - b $\lim_{|s| \rightarrow +\infty} G(s)H(s) = \text{constant}$.
- ② Find out $P = \#$ of open-loop poles of $G(s)H(s)$ on RHP.
- ③ Check whether the Nyquist-plot $G(j\omega)H(j\omega)$ encircles the -1 point (or $-\frac{1}{K}$ point) counter-clockwise P times.

(9)

Ex)



The closed-loop poles are roots of

$$1 + \frac{K}{s-1} = 0$$

$$\Leftrightarrow \frac{1}{K} + \frac{1}{s-1} = 0$$

Let us work on two special cases first.

① Case 1: $K = \frac{1}{2}$

$$2 + \frac{1}{s-1} = 0$$

$\Rightarrow G(s)$

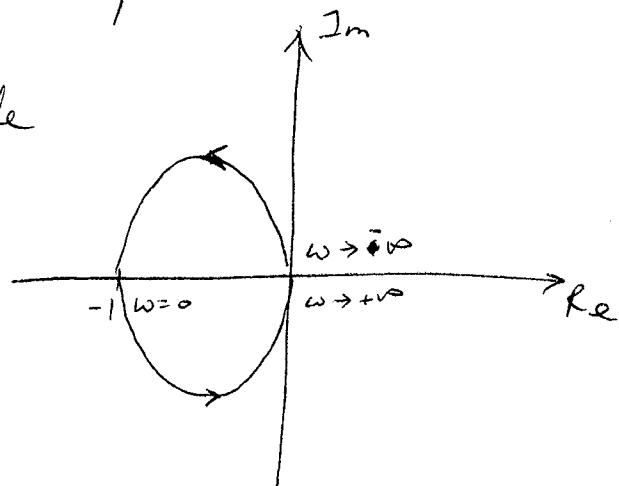
Step ①: no poles on $j\omega$ -axis ✓

$$\underset{s \rightarrow +\infty}{\lim} G(s) = 0 \quad \checkmark$$

Step ②: $P = 1$

Step ③: The Nyquist-plot is

It does not encircle
the point -2



Hence, the system is
unstable.

Verify: $2 + \frac{1}{s-1} = 0 \Rightarrow s = \frac{1}{2}$ UNSTABLE!

(16)

② Case 2: $K = 2$

$$\frac{1}{2} + \frac{1}{s-1} = 0$$

$\Rightarrow G(s)$

Step ①: ok

Step ②: $P = 1$ Step ③: The Nyquist-plot of $G(j\omega)$ encircles
 $-\frac{1}{2}$ once counter-clockwise.

The system is stable

Verify:

$$\frac{1}{2} + \frac{1}{s-1} = 0$$

$$\Rightarrow s = -1 \quad \text{STABLE!}$$

In fact, we can determine the range of K for stability. For stability, the Nyquist-plot of $G(j\omega)$ must encircle $-\frac{1}{K}$ once counter-clockwise. This only happens when $-\frac{1}{K}$ is between -1 and 0

$$-1 < -\frac{1}{K} < 0$$

$$\Rightarrow K > 1 \quad \text{for stability}$$

(11)

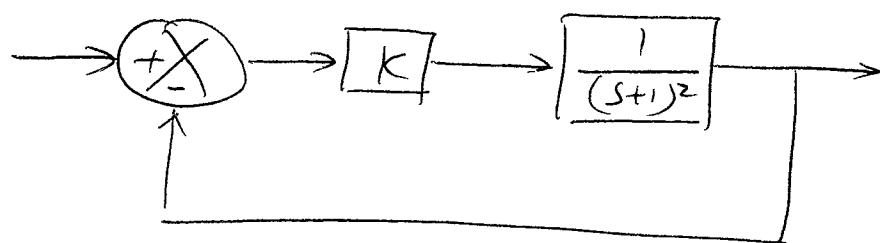
Question: What if the Nyquist-plot passes through the -1 point (e.g. $K=1$)

$$\Rightarrow KG(j\omega)H(j\omega) = -1 \text{ for some } \omega$$

\Rightarrow closed-loop pole is on the $j\omega$ -axis

\Rightarrow UNSTABLE.

Ex)



Determine the range of K for Stability.

Step ①: no poles on $j\omega$ -axis ✓

$$\lim_{s \rightarrow +\infty} G(s) = 0 \quad \checkmark$$

Step ②: $P = 0$

Step ③: For stability, we require zero encirclement of the $-1/K$ point.

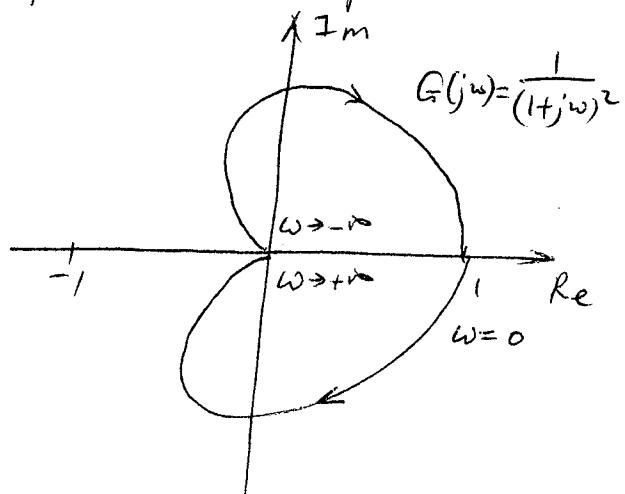
This is true for

all $K > 0$

\Rightarrow The system is always stable.

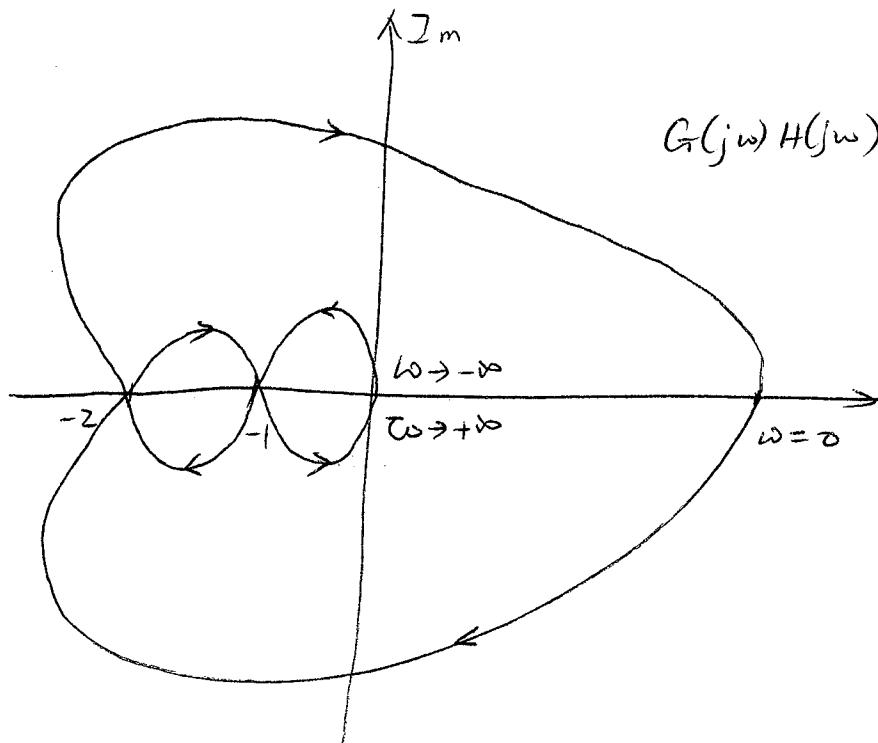
$$\text{Verify: } 1 + K \frac{1}{(s+1)^2} = 0$$

$$s = -1 \pm \sqrt{k}j$$



(12)

Ex)



Assume that $G(s)H(s)$ is stable (i.e., $P=0$)

Determine the range of k for stability of the closed-loop system.

Step ①: no poles on $j\omega$ -axis ✓

(otherwise the Nyquist-plot will blow up).

$$\underset{s \rightarrow +\infty}{\text{Lim}} G(s)H(s) = 0 \quad \checkmark$$

Step ②: $P = 0$

Step ③: For stability, the Nyquist-plot must not encircle the $-k$ point.

encirclement
counter-clockwise

$$-1 < -\frac{k}{k} < 0 \quad 0 \quad \text{STABLE} \quad k > 1$$

$$-2 < -\frac{k}{k} < -1 \quad -2 \quad \text{UNSTABLE} \quad \frac{1}{2} < k < 1$$

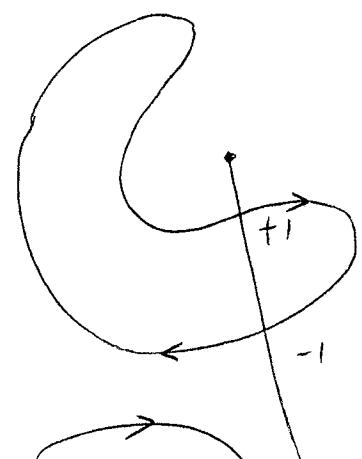
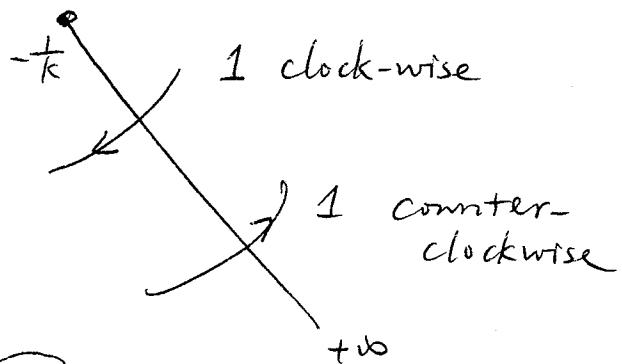
$$-\frac{1}{k} < -2 \quad 0 \quad \text{STABLE} \quad k < \frac{1}{2}$$

⇒ For stability, need $k > 1$ or $k < \frac{1}{2}$.

(13)

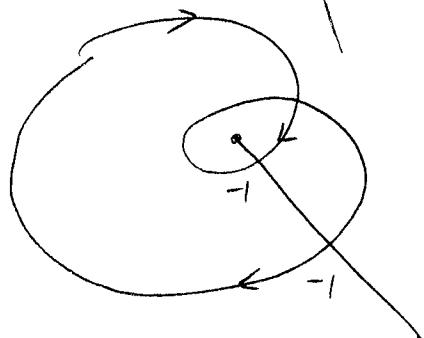
Counting the number of encirclement can be confusing:

Follow this easy procedure. To find the number of encirclement of any point $-k$, draw a ray from that point $-k$ to $+\infty$ (in any direction). Count the # of intersections with the Nyquist plot, paying attention to the direction with which the Nyquist-plot crosses the ray.



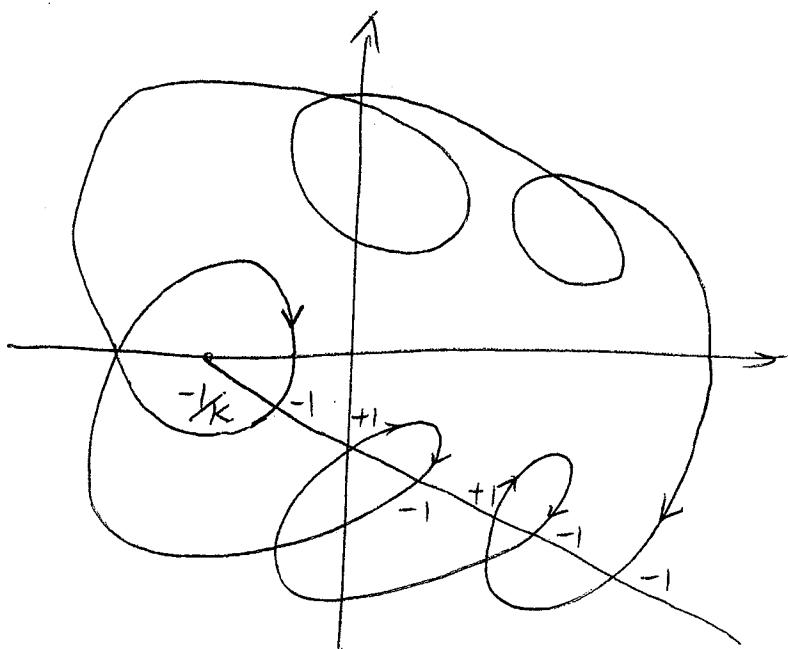
If we take counter-clockwise as the positive-direction

$$(+1) + (-1) = 0 \quad \text{encirclement counter-clockwise}$$



$$(-1) + (-1) = -2 \quad \text{encirclement counter-clockwise}$$

(14)



$$(-1) + (+1) + (-1) + (+1) + (-1) + (-1) = -2$$

encirclement
counter-clockwise.