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Routh's Stability Criterion P 275-281

We know that a LTI system is stable if and only if all poles of the transfer function $H(s)$ have negative real parts.

$$\text{Ex) } H(s) = \frac{s^2 + 1}{(s+1)(s+3)(s+1+j)(s+1-j)}$$

poles: $-1, -3, -1 \pm j$ STABLE

$$\text{Ex) } H(s) = \frac{1}{s(s+1)(s+2)}$$

poles: $0, -1, -2$ UNSTABLE

$$\text{Ex) } H(s) = \frac{1}{(s^2 + 1)(s + 1)}$$

poles $\pm j, -1$ UNSTABLE

Routh's Stability Criterion: Enables us to compute the # of poles with nonnegative real parts, without having to factor out the denominator polynomial.

A technique due to Routh (1874) and Hurwitz (1895). Sometimes called Routh-Hurwitz Criterion.

A polynomial all of whose roots have negative real parts is called a Hurwitz Polynomial.

Steps in using the Routh's Criterion:

- ① Write the denominator polynomial of the transfer function (also known as the "characteristic polynomial") as

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

↑

leading coefficient is 1

If any of the coefficients a_1, a_2, \dots, a_n are zero or negative, then at least one of the roots of $a(s)$ has non-negative real parts
 \Rightarrow The system must be unstable!

Reason:

$$a(s) = (s+p_1) \cdots (s+p_q) (s^2+b_1 s+c_1) \cdots (s^2+b_k s+c_k)$$

If all poles have ~~no~~ negative real parts, then $p_1, \dots, p_q, b_1, \dots, b_k, c_1, \dots, c_k$ should all be positive.

\Rightarrow The coefficients in the product must also be positive.

- Ex) ① $s^2 - s + 1$ UNSTABLE
- ② $s^4 + 4s^3 + s^2 + 1$ UNSTABLE (missing s)
- ③ $s^6 + 3s^5 + 4s^4 + s^3 + s^2 + s + 1$ CAN'T TELL.

If all the coefficients a_i are positive (as in (c)) or if we need to know the exact # of poles of $a(s)$ with nonnegative real parts (as in (d) & (e)), we need to form the Routh array.

(d) The Routh array:

Arrange the coefficients of the characteristic polynomial in two rows, each beginning with the first and second coefficients, and followed by the even-numbered and odd-numbered coefficients.

$$\begin{array}{cccc} s^n : & 1 & a_2 & a_4 \\ s^{n-1} : & \downarrow a_1 & \downarrow a_3 & \downarrow a_5 \end{array}$$

Next, form the third row as

$$s^{n-2} : b_1 \ b_2 \ b_3 \dots$$

where

$$b_1 = \frac{a_1 a_2 - a_3}{a_1},$$

$$b_2 = \frac{a_1 a_4 - a_5}{a_1}, \text{ etc}$$

Next, form the 4th row

$$s^{n-3} : c_1 \ c_2 \ c_3$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1},$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, \text{ etc}$$

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Continue until the procedure terminates
 The last row should be s^0

$$\text{Ex) } a(s) = s^6 + 3s^5 + 4s^4 + s^3 + s^2 + s + 1$$

$$\begin{array}{r} s^6 \quad 1 \quad 4 \quad 1 \quad 1 \\ s^5 \quad 3 \quad 1 \quad 1 \quad 0 \\ s^4 \quad \frac{11}{3} \quad \frac{2}{3} \quad \frac{3}{3} \\ \hline \end{array}$$

$$\begin{array}{r} s^3 \quad \frac{11}{3} - \frac{6}{3} \quad \frac{11}{3} - 3 \\ \hline \frac{11}{3} \quad \frac{11}{3} \\ \hline \end{array}$$

$$\begin{array}{r} " \quad \frac{5}{11} \quad " \quad \frac{2}{11} \\ \hline \end{array}$$

$$\begin{array}{r} s^2 \quad \frac{10}{33} - \frac{22}{33} \quad \frac{5}{11} - 0 \\ \hline \frac{5}{11} \quad \frac{5}{11} \\ \hline " - \frac{12}{33} \quad " \quad 1 \\ \hline \frac{5}{11} \quad \frac{5}{11} \\ \hline " - \frac{4}{5} \end{array}$$

$$\begin{array}{r} s^1 \quad - \frac{3}{55} - \frac{5}{11} \\ \hline - \frac{4}{5} \\ \hline " - \frac{33}{55} \\ \hline - \frac{4}{5} \\ \hline " \frac{3}{4} \end{array}$$

$$\begin{array}{r} s^0 \quad \frac{3}{4} - 0 \\ \hline \frac{3}{4} \\ \hline " 1 \end{array}$$

- ③ Write down the sign pattern of the first column in the Routh array.
Then

$$\begin{array}{c} \# \text{ of roots of } a(s) \\ \text{with non-negative real parts} \end{array} = \begin{array}{c} \# \text{ of sign changes} \end{array}$$

(Right-Half-plane (RHP) roots)

Ex) Sign changes are

+ + + + - + +
 ↓↓

$$\# \text{ of RHP roots} = 2$$

Using MATLAB, we can check that the roots are in fact

$$-1.43 \pm 1.18j, -0.51 \pm 0.54j$$

$$0.44 \pm 0.57j$$

③ Write down the sign pattern of the first column in the Routh array.

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$$\begin{array}{c} \# \text{ of roots of } a(s) \\ \text{with non-negative real parts} \end{array} = \begin{array}{c} \# \text{ of sign changes} \end{array}$$

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$$0.44 \pm 0.57j$$

Trick: Since we only care about the signs, an entire row of the Routh array can be multiplied by a positive constant, without affecting the # of sign changes.

$$\text{Ex) } a(s) = s^6 + 3s^5 + 4s^4 + s^3 + s^2 + s + 1$$

$$\begin{array}{r}
 + \quad s^6 \quad 1 \quad 4 \quad 1 \quad 1 \\
 + \quad s^5 \quad 3 \quad 1 \quad 1 \\
 s^4 \quad \cancel{\frac{11}{3}} \quad \cancel{\frac{2}{3}} \quad \cancel{\frac{3}{3}} \quad \times 3 \\
 + \quad 11 \quad 2 \quad 3
 \end{array}$$

$$\begin{array}{r}
 s^3 \quad \cancel{\frac{5}{1}} \quad \cancel{\frac{2}{1}} \quad \times 11 \\
 + \quad 5 \quad 2
 \end{array}$$

$$\begin{array}{r}
 s^2 \quad \cancel{\frac{-12}{5}} \quad \cancel{\frac{15}{5}} \quad \times \frac{5}{3} \\
 - \quad -4 \quad 5
 \end{array}$$

$$\begin{array}{r}
 s^1 \quad \cancel{\frac{-33}{4}} \quad \times \frac{4}{33} \\
 + \quad 1
 \end{array}$$

$$+ \quad s^0 \quad 5$$

Sign changes : + + + + - + +

RHP roots = 2.

Special cases:

- ① If the first element of a row is zero, and there is some element that is non-zero in the same row.

$$\text{Ex) } a(s) = s^4 + s^2 + s + 1$$

$$\varepsilon \rightarrow 0^+ \quad \varepsilon \rightarrow 0^-$$

$$\begin{array}{cccccc} + & + & s^4 & 1 & 1 & 1 \\ + & - & s^3 & \cancel{\varepsilon} & 1 \\ - & + & s^2 & \frac{s-1}{\varepsilon} & \cancel{\frac{\varepsilon}{\varepsilon}} & 1 \end{array}$$

$$\begin{array}{cccccc} + & + & s^1 & \frac{s-1}{\varepsilon} - \varepsilon & & \\ & & & \frac{s-1}{\varepsilon} & \swarrow & \\ & & & 1 - \frac{\varepsilon^2}{s-1} & & \\ + & + & s^0 & \frac{1 - \frac{\varepsilon^2}{s-1}}{1 - \frac{\varepsilon^2}{s-1}} & \swarrow & 1 \end{array}$$

- ① Replace the zero with a non-zero, small ε and complete the rest of the Routh array.
- ② Let $\varepsilon \rightarrow 0^+$, (i.e., assume $\varepsilon > 0$ and let $\varepsilon \rightarrow 0$). Count the # of sign changes in the first column, and let this number be N_+
- ③ Let $\varepsilon \rightarrow 0^-$, count the # of sign changes in the first column. Let this number by N_- . Then $\max\{N_+, N_-\} = \text{Total # of RHP roots of } a(s)$
(including imaginary roots)
 $|N_+ - N_-| = \# \text{ of roots on the imaginary axis.}$

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Return to the example:

$$N_+ = 2$$

$$N_- = 2$$

\therefore 2 roots in RHP, no imaginary roots.

(MATLAB gives roots at

$$\begin{aligned} & 0.55 \pm j1.12 \\ & -0.55 \pm j0.59 \end{aligned} \quad)$$

Special cases

② An entire row is zero

FACT: This implies that the system has poles of equal magnitude lying radially opposite in the complex plane - e.g. $j\omega, -j\omega$.

$$\text{Ex) } a(s) = s^5 + 6s^4 + 12s^3 + 12s^2 + 11s + 6$$

$$+ \quad s^5 \quad 1 \quad 12 \quad 11$$

$$+ \quad s^4 \quad 6 \quad 10 \quad 6$$

$$\quad \quad \quad 1 \quad 2 \quad 1$$

$$+ \quad s^3 \quad 10 \quad 10$$

$$\quad \quad \quad 1 \quad 1$$

$$+ \quad s^2 \quad 1 \quad 1$$

$$+ \quad s^1 \quad 0 \quad 2$$

$$+ \quad s^0 \quad 1$$

$$a_1(s) = s^2 + 1 \quad a'(s) = 2s$$

- ① Suppose i th row is zero. Form an auxiliary equation from the previous row as follows

$$a_1(s) = \beta_1 s^{i+1} + \beta_2 s^{i-1} + \beta_3 s^{i-3} + \dots$$

where β_1, β_2, \dots are coefficients of the $(i-1)$ -th row.

- ② Replace i th row by the derivative of the auxiliary polynomial, and complete the rest of the Routh array. Perform Routh test.
- ③ Roots of the auxiliary polynomial $a_1(s)$ are also roots of $a(s)$. Test $a_1(s)$ separately.

Combine results from step ② & ③.

Return to the example

- No RHP roots according to Routh test
- Roots of $a_1(s)$ is $\pm j$

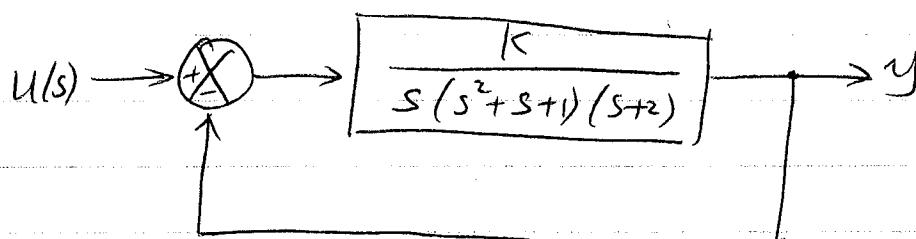
$\Rightarrow a(s)$ has two roots on the imaginary axis.

Note: In both of these two special cases, the system is immediately unstable.

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Application of the Routh array test

Ex) P280 of text



Determine the range of K for stability.

$$\frac{Y(s)}{U(s)} = \frac{K}{K + s(s^2 + s + 1)(s + 2)}$$

$$a(s) = s^4 + 3s^3 + 3s^2 + 2s + K$$

$$\begin{array}{r} s^4 \quad 1 \quad 3 \quad K \\ s^3 \quad 3 \quad 2 \\ s^2 \quad \frac{7}{3} \quad \cancel{\frac{3K}{3}} \\ \hline s^1 \quad \cancel{\frac{14}{3} - 3K} \\ \hline s^0 \quad 2 - \frac{9}{7}K \end{array}$$

$$\begin{array}{r} s^4 \quad 1 \quad 3 \quad K \\ s^3 \quad 3 \quad 2 \\ s^2 \quad \frac{7}{3} \quad \cancel{\frac{3K}{3}} \\ \hline s^1 \quad \cancel{\frac{14}{3} - 3K} \\ \hline s^0 \quad 2 - \frac{9}{7}K \end{array}$$

$$s^0 \quad K$$

For stability, we require no RHP roots for $a(s)$ or no sign changes in the first column.

$$\Rightarrow 2 - \frac{9}{7}K > 0, \quad K > 0$$

$$\Rightarrow 0 < K < \frac{14}{9} \quad \text{for stability}$$