

(1)

The Root-Locus Method P337-358

Motivation: The transient response of the system is closely related to the location of the poles.

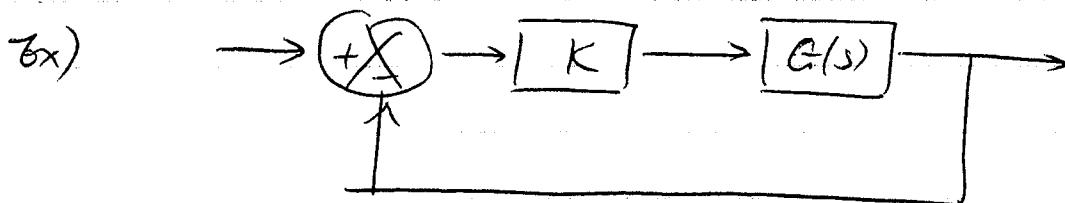
To ensure better transient response, we want the dominant poles to be at a position that corresponds to damping ratio between 0.5 ~ 0.8

The Routh Stability Criterion

only allows us to determine whether the poles are all on the LHP.

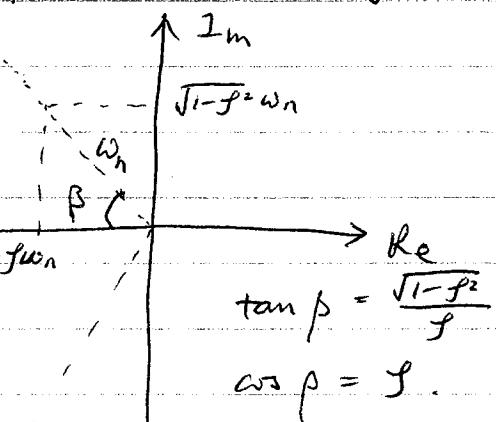
For more refined analysis of the position of poles, we can use the root-locus method.

Root-locus: Locus of the closed-loop poles as a function of a single parameter



The end-to-end transfer function is

$$H(s) = \frac{KG(s)}{1 + KG(s)}$$



(2)

Hence, the closed-loop poles are roots of

$$1 + KG(s) = 0$$

$$\text{If } G(s) = \frac{1}{s^2 + s + 1}$$

(standard form
of characteristic
equation)

$$\text{then } s^2 + s + (K+1) = 0$$

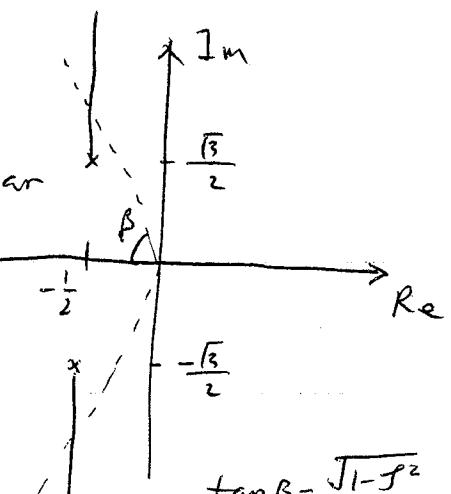
The closed-loop poles are

$$s = -\frac{1}{2} \pm j \frac{\sqrt{3+4K}}{2}$$

If we want to achieve a particular damping ratio, ξ , we can draw half-lines with angle $\beta = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$ with the negative real axis.

The intersection point is where the closed-loop poles should be.

\Rightarrow also allows us to calculate the correct value of K .



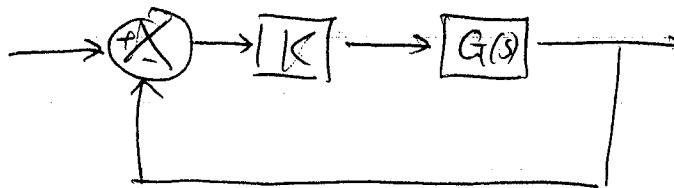
$$\tan \beta = \frac{\sqrt{1-\xi^2}}{\xi}$$

To apply the root-locus method, we first write down the ~~char~~ equation that the characteristic polynomial equal to zero. We then convert it into the standard form

$$1 + K G(s) = 0$$

(3)

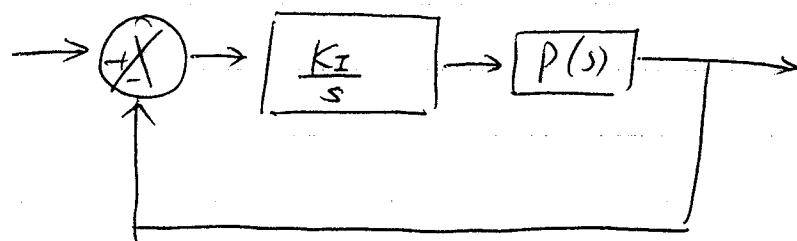
Ex) ①



The characteristic equation is

$$1 + KG(s) = 0$$

②



The characteristic equation is

$$1 + \frac{K_I}{s} P(s) = 0$$

$$\Rightarrow 1 + \underbrace{K_I}_{K} \underbrace{\frac{P(s)}{s}}_{G(s)} = 0$$

③



The characteristic equation is

$$1 + \frac{1}{s+\alpha} P(s) = 0$$

$$\Rightarrow (s + P(s)) + \alpha = 0$$

$$\Rightarrow \underbrace{1 + \alpha \cdot \frac{1}{s + P(s)}}_{K G(s)} = 0$$

(4)

There is a matlab command to generate the root-locus :

$$\text{num} = [1]$$

$$\text{den} = [1 \ 1 \ 1]$$

$$\text{sys} = \text{tf}(\text{num}, \text{den})$$

$$\text{rlocus}(\text{sys})$$

We will study how to sketch the root-locus by hand,
(crucial both for understanding and for design.)

A few remarks before we start

- ① You should think of the root-locus as a "trajectory" of the poles as K varies from 0 to $+\infty$. (Think about a missile flying through the sky.)
- ② Since an equation with order n has exactly n roots, there will be n such trajectories.
- ③ Each trajectory must come from somewhere (as $K=0$), and end at somewhere (as $K \rightarrow +\infty$)
- ④ Trajectories (or branches) can meet (collide); then they must also depart from each other.
- ⑤ We will use a lot of approximations

$$as^2 + bs + c \approx as^2 \text{ when } s \text{ is large}$$

$$\approx c \text{ when } s \text{ is close to zero.}$$

(5)

- ⑥ Keep in mind that the poles of $G(s)$ are different from the closed-loop poles of the overall system.

Basic Relationships

$$1 + KG(s) = 0 \quad m \text{ zeros}$$

$$G(s) = \frac{(s-z_1)(s-z_2) \dots (s-z_m)}{(s-p_1)(s-p_2) \dots (s-p_n)} \quad n \text{ poles}$$

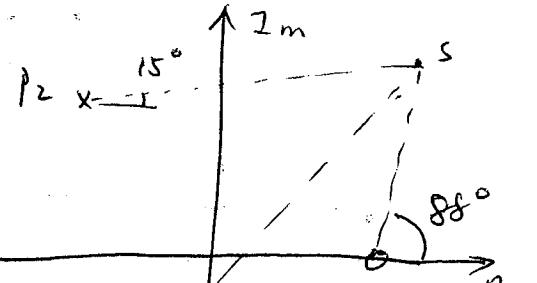
- ① Angle condition: A complex number s is on the locus if and only if

$$G(s) = -\frac{1}{K} \quad \text{for some } K > 0$$

In other words, iff

$$\angle G(s) = \sum_{j=1}^m \angle(s-z_j) - \sum_{j=1}^n \angle(s-p_j)$$

$$= 180^\circ + l \cdot 360^\circ, \quad l = 0, \pm 1, \dots$$



We will mainly use the angle condition to sketch the root-locus.

$$88^\circ - 60^\circ - 15^\circ = 3^\circ$$

s is NOT on the root-locus

- ② Magnitude condition: Once a point s is on the root-locus, the corresponding value of K can be determined from

$$K = \left| \frac{1}{G(s)} \right| = \frac{\prod_{i=1}^n |s - p_i|}{\prod_{j=1}^m |s - z_j|} \quad \begin{matrix} \text{distance to the} \\ \text{poles} \end{matrix} \quad \begin{matrix} \text{distance to the} \\ \text{zeros} \end{matrix}$$

Magnitude condition allow us to find the gain on a specific point of the root-locus.

- ③ Rewrite the characteristic equation as

$$(s - p_1)(s - p_2) \cdots (s - p_n) + K(s - z_1)(s - z_2) \cdots (s - z_m) = 0$$

If $K=0$, the roots are the poles p_1, \dots, p_n

If $K=+\infty$, the roots are the zeros z_1, \dots, z_m .

Hence, assuming that $n \geq m$, there are n branches / trajectories of the locus

- each branch begins from a pole ($K=0$), and ends at a zero ($K=+\infty$)
- If $n > m$, $G(s)$ is said to have $(n-m)$ zeros at ∞ . In this case, $(n-m)$ branches begin from poles and end at ∞ .
- The branches are symmetric w.r.t. the real axis
- Remember that each branch is a trajectory: It must be continuous.

Questions:

- ① Which pole goes to which zero?
- ② How? In what shape?

We are not going to solve the roots for all K . Instead, we will use the following set of simple rules by W. R. Evans in 1948.

Procedures for Drawing Root-Locus
(Based mainly on the angle condition)

Step 1: Write the characteristic equation as

$$1 + K G(s) = 0$$

Find the poles & zeros of $G(s)$

$$1 + K \frac{\prod_{j=1}^m (s - z_j)}{\prod_{i=1}^n (s - p_i)} = 0$$

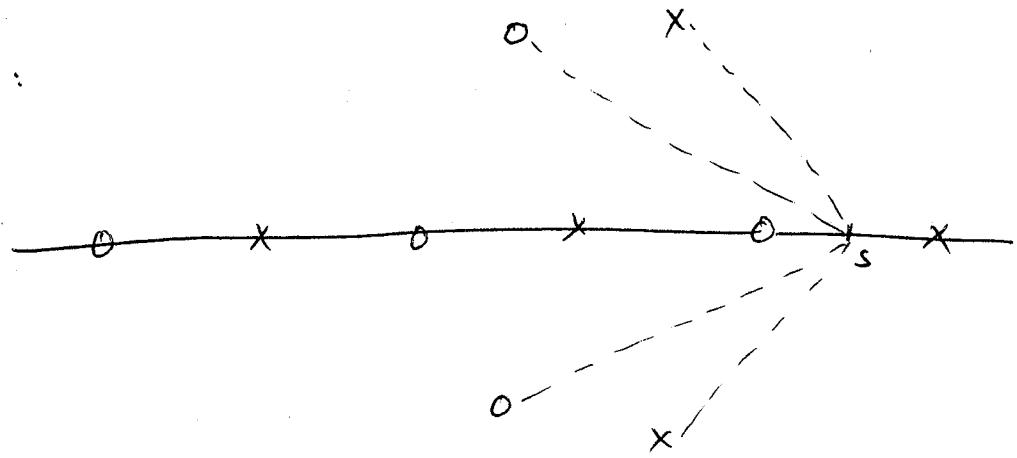
Step 2: Locate the poles & zeros on the complex plane.

Step 3: We can now determine the part of the root-locus that lies on the real-axis.

The root-locus on the real-axis lies in a segment of the real axis to the left of an odd number of real poles & zeros.

(8)

Reason:



$$\angle G(s) = \sum_{j=1}^m \angle(s - z_j) - \sum_{i=1}^n \angle(s - p_i)$$

- The angle to a pole/zero on the right of s is 180°
- The angle to a pole/zero on the left of s is 0°
- The angles to two complex-conjugate poles/zeros add up to 360°

Hence

$$\begin{aligned} \angle G(s) &= 180^\circ (\# \text{ of zeros to the right of } s \\ &\quad - \# \text{ of poles to the right of } s) \\ &\quad + 360^\circ (\# \text{ of complex-conjugate pairs}) \end{aligned}$$

In order for

$$\angle G(s) = 180^\circ + l \cdot 360^\circ$$

s must lie in a segment to the left of an odd number of real poles & zeros.

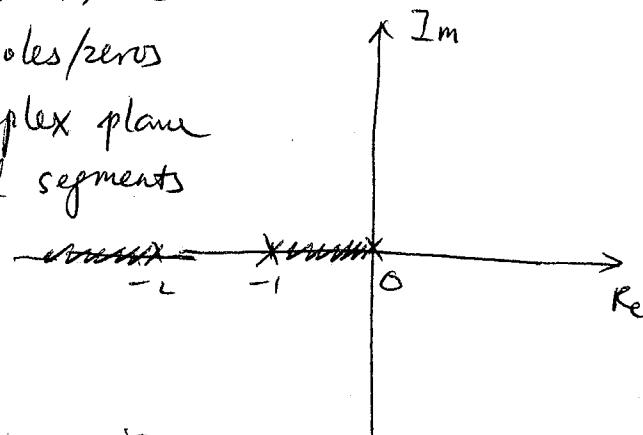
(9)

Ex) $G(s) = \frac{1}{s(s+1)(s+2)}$

Step 1: poles are 0, -1, -2

Step 2: Locate the poles/zeros
on the complex plane

Step 3: Determine real segments
of the locas



Observations

① One branch from $-2 \rightarrow -\infty$

② Two branches from 0, -1, respectively.

They must meet at somewhere between 0, -1
and then break ~~into~~ away from the real-axis.
They will then go to ∞ .

Step 4. Find Asymptotes to Zeros at Infinity.

$(n-m)$ branches proceed to zeros at infinity
along asymptotes centered at Γ_a and
with angle ϕ_a :

$$\Gamma_a = \frac{\sum_{i=1}^n p_i - \sum_{j=1}^m z_j}{n-m}$$

$$\phi_a = \frac{180^\circ}{n-m} + l \cdot \frac{360^\circ}{n-m}, \quad l = 0, 1, \dots, n-m-1$$

(10)

$$Ex) \quad Q(s) = \frac{1}{s(s+1)(s+2)}$$

3-0 branches proceed to infinity

$$\Gamma_a = \frac{\sum_{i=1}^3 p_i - 0}{3} = -1$$

$$\phi_a = \frac{180^\circ}{3} + 1 \cdot \frac{360^\circ}{3}$$

$$= 60^\circ, 180^\circ, -60^\circ$$

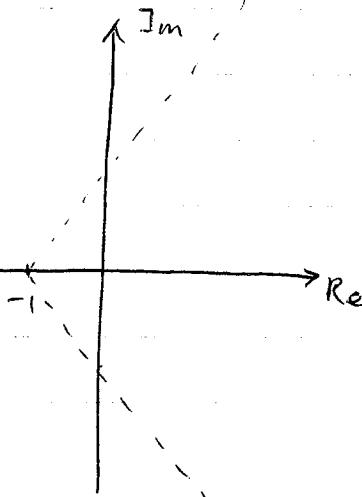
Reasons:

① When s is large,

$$G(s) \approx \frac{1}{s^{n-m}}$$

$$\angle G(s) \approx -(n-m) \angle s = 180^\circ + 1 \cdot 360^\circ$$

$$\Rightarrow \angle s = \frac{180^\circ}{n-m} + 1 \cdot \frac{360^\circ}{n-m}$$



② Consider a polynomial

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$$

$$= (s - r_1)(s - r_2)(s - r_3) \dots (s - r_n)$$

$$= s^n - \sum_{i=1}^n r_i \cdot s^{n-1} + \dots$$

$$\therefore a_1 = - \sum_{i=1}^n r_i$$

In other words, the ratio between the coefficient of s^{n-1} to the coefficient of s^n equal to the negative of the sum of the roots.

Now, note that the characteristic equation is

$$(s - p_1)(s - p_2) \dots (s - p_n) + K(s - z_1) \dots (s - z_m) = 0$$

If $n - m \geq 2$, then the coefficients of s^n , s^{n-1} is 1 and $-\sum_{i=1}^n p_i$, respectively, which are independent of K .

(11)

Let r_1, \dots, r_n denote the roots (i.e., closed-loop poles)
Then

$$\sum_{i=1}^n r_i = \sum_{i=1}^n p_i \quad \text{for all } k$$

When $k \rightarrow +\infty$, let us denote r_{m+1}, \dots, r_n to be the poles that go to ∞ . The rest of the poles r_1, \dots, r_m go to z_i . Hence, when $k \rightarrow +\infty$

$$\sum_{i=m+1}^n r_i = \sum_{i=1}^n p_i - \sum_{j=1}^m z_j$$

\Rightarrow The center of these poles (that go to infinity) is

$$J_a = \frac{\sum_{i=m+1}^n r_i}{n-m} = \frac{\sum_{i=1}^n p_i - \sum_{j=1}^m z_j}{n-m}$$

Step 5. Find the point at which the root-locus crosses the imaginary axis.

Find K such that the Routh Array has a row that is entirely zero.

Solve the ~~the~~ corresponding auxillary polynomial to get the imaginary closed-loop poles.

(12)

$$\text{Ex) } G(s) = \frac{1}{s(s+1)(s+2)}$$

The characteristic equation is
 $s^3 + 3s^2 + 2s + k = 0$

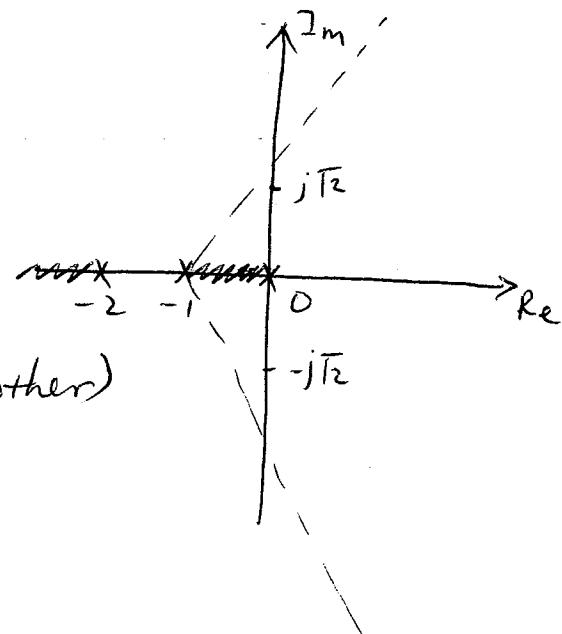
$$\begin{matrix} s^3 & 1 & 2 \\ s^2 & 3 & 1k \\ s^1 & \frac{6-k}{3} & \\ s^0 & k & \end{matrix}$$

If $k=6$, then this row will be all zero.

The auxiliary polynomial is

$$3s^2 + 6 = 0$$

$$s = \pm j\sqrt{2}$$



Step 6. Determine the break-in/break-away points (where branches touch each other) by solving

$$\frac{dk}{ds} = 0$$

$$\text{Ex) } G(s) = \frac{1}{s(s+1)(s+2)}$$

The characteristic equation is

$$1 + \frac{k}{s(s+1)(s+2)} = 0$$

$$k = -(s^3 + 3s^2 + 2s)$$

$$\frac{dk}{ds} = -(3s^2 + 6s + 2) = 0$$

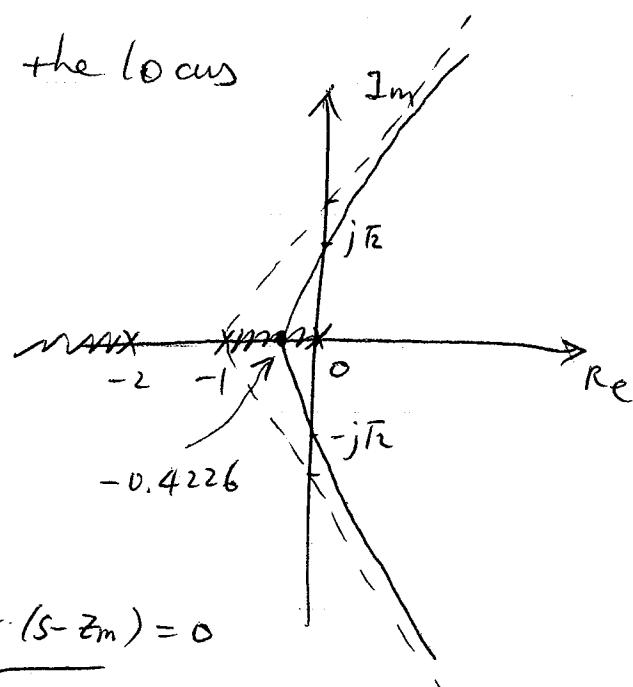
$$\Rightarrow s_1 = -0.4226$$

$$s_2 = -1.5774$$

Only s_1 can be on the locus

Reason: When branches touch each other, we have repeated closed-loop poles. Recall the characteristic equation is

$$f(s) = \underbrace{(s-p_1) \cdots (s-p_n)}_{A(s)} + k \underbrace{(s-z_1) \cdots (s-z_m)}_{B(s)} = 0$$



If $f(s)$ has repeated poles at r , i.e.
 $f(s) = (s-r)^2 \cdot (\dots) \cdot (\dots) \cdot (\dots)$

then

$$\frac{df}{ds} \Big|_{s=r} = 0 \quad \text{and} \quad f(r) = 0$$

or, equivalently

$$\frac{dA}{ds} \Big|_{s=r} = -k \frac{dB}{ds} \Big|_{s=r} \quad \text{and} \quad A(r) + kB(r) = 0$$

$$\Rightarrow \frac{dk}{ds} = -\frac{d}{ds} \frac{A(s)}{B(s)} = \frac{\frac{dB}{ds} A(s) - \frac{dA}{ds} B(s)}{[B(s)]^2}$$

$$= 0 \quad \text{when } s=r.$$

However, not all points that satisfy $\frac{dk}{ds} = 0$ belong to the root-locus. We need to solve $\frac{dk}{ds} = 0$, then check whether each of the solution is on the root locus or not.

Step 7: If $G(s)$ has complex poles/zeros or repeated ~~non-repeating~~ real poles/zeros, determine the angle of departure of locus from a pole p_i by

$$\theta_{p_i} = 180^\circ + \left(\sum_{j=1, j \neq i}^m \angle(p_i - z_j) - \sum_{j=1}^n \angle(p_i - p_j) \right)$$

Determine the angle of arrival of locus at a zero z_i by

$$\theta_{z_i} = 180^\circ - \left(\sum_{j=1, j \neq i}^m \angle(z_i - z_j) - \sum_{j=1}^n \angle(z_i - p_j) \right)$$

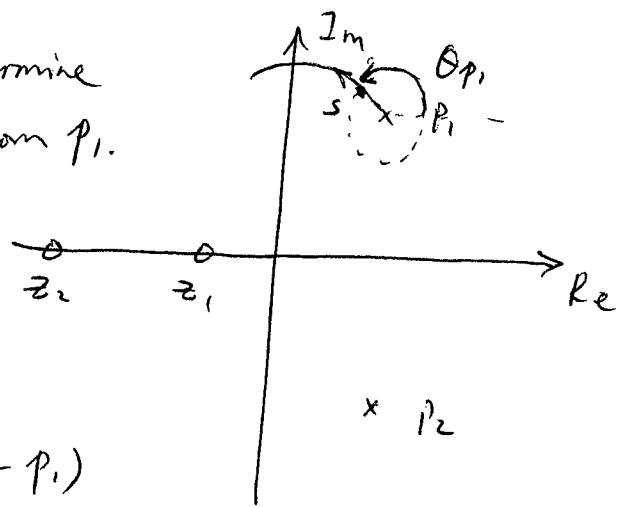
Note: If poles/zeros are non-repeating and real, the angle of departure/arrival is either 0° or 180° as determined by the real segments of the locus.

Both equations can be easily derived from the angle condition:

Suppose we want to determine the angle of departure from p_1 .

Look at a very small circle around p_1 .

Let s be the point the root-locus intersect the circle. Then $\theta_{p_1} = \angle(s - p_1)$



(15)

By angle condition

$$\angle(s-z_1) + \angle(s-z_2) - \angle(s-p_1) - \angle(s-p_2) = 180^\circ$$

Since s is close to p_1 , we have

$$\angle(p_1-z_1) + \angle(p_1-z_2) - \theta_{p_1} - \angle(p_1-p_2) = 180^\circ$$

$$\Rightarrow \theta_{p_1} = -180^\circ + (\angle(p_1-z_1) + \angle(p_1-z_2)) - \angle(p_1-p_2)$$

In this example

$$\angle(p_1-z_1) = 60^\circ$$

$$\angle(p_1-z_2) = 30^\circ$$

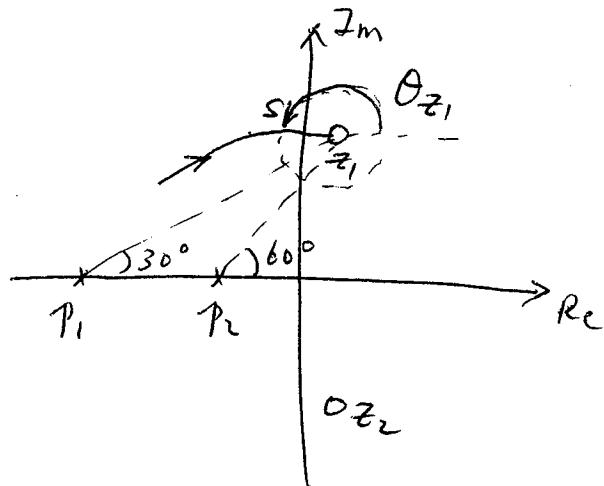
$$\angle(p_1-p_2) = 90^\circ$$

$$\therefore \theta_{p_1} = 180^\circ$$

Similar derivation for the angle of arrival.

Ex) Find the angle of arrival for z_1 ,

Let s denote the intersection of the root-locus with a small circle around z_1 .

Then $\angle(s-z_1) = \theta_{z_1}$ 

By angle condition

$$\angle(s-z_1) + \angle(s-z_2) - \angle(s-p_1) - \angle(s-p_2) = 180^\circ$$

As $s \rightarrow z_1$, \downarrow \downarrow \downarrow

$$\theta_{z_1} + 90^\circ - 30^\circ - 60^\circ = 180^\circ$$

$$\Rightarrow \theta_{z_1} = 180^\circ$$

Summary of Steps for Sketching Root-locus

Step 1: Write the characteristic equation as

$$1 + KG(s) = 1 + K \frac{\prod_{j=1}^m (s - z_j)}{\prod_{i=1}^n (s - p_i)} = 0$$

Step 2: Locate the poles & zeros on the complex plane.

Step 3: Find real segments of the root-locus:
to the left of an odd number of real poles and zeros

Step 4: Find asymptotes to infinity

$$\sigma_a = \frac{\sum_{i=1}^n p_i - \sum_{j=1}^m z_j}{n-m}$$

$$\phi_a = \frac{180^\circ}{n-m} + l \cdot \frac{360^\circ}{n-m}, \quad l = 0, 1, \dots, n-m-1$$

Step 5: Find imaginary closed-loop poles: when an entire row of Routh array is zero.

Solve roots of auxiliary polynomial.

Step 6: Find break-in/break-away points by solving $\frac{dk}{ds} = 0$.

Step 7: Find angle of departure from poles or angle of arrival to zeros using the angle condition.

(17)

A Full Example : P347 of text

$$1 + K G(s) = 0$$

where

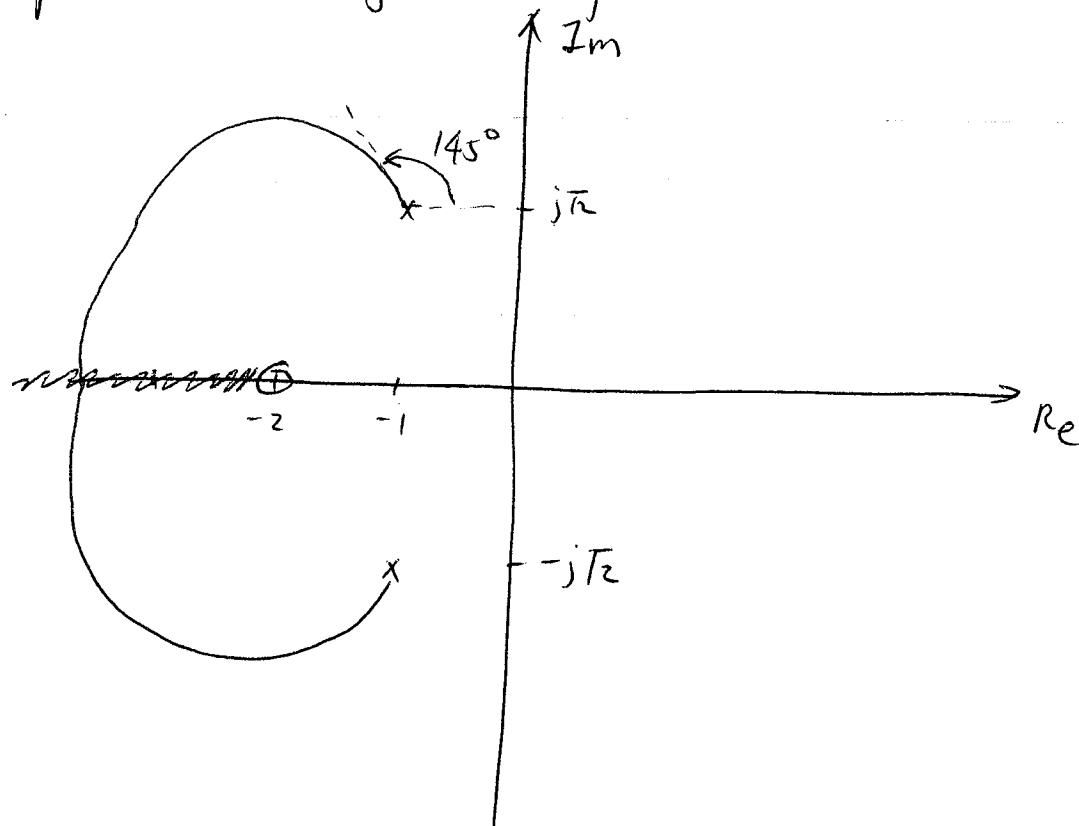
$$G(s) = \frac{s+2}{s^2 + 2s + 3}$$

Step 1: poles of $G(s)$ are $s = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm j\sqrt{2}$
 The standard form is

$$1 + K \cdot \frac{s+2}{[s - (-1+j\sqrt{2})][s - (-1-j\sqrt{2})]} = 0$$

Step 2: locate poles & zeros

Step 3: real segments of the root-locus



(18)

Step 4: find asymptotes to infinite,

At large s

$$\angle G(s) \approx \angle -\frac{1}{s} = -\angle s = 180^\circ + 1.38^\circ \\ \Rightarrow \angle s = +180^\circ$$

No intersections.

Step 5: find imaginary closed-loop poles

$$s^2 + (k+2)s + (2k+3) = 0$$

$$\begin{array}{ccc} s^2 & 1 & 2k+3 \\ s^1 & k+2 \\ s^0 & 2k+3 \end{array}$$

No rows are entirely zero for any $k > 0$.

\Rightarrow No purely imaginary closed-loop poles.

Step 6: find break-in points

$$K = -\frac{s^2 + 2s + 3}{s+2} = -s - \frac{3}{s+2}$$

$$\frac{dk}{ds} = -1 + \frac{3}{(s+2)^2} = 0$$

$$(s+2)^2 = 3$$

$$s = -2 \pm \sqrt{3} = \begin{cases} -0.2680 \\ -3.7320 \end{cases}$$

Only $-2 - \sqrt{3}$ is on the locus.

Step 7: Find the angle of departure at P_1

$$\angle(s-z_1) - \angle(s-P_1) - \angle(s-P_2) = 180^\circ$$

When s is close to P_1

$$\angle(P_1-z_1) - \theta_{P_1} - \angle(P_1-P_2) = 180^\circ$$

$$\begin{aligned} & \parallel \\ & \tan^{-1}(T_2) \\ & = 55^\circ \end{aligned}$$

$$\begin{aligned} & \parallel \\ & 90^\circ \end{aligned}$$

$$\Rightarrow \theta_{P_1} = 180^\circ + 55^\circ - 90^\circ = 145^\circ$$

Angle of departure from P_2 can be obtained by symmetry.

Questions:

- ① Is there a value of K that leads to $\vartheta = 0.707$
Yes. Can determine K from magnitude

Condition:

$$K = \frac{\prod_{i=1}^n |s-P_i|}{\prod_{j=1}^m |s-z_j|}$$

- ② Is there a value of K that leads to $\vartheta = 0.5$
No.

See typical root-locus shapes in text p357

Also, more examples in A-6-1 to A-6-10.

(20)

Note: Sometimes small change in pole-zero configuration may cause significant changes in root-locus

