Recall that, from the root-locus, we can determine:

1. What are the locations of closed-loop poles?
2. What is the range of $K$ for stability?
3. What is the value of $K$ that corresponds to a given spec?

\[ \beta = \tan^{-1} \frac{1/f^2}{g} \]

Given $f$

Given $\omega_n$

4. Or, if the design specs can be not be achieved by adjusting $K$ alone, how to design a compensator to move the closed-loop poles to the desired location?

Example: Specifying both $f$ & $\omega_n$

\[ \Rightarrow \text{desired pole location } C \text{ is not on the locus} \]

\[ \Rightarrow \text{Need compensation.} \]
Compensation: Redesign the controller (typically by adding open-loop poles & zeros) such that the modified root-locus passes through the desired closed-loop pole location.

\[\Rightarrow\text{ such a controller is called a compensator.}\]

\[\Rightarrow\text{ Basic compensators are}\]
- Lead-compensator
- Lag-compensator
- Lead-Lag-lead compensator.

Basics:

If we add an open-loop pole/zero, what is the effect on the root-locus? P423

1. Addition of poles:

Note:
- \# of Asymptotes = \# of poles - \# of zeros
- pull the root-locus to the right
- tend to reduce stability, and slow down the settling of the response.
Recall that the addition of integral control adds a pole at the origin, thus makes the system less stable.

2. Addition of zeros:

- pull the root-locus to the left
- tend to improve stability, and speed up the settling of the response.

Recall that the addition of derivative control adds a zero at the origin, and makes the system more stable.

In summary: zeros are the "good" guys, poles are the "bad" guys.
The idea of Compensation:

If a point s is not on the existing root-locus, the angle of \( G(s) \) is not 180°:

\[
\angle G(s) = \sum_{j=1}^{m} \angle (s-z_j) - \sum_{i=1}^{n} \angle (s-p_i) \neq 180^\circ
\]

The difference \( 180^\circ - \angle G(s) \) is the amount of compensation required to adjust the root-locus so that it passes through s.

\[
\angle G(s) = -135^\circ - 90^\circ = -225^\circ = 135^\circ
\]

Compensation required:
add 45°.

Possible solution: add a zero at -2. New \( \angle G(s) \) is
\[45^\circ - 135^\circ - 90^\circ = 180^\circ\]

In summary, for points above the real axis, an additional zero adds angle to \( \angle G(s) \), and an additional pole subtracts angle from \( \angle G(s) \).

The example is typical: often we need to add an angle to move the root-locus to the left (i.e., make the system more stable and have better transient response). We then need to add a zero.
However, in practice it is difficult to only add a zero (involving differentiation). Instead, it is easier to add a pair of pole & zero.

\[
\frac{E_o(s)}{E_i(s)} = \frac{S + \frac{1}{R_1C_1}}{S + \frac{1}{R_2C_2}} \cdot \frac{R_4C_1}{R_3C_2}
\]

\[
= K \cdot \frac{S + \frac{1}{T}}{S + \frac{1}{\alpha T}}
\]

where \( K = \frac{R_4C_1}{R_3C_2} \), \( T = R_1C_1 \), \( \alpha T = R_2C_2 \)

① If \( \alpha < 1 \), lead- compensator

For \( s \) above the real axis,
a lead compensator adds
an angle to \( \angle G(s) \)

"zero leads the pole" unlike adding a zero."
Use of a lead-compensator
- Provide the amount of compensation required to move the root-locus to the left so that it passes through the desired location.
  \( \alpha \) - determined by the angle of compensation
  \( \beta \) - determined by magnitude condition.

Potential drawback
- When \( \alpha \) is small, the DC-gain decreases
  \( \Rightarrow \) larger steady-state error

(2) If \( \alpha > 1 \) - lag-compensator.

For \( s \) above the real-axis,
a (lag-compensator) subtracts an angle from \( \angle G(s) \).

We often pick \( \beta \) very small,
so that this angle is close to zero.
Thus the root-locus is not changed much.

Use of a lag-compensator
- Pull up the DC-gain by \( \alpha \)
  \( \Rightarrow \) smaller steady-state error

Potential drawback
- Add a slowly-decaying transient \( e^{-\alpha t} \).
Design with Lead & Lag compensators P421-450

Basic idea:
- Use lead compensator to move the closed-loop pole to the desired location
  ⇒ improve transient response
- Use lag compensator with small zeros & poles to improve the DC-gain
  ⇒ improve steady-state performance.

Lead-compensator P421-428

\[ G_c(s) = K_c \cdot \frac{5 + \frac{1}{\alpha}}{s + \frac{1}{\alpha}} , \quad 0 < \alpha < 1 \]

Used to move the root-locus to the desired location.

Steps:
1. Determine the desired location for the dominant closed-loop poles from the performance specifications.
(2) By sketching the root-locus of the uncompensated system, check whether adjusting the gain $k$ alone can yield the desired closed-loop poles.

If not, calculate the angle deficiency

$$\phi = 180^\circ - \angle G(j\omega)$$

(3) Determine the value of $2\pi T$ of the lead-compensator that compensates the angle deficiency $\phi$

(4) Determine the value of $K_c$ through the magnitude condition

(5) Check steady-state error.
May add a lag-compensator if the steady-state error is too large.

Ex) P424 in text
Original System is

$$\rightarrow (\times) \rightarrow K \rightarrow \frac{1}{s(s+u)} \rightarrow$$

Design the controller such that

$\omega_n = 4 \quad \xi = 0.5$.
Step 1: The desired location of the closed-loop pole is
\[ s = (-9 \pm \sqrt{1-9^2}) \omega_n = -2 \pm j2\sqrt{3} \]

Step 2: Sketch the root locus of the uncompensated system or check \( G(s) \)

From the root locus, we can determine that adjusting \( K \) alone cannot meet the design spec.

Find angle deficiency from lead-compensator
\[ \angle G(s) = -120^\circ - 90^\circ = -210^\circ \]
\[ = 150^\circ \]
\[ \phi = 180^\circ - \angle G(s) = 30^\circ \]

Step 3: Determine the value of \( \alpha, T \) in the lead-compensator \( K \frac{s + \frac{\alpha}{T}}{s + \frac{\alpha}{T}} \) that compensates the angle deficiency \( \phi = 30^\circ \).

There are many choices. The textbook has a procedure for choosing \((\alpha, T)\) with the largest \( \alpha \) (p423).

We use a simpler procedure. Assume that you have been given the additional zero at \(-4\). Where should be the pole?
Since the additional zero at -1 produces an angle 60°, we need an additional pole to produce -30°.

\[ \Rightarrow \text{the pole should be at} \]
\[ -2 - \frac{2\sqrt{3}}{\tan 30°} = -8 \]

The lead-compensator is then

\[ K = \frac{s+4}{s+8} \]

where \( T = \frac{4}{4}, \; \alpha = \frac{1}{2} \)

Step 4: Determine the value of \( K \)

\[ K = \frac{\text{distances to the poles}}{\text{distances to the zeros}} \]
\[ = \frac{2\sqrt{3} \times 4 \times 4\sqrt{3}}{4} = 24 \]

The lead-compensator is then

\[ 24 \; \frac{s+4}{s+8} \]

Step 5: Check steady-state error constants

\[ K_v = \lim_{s \to 0} s \; G_c(s) \; G(s) \]
\[ = \lim_{s \to 0} \frac{24 \; s + 4}{s + 8} \cdot \frac{1}{s + 2} \]
\[ = \frac{24 \times 4}{2} - \frac{1}{2} = 6 \]
We can use MATLAB to compare the root-locus & transient response of the new & old system.

Original system:
Need $K = 4$ for $s = 0.5$. Root-locus $1 + \frac{K}{s(s+2)} = 0$

Closed-loop transfer function is

$$\frac{4}{s(s+2)} \left(1 + \frac{4}{s(s+2)}\right) = \frac{4}{s^2 + 2s + 4}$$

New system:
Root-locus

$$1 + K - \frac{s + 4}{s(s+8)(s+2)} = 0$$

Closed-loop transfer function ($K = 24$)

$$\frac{24(s+4)}{s(s+8)(s+2)} \left(1 + \frac{24(s+4)}{s(s+8)(s+2)}\right) = \frac{24(s+4)}{s^3 + 10s^2 + 40s + 76}$$
Worksheet for designing a lead compensator

Desired closed-loop pole: $-2 \pm j2\sqrt{3}$

Open loop zeros: none

Open loop poles:
- $0$
- $-2$
- $-8$

New poles:
- $-4$

Open loop angles:
- Sum: $0^\circ$ $+60^\circ$
- $120^\circ$
- $90^\circ$
- $+30^\circ$

New angles:
- Sum: $210^\circ$ $240^\circ$

Total angle:
$\angle G(s) = -210^\circ + 180^\circ = 390^\circ - 30^\circ = 30^\circ$

$\gamma = 180^\circ - 2\angle G(s) = 390^\circ - 30^\circ = 30^\circ$

$K = \frac{4 \times 2\sqrt{3} \times 4\sqrt{3}}{1 \times 4} = 24$. 