

Linearization of Nonlinear Systems p112-114

A linear function satisfies the following superposition property:

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

for all x_1, x_2 .

Examples of linear functions: 1, c, ax , $a+bx$, $a+bx+cy$, etc.

Linear systems are those where the principle of superposition applies

$$a_0 y + a_1 \dot{y} + \dots + a_n y^{(n)} = b_0 u + b_1 \dot{u} + \dots + b_m u^{(m)}$$

If $(y_1(t), u_1(t))$ and $(y_2(t), u_2(t))$ both satisfy the differential equation, then so does $(y_1(t) + y_2(t), u_1(t) + u_2(t))$.

We can also rewrite such a linear system as

$$f(y, \dot{y}, \dots, y^{(n)}, u, \dot{u}, \dots, u^{(m)}) = 0$$

where

$$f(\dots) = a_0 y + a_1 \dot{y} + \dots + a_n y^{(n)} - (b_0 u + b_1 \dot{u} + \dots + b_m u^{(m)})$$

Then $f(\cdot)$ is a linear function of its variables.

IMPORTANT: We can only take Laplace transform only for linear systems.

Non-linear systems are those where the principle of superposition does not apply.

Examples of non-linear systems

① Inverted pendulum:

$$U = (M+m)\ddot{x} - m \underbrace{\sin\theta}_{\text{gravitational force}} \cdot (\dot{\theta})^2 + m \underbrace{\cos\theta}_{\text{centrifugal force}} \cdot \ddot{\theta}$$

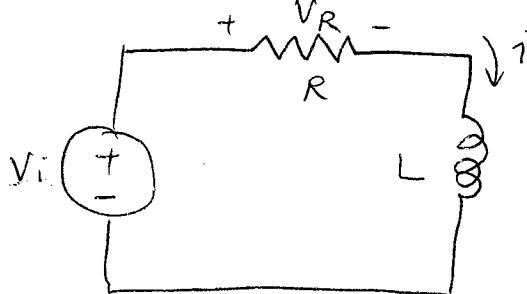
$$\underbrace{mgl \sin\theta}_{\text{gravitational force}} - m \underbrace{\cos\theta}_{\text{centrifugal force}} \cdot \ddot{x} = (I + m l^2) \ddot{\theta}$$

If we rewrite these equations as

$$f(\theta, \dot{\theta}, \ddot{\theta}, x, \dot{x}, \ddot{x}, u) = 0$$

The function f is NOT a linear function of its variables.

② Non-linear circuit



Non-linear resistor

$$V_R = i^3 R$$

$$V_i = i^3 R + L \frac{di}{dt}$$

WE CANNOT TAKE LAPLACE TRANSFORM FOR NON-LINEAR SYSTEMS.

We could, however, derive a linear system that is an approximation of the original system, then take L.T.

Linearization

Basic idea: If the system operates only around a small neighborhood of an equilibrium point, by expanding the nonlinear functions into a Taylor series about the equilibrium point, we can retain the linear ~~fractional~~ terms, and neglect the higher-order terms. In this way we obtain linear approximation of the original system.

Such a linear model would "work" if the deviation of the signals from the equilibrium point is small.

We can then study the linear system by taking Laplace transform and so on.

Linearization of a function

Suppose $y = f(x)$, f is nonlinear. Take Taylor series expansion, assuming all derivatives exist.

$$\begin{aligned} y &= f(x) \\ &= f(\bar{x}) + \frac{df}{dx} \Big|_{x=\bar{x}} (x-\bar{x}) + \frac{1}{2!} \frac{d^2f}{dx^2} \Big|_{x=\bar{x}} (x-\bar{x})^2 + \dots \end{aligned}$$

↑ derivatives evaluated at \bar{x}
(constant).

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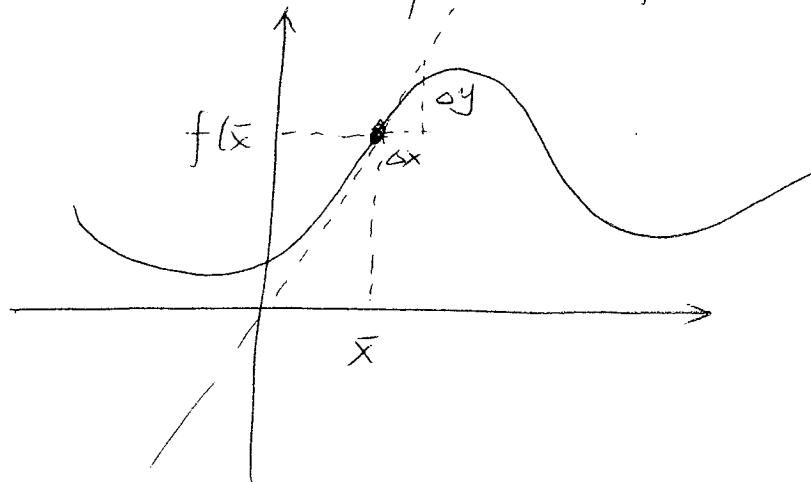
When $|x - \bar{x}|$ is small,

$$y \approx f(\bar{x}) + \frac{df}{dx} \Big|_{x=\bar{x}} (x - \bar{x}) \quad \text{Linear!}$$

Letting $\Delta y = y - f(\bar{x})$, $\Delta x = x - \bar{x}$, we have

$$\Delta y = \frac{df}{dx} \Big|_{x=\bar{x}} \cdot \Delta x$$

Deviations from $f(\bar{x})$ of the output y are linear in deviations from \bar{x} of the input x .



⑥ What if f is already linear?

⑦ Its 2nd & higher-order derivatives are zero
 \Rightarrow The linear approximation is simply itself.

Linearization of

a function with multiple variables

$$y = f(x_1, x_2, u_1, u_2)$$

$$y \approx f(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2) + \frac{\partial f}{\partial x_1} \Big|_{x_1=\bar{x}_1} (x_1 - \bar{x}_1) + \frac{\partial f}{\partial x_2} \Big|_{x_2=\bar{x}_2} (x_2 - \bar{x}_2) + \frac{\partial f}{\partial u_1} \Big|_{u_1=\bar{u}_1} (u_1 - \bar{u}_1) + \frac{\partial f}{\partial u_2} \Big|_{u_2=\bar{u}_2} (u_2 - \bar{u}_2)$$

$x_1 = \bar{x}_1$
 $u_1 = \bar{u}_1$
 $u_2 = \bar{u}_2$

The coefficient in front of each variable is simply the partial derivative evaluated at the equilibrium point.

Procedures for linearization

Given the differential equation

- ① Find equilibrium point by setting all $\dot{x}, \ddot{x}, \ddot{u}, \ddot{u}, \dots$ to zero
 \Rightarrow solve for the equilibrium point.
- ② For each term that is non-linear, view it as a non-linear function of the variables $x, \dot{x}, \ddot{x}, u, \dot{u}, \ddot{u} \dots$.
Evaluate the partial derivative at the equilibrium point:
 $\bar{x}, 0, 0, \bar{u}, 0, 0 \dots$
- ③ Write down the linear approximation
- ④ Simplify the linear equations. Replace each variable by its deviation from the equilibrium.

Linearization of a differential equation of one variable

Suppose $\frac{dx}{dt} = f(x(t))$ f : nonlinear

- ① Find the equilibrium point of the system by Setting $\dot{x} = 0 \Rightarrow f(\bar{x}) = 0$.

The set of solutions that satisfy the above equation is called the set of equilibrium points of the system. (Idea: If $x(0) = \bar{x}$, then $x(t) = \bar{x}$ for all t . \Rightarrow "The system is at equilibrium." Hopefully, if $x(0)$ is close to \bar{x} , $x(t)$ will stay close to \bar{x} for at least some interval $(0, T)$.)

- ② Evaluate partial derivatives for each non-linear term.

$$\frac{dx}{dt} \approx f(\bar{x}) + \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}} (x - \bar{x})$$

- ③ Simplify the linear equation

Note that $f(\bar{x}) = 0$ by the definition of equilibrium points \bar{x} .

Let $\delta x = x - \bar{x}$. Then $\frac{d}{dt} \delta x = \frac{d}{dt} x$

$$\Rightarrow \dot{\delta x} = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}} \delta x$$

Linearization of a differential equation with
inputs & higher order derivatives

$$\frac{d^2}{dt^2}x(t) = f(x(t), \dot{x}(t), u(t))$$

- ① Find the equilibrium points. For a reference (constant) input u_{ref} , the equilibrium points are solutions of

$$f(x_{eq}, 0, u_{eq}) = 0$$

- ② Evaluate partial derivatives for non-linear terms

$$\frac{d^2}{dt^2}x(t) = f(x_{eq}, 0, u_{eq}) + \left. \frac{\partial f}{\partial x} \right|_{\begin{array}{l} x=x_{eq} \\ \dot{x}=0 \\ u=u_{eq} \end{array}} (x(t) - x_{eq})$$

$$+ \left. \frac{\partial f}{\partial x} \right|_{\begin{array}{l} x=x_{eq} \\ \dot{x}=0 \\ u=u_{eq} \end{array}} (\dot{x}(t) - 0) + \left. \frac{\partial f}{\partial u} \right|_{\begin{array}{l} x=x_{eq} \\ \dot{x}=0 \\ u=u_{eq} \end{array}} (u(t) - u_{eq})$$

- ③ Simplify. Note that $f(x_{eq}, 0, u_{eq}) = 0$

Let $\delta x = x - x_{eq}$, $\delta u = u - u_{eq}$

$$\ddot{\delta x} = \left. \frac{\partial f}{\partial x} \right|_{\begin{array}{l} x=x_{eq} \\ \dot{x}=0 \\ u=u_{eq} \end{array}} \delta x + \left. \frac{\partial f}{\partial x} \right|_{\begin{array}{l} x=x_{eq} \\ \dot{x}=0 \\ u=u_{eq} \end{array}} \dot{\delta x} + \left. \frac{\partial f}{\partial u} \right|_{\begin{array}{l} x=x_{eq} \\ \dot{x}=0 \\ u=u_{eq} \end{array}} \delta u$$

Example: $V = i^3 R + L \frac{di}{dt}$

$$\frac{di}{dt} = +\frac{V}{L} - \frac{R}{L} i^3$$

\uparrow
non linear

- ① Find equilibrium points. Suppose $V_{ref} = 1$
Set $\frac{di}{dt} = 0$, we have

$$\frac{V_{ref}}{L} - \frac{R}{L} i^3 = 0$$

$$i_{eq} = \sqrt[3]{\frac{V_{ref}}{R}} = \sqrt[3]{\frac{1}{R}}$$

- ② Evaluate partial derivatives. Let $f(i) = -\frac{R}{L} i^3$

$$f(i) \approx f(i_{eq}) + \left. \frac{\partial f}{\partial i} \right|_{i=i_{eq}} (i - i_{eq})$$

$$\left. \frac{\partial f}{\partial i} \right|_{i=i_{eq}} = -\frac{3R}{L} i^2 \Big|_{i=i_{eq}} = -\frac{3}{L} \sqrt[3]{R}$$

Hence, the linear approximation is

$$\frac{di}{dt} = \frac{V}{L} + f(i_{eq}) - \frac{3}{L} \sqrt[3]{R} (i - i_{eq})$$

- ③ Simplify. Let $\delta V = V - V_{ref}$, $\delta i = i - i_{eq}$

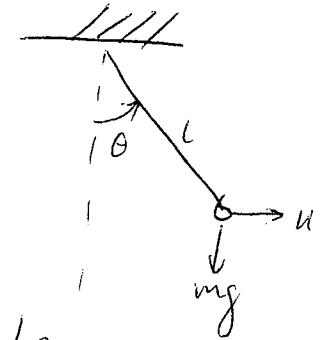
$$\delta \overset{\circ}{i} = \frac{\delta V}{L} + \underbrace{\frac{V_{ref}}{L} + f(i_{eq})}_{=0} - \frac{3}{L} \sqrt[3]{R} \delta i$$

$$\Rightarrow \delta \overset{\circ}{i} = \frac{\delta V}{L} - \frac{3}{L} \sqrt[3]{R} \delta i$$

Take L.T. $\frac{\delta i(s)}{\delta V(s)} = \frac{\frac{1}{L}}{s + \frac{3}{L} \sqrt[3]{R}}$

Example: pendulum

$$I\ddot{\theta} = u \cos \theta - mgl \sin \theta - b\dot{\theta}$$



- ① Find equilibrium points.

Suppose $u_{ref} = 0$. Set $\dot{\theta}, \ddot{\theta} = 0$. We have

$$mgl \sin \theta = 0$$

$$\Rightarrow \bar{\theta} = 0, \quad \bar{\theta} = \pi \text{ discard}$$

- ② Evaluate partial derivatives

$$\text{let } f(\theta, u) = u \cos \theta - mgl \sin \theta$$

$$f(\theta, u) = f(\bar{\theta}, u_{ref}) + \left. \frac{\partial f}{\partial \theta} \right|_{\substack{u=u_{ref} \\ \theta=\bar{\theta}}} (\theta - \bar{\theta})$$

$$+ \left. \frac{\partial f}{\partial u} \right|_{\substack{u=u_{ref} \\ \theta=\bar{\theta}}} (u - u_{ref})$$

$$\left. \frac{\partial f}{\partial \theta} \right|_{\substack{u=u_{ref} \\ \theta=0}} = -u \sin \theta - mgl \cos \theta \Big|_{\substack{u=u_{ref} \\ \theta=0}} = -mgl$$

$$\left. \frac{\partial f}{\partial u} \right|_{\substack{u=u_{ref} \\ \theta=0}} = \cos \theta \Big|_{\substack{u=u_{ref} \\ \theta=0}} = 1$$

Hence, the linear approximation is

$$I\ddot{\theta} = f(\bar{\theta}, u_{ref}) - mgl(\theta - \bar{\theta}) + ((u - u_{ref}) - b\dot{\theta})$$

- ③ Simplify. Note that $f(\bar{\theta}, u_{ref}) = 0$

$$\text{let } \delta\theta = \theta - \bar{\theta}$$

$$\delta u = u - u_{ref}$$

$$I\ddot{\delta\theta} = -mgl \cdot \delta\theta + (\delta u - b\dot{\delta\theta})$$