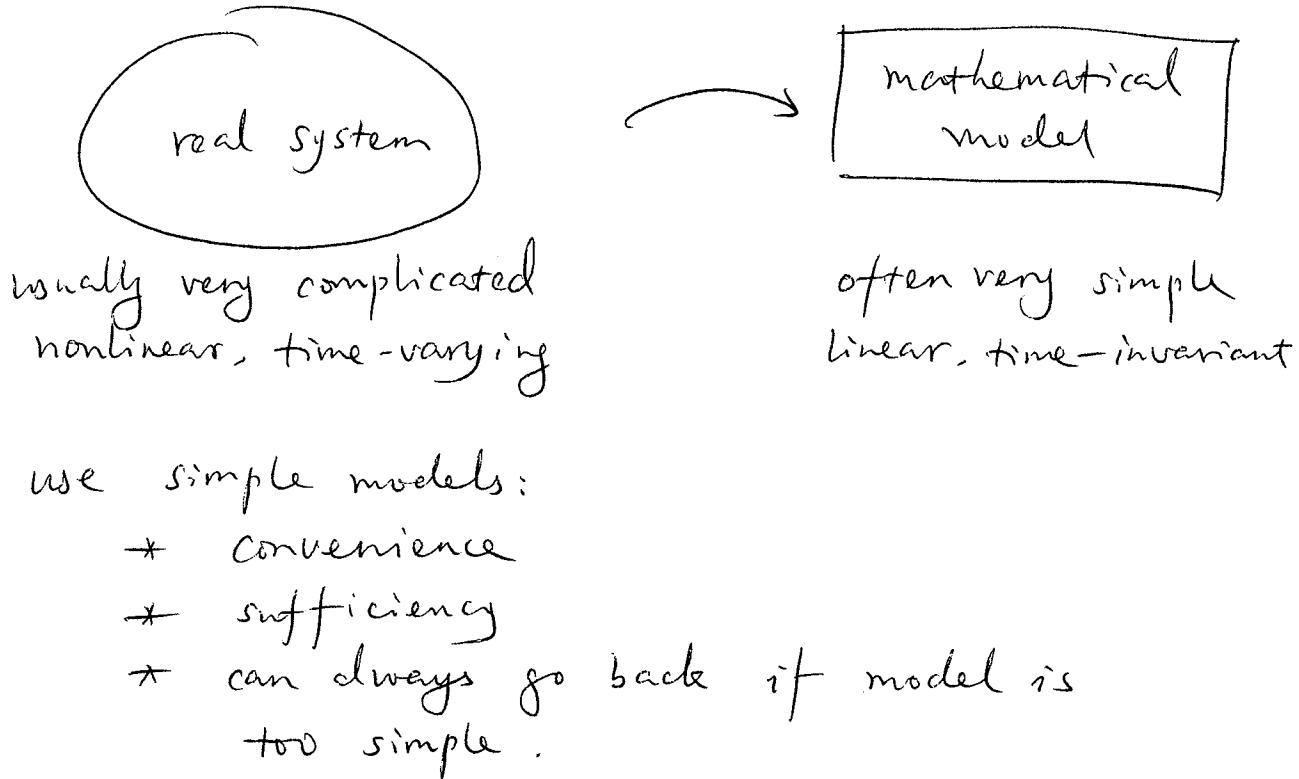


(6)

## Mathematical Modeling of Mechanical & Electrical Systems



### Mechanical Systems p80-81, p85-86

We need to know how to model mechanical systems because in <sup>many</sup> real systems the control rules are eventually applied through mechanical actuators such as motors.

### Translational Systems

Simplest models are based on Newton's Second Law.

$$\vec{F} = m\vec{a} \quad \text{for each free-body}$$

$\vec{F}$  : vector sum of all forces applied to each body in the system.  
in Newtons (N)

$\vec{a}$  : vector acceleration of each body  
in  $m/s^2$

$m$  : mass of the body, in kg

To use Newton's Second Law

① Define coordinates.

In particular, define the positive direction.

The displacement, velocity, acceleration must be consistent in their positive direction.

② Use a free-body diagram to find all forces



Diagram used to show the relative magnitude & directions of all forces acting upon an object.

- external forces (input, disturbance, etc)

- spring  $k \cdot \Delta x$

$\uparrow$  strong constant

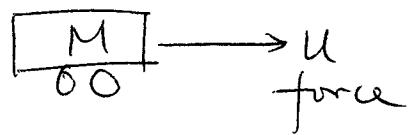
- friction (e.g. piston in a cylinder)

friction coefficient  $\begin{cases} b \cdot m \\ b \cdot \dot{x} \end{cases}$

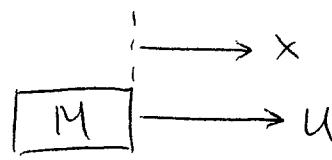
(63)

- ③ Use  $\vec{F} = m\vec{a}$  to get a differential equation that describe the dependencies between coordinates & forces
- ④ Get the transfer function of interest.

Ex) One-mass system



Neglect rotational inertia of wheels, friction, etc.,



freebody diagram

Apply Newton's Law

$$u = M \ddot{x} \quad (*)$$

Take Laplace transform

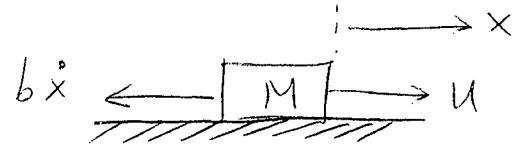
$$U(s) = M \cdot s^2 X(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{Ms^2}$$

Compared with (\*),  $\frac{1}{s^2}$  corresponds to a double integrator, i.e.,  $x(t)$  is much smoother than  $u(t)$ .

(64)

Ex) one-mass system with friction



we model friction as proportional to velocity.

$$u - b\dot{x} = M\ddot{x}$$

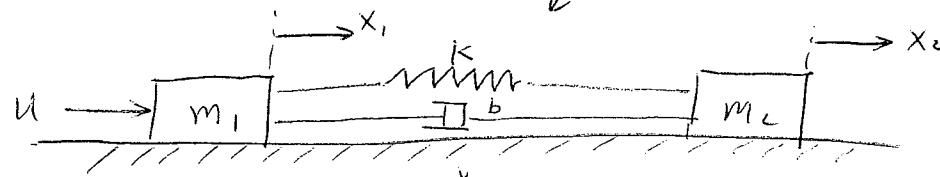
Take Laplace transform

$$U(s) - bsX(s) = Ms^2X(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{Ms^2 + bs}$$

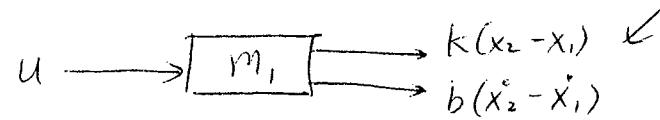
Ex) Two-mass system

spring (force proportional to extension/compression)



↑ damper or dashpot  
(decelerating force proportional to velocity)

Free-body diagram

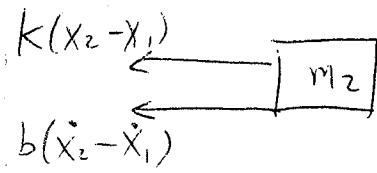
How to determine sign?

If  $x_2 > x_1$ , spring is extended  $\Rightarrow$  force pulling to the right

↖ If  $x_2 > x_1$ ,  $m_2$  is leaving away from  $m_1$   $\Rightarrow$  force pulling to the right.

$$u + k(x_2 - x_1) + b(x_2 - x_1) = m_1 \ddot{x}_1$$

(65)



$$-k(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) = m_2 \ddot{x}_2$$

↑  
don't need to find  
the sign again; simply  
use "reaction force  
opposite to the acting force"

Take Laplace transforms

$$X_1(s) [m_1 s^2 + bs + k] - X_2(s) [bs + k] = U(s)$$

$$-X_1(s) [bs + k] + X_2(s) [m_2 s^2 + bs + k] = 0$$

Solve for  $X_1, X_2$ . If we are only interested in  $X_2$

$$\frac{X_2(s)}{U(s)} = \frac{bs + k}{m_1 m_2 s^4 + (m_1 + m_2)bs^3 + (m_1 + m_2)ks^2}$$

Rotational Systems p 86-90

Newton's second Law now reads

$$M = I\alpha$$

where:

M: sum of all torques on body, in N·m

torque = force  $\times$  (distance from the axis of rotation)

I: moment of inertia of body around its axis of rotation. in  $\text{kg} \cdot \text{m}^2$

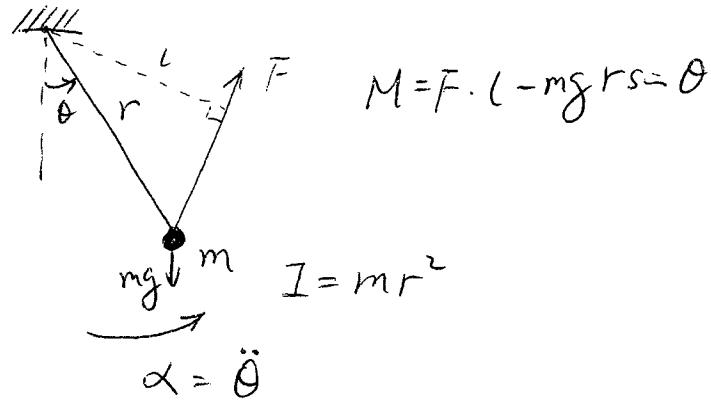
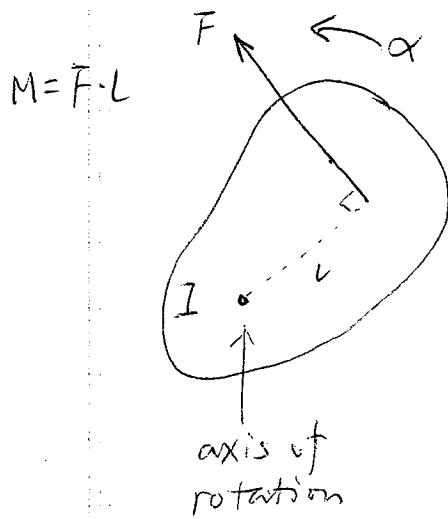
I of a point mass

$$= \text{mass} \times (\text{distance from the axis of rotation})^2$$

I of a more complex mass

$$= \int_0^M r^2 dm$$

$\alpha$ : angular acceleration ( $\text{rad/s}^2$ ) around the axis of rotation



Note: To use  $M = I\alpha$  correctly, need to choose the right axis of rotation

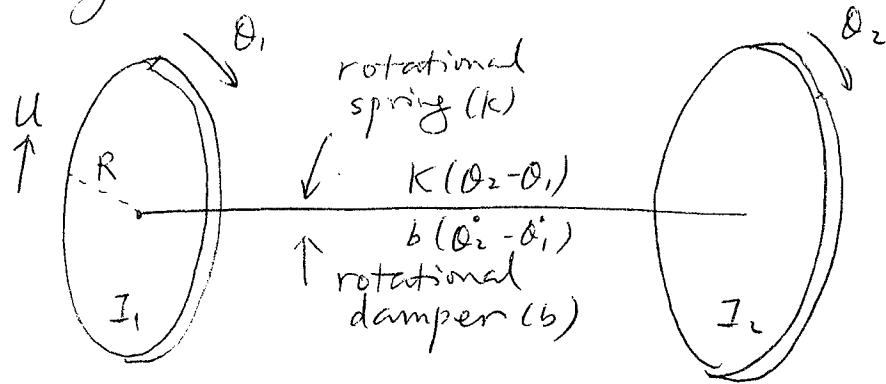
① If the axis is static/fixed, use it

② use the body's center of mass as the axis

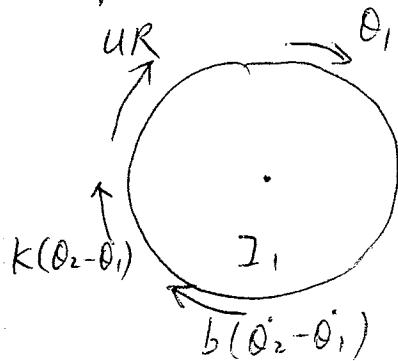
Caution: do not use an axis that is moving unless it is the center of mass.

(67)

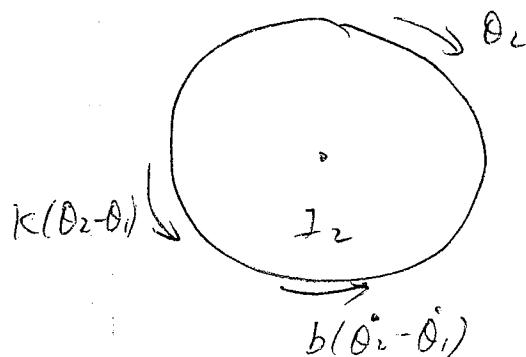
Ex) The rotational equivalent of the two-mass system we studied earlier



$$\text{applied torque} = U \cdot R$$



$$I_1 \ddot{\theta}_1 = UR + k(\theta_2 - \theta_1) + b(\dot{\theta}_2 - \dot{\theta}_1)$$

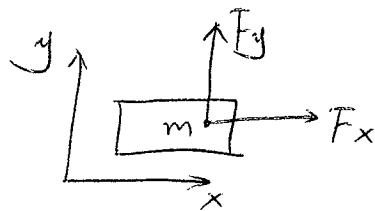


$$I_2 \ddot{\theta}_2 = -k(\theta_2 - \theta_1) - b(\dot{\theta}_2 - \dot{\theta}_1)$$

## Dimensions of Freedom



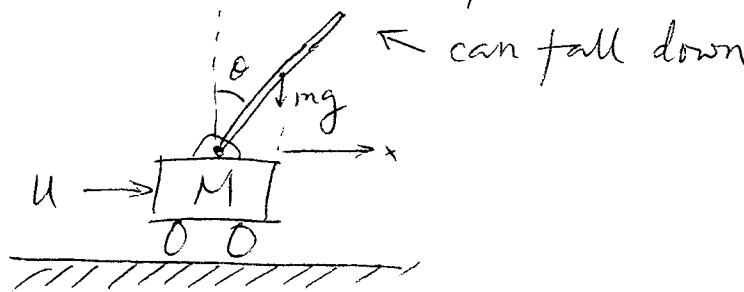
one dimension of freedom  
 $F = m \ddot{x}$



Two dimensions of freedom  
 $F_x = m \ddot{x}$   
 $F_y = m \ddot{y}$

## Translational + Rotational Systems

Example 3-8 in textbook p86

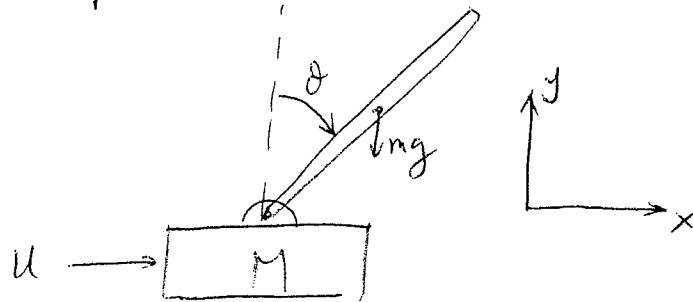


"Inverted pendulum": Segway HT, Space shuttle  
 Objective: Apply force  $u$  to keep the inverted pendulum in the upright position.

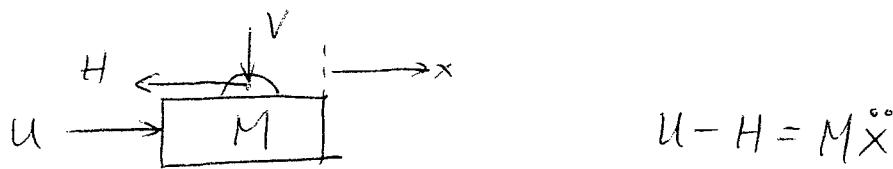
For this system, the object  $m$  has both translational motion & rotational motion.  
 We can consider it as the combination of translational motion at the center of mass and

rotational motion around the center of mass  
 $\Rightarrow$  two sets of equations for object m.

① Set up coordinates

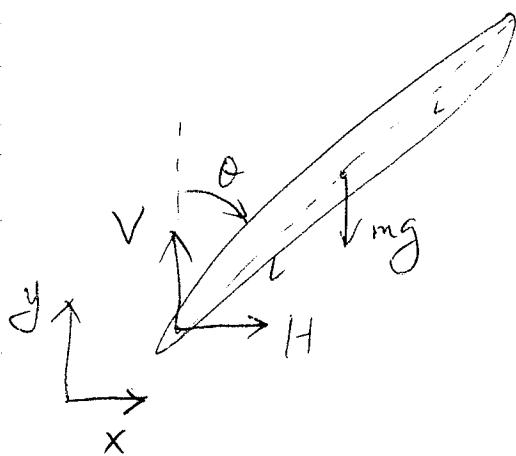


② Free-body diagram for M



$$u - H = M\ddot{x}$$

③ Free-body diagram for m



Apply Newton's Law on the horizontal direction.

④ What is the displacement of m in the horizontal direction?

(A) relative to M:  $l \cdot \sin \theta$   
 relative to earth:  $x + l \cdot \sin \theta$

$$H = m \frac{d^2}{dt^2} (x + l \cdot \sin \theta)$$

(76)

On the vertical direction

$$V - mg = m \frac{d^2}{dt^2} (l \cdot \cos \theta)$$

Around the center of mass

$$V \cdot l \sin \theta - H \cdot l \cdot \cos \theta = I \ddot{\theta}$$

4 equations

Quantities of interest:  $\theta, x$

Additional unknowns:  $V, H \leftarrow$  To eliminate

Use

$$\frac{d}{dt} (\sin \theta) = \cos \theta \cdot \dot{\theta}$$

$$\frac{d^2}{dt^2} (\sin \theta) = \frac{d}{dt} (\cos \theta \cdot \dot{\theta}) = -\sin \theta \cdot \dot{\theta} \cdot \ddot{\theta} + \cos \theta \cdot \ddot{\theta}$$

$$\frac{d}{dt} (\cos \theta) = -\sin \theta \cdot \dot{\theta}$$

$$\frac{d^2}{dt^2} (\cos \theta) = \frac{d}{dt} (-\sin \theta \cdot \dot{\theta}) = -\cos \theta \cdot \dot{\theta} \cdot \ddot{\theta} - \sin \theta \cdot \ddot{\theta}$$

Hence

$$\begin{cases} U - H = M \ddot{x} & \textcircled{a} \\ H = m \ddot{x} - ml \cdot \sin \theta \cdot (\dot{\theta})^2 + ml \cdot \cos \theta \cdot \ddot{\theta} & \textcircled{b} \\ V - mg = -ml \cdot \cos \theta \cdot (\dot{\theta})^2 - ml \cdot \sin \theta \cdot \ddot{\theta} & \textcircled{c} \\ Vl \sin \theta - Hl \cdot \cos \theta = I \ddot{\theta} & \textcircled{d} \end{cases}$$

$\textcircled{a} + \textcircled{b}$ , eliminate  $H$

$$U = (M+m) \ddot{x} - ml \cdot \sin \theta \cdot (\dot{\theta})^2 + ml \cdot \cos \theta \cdot \ddot{\theta} \quad \textcircled{e}$$

(71)

$\theta, \dot{\theta} \rightarrow \theta$ , eliminate  $H, V$

$$mgl(s\cos\theta - ml^2\ddot{\theta}) - m(l\cos\theta)\ddot{x} = I\ddot{\theta}$$

$$\Rightarrow mgl(s\cos\theta - m(l\cos\theta)\ddot{x}) = (I + ml^2)\ddot{\theta} \quad \text{④}$$

② & ④ describe the relationship between  $u$  and  $x, \theta$ .  
There are non-linear terms

$$s = \theta \cdot (\dot{\theta})^2, \cos\theta \cdot \ddot{\theta}, s\dot{\theta}, \cos\theta \cdot \ddot{x}$$

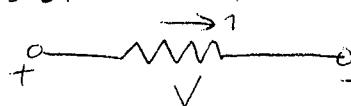
We will study how to linearize these equations  
later on.

# Electrical Systems

P 90-102 in text

## Models of some common elements

Resistor



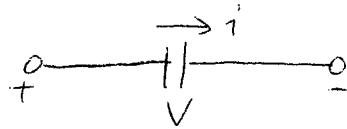
$$V = R i$$

L.T.

Complex impedance

$$R$$

Capacitor



$$i = C \frac{dV}{dt}$$

$$v(s) = R i(s)$$

$$\frac{1}{sC}$$

Inductor

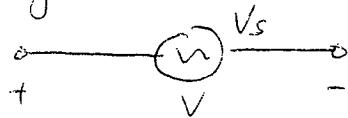


$$V = L \frac{di}{dt}$$

$$v(s) = L s i(s)$$

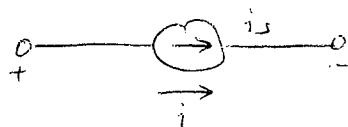
$$sL$$

Voltage source



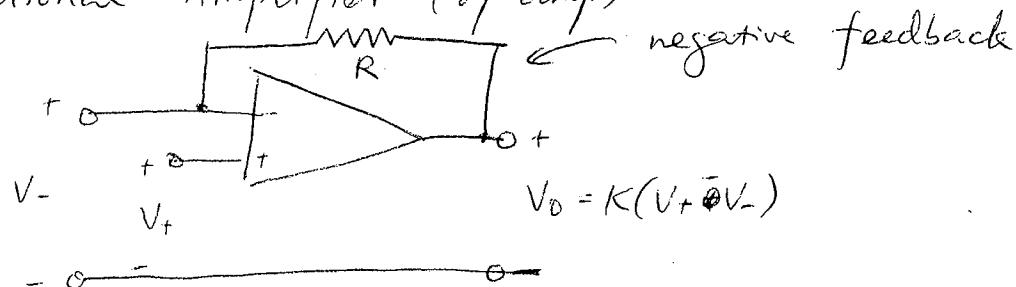
$$V = V_s$$

Current source



$$i = i_s$$

Operational Amplifier (op-amp)



$$V_0 = K(V_+ + V_-)$$

We usually assume that the op-amp is ideal:

- infinite input impedance  $\Rightarrow$  zero current drain by input
- zero output impedance  $\Rightarrow$   $V_0$  does not change due to load

(73)

- gain  $K = +\infty$

$\Rightarrow$  when there is negative feedback, then

$$V_+ = V_-$$

## Kirchhoff's Laws

governs the relationship between the various current and voltages

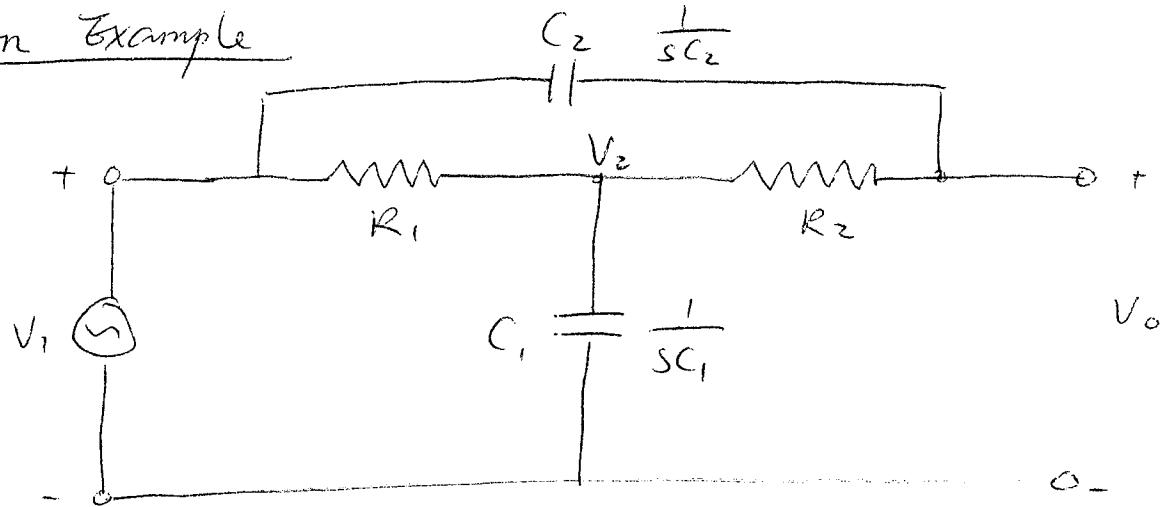
### Kirchhoff's current law (KCL)

The algebraic sum of all currents leaving a junction or node in the circuit is zero

### Kirchhoff's voltage law (KVL)

The algebraic sum of all voltages taken around a closed path in a circuit is zero

### An Example



Use KCL. Define voltages at every junction/node.  
Write down the current equation at each node.

$$\frac{V_2 - V_i}{R_1} + \frac{V_2 - V_o}{R_2} + C_1 \frac{dV_2}{dt} = 0$$

$$\frac{V_o - V_2}{R_2} + C_2 \frac{d}{dt}(V_o - V_i) = 0$$

Assume zero initial condition, we can take Laplace transform

$$\frac{V_2(s) - V_i(s)}{R_1} + \frac{V_2(s) - V_o(s)}{R_2} + SC_1 V_2(s) = 0$$

$$\frac{V_o(s) - V_2(s)}{R_2} + SC_2 (V_o(s) - V_i(s)) = 0$$

Note: We could have directly written down these equations using the complex impedance of each component (assuming zero initial ~~con~~ conditions)  
 $\Rightarrow$  impedance approach.

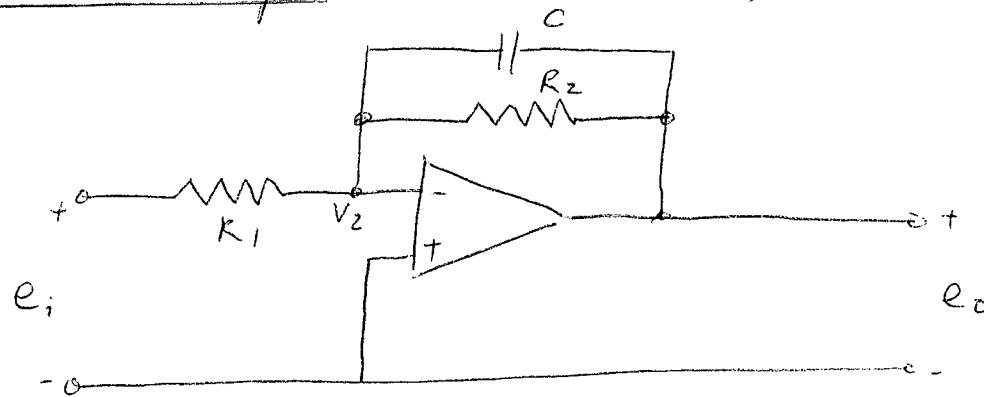
Eliminating  $V_2(s)$ , we can obtain the relationship between input  $V_i(s)$  & output  $V_o(s)$ .

Rearrange terms

$$\left\{ \begin{array}{l} -\frac{V_i(s)}{R_1} + \left( \frac{1}{R_1} + \frac{1}{R_2} + SC_1 \right) V_2(s) - \frac{1}{R_2} V_o(s) = 0 \\ -SC_2 V_i(s) - \frac{1}{R_2} V_2(s) + \left( \frac{1}{R_2} + SC_2 \right) V_o(s) = 0 \end{array} \right.$$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{R_1 R_2} + SC_2 \left( \frac{1}{R_1} + \frac{1}{R_2} + SC_1 \right)}{\left( \frac{1}{R_2} + SC_2 \right) \left( \frac{1}{R_1} + \frac{1}{R_2} + SC_1 \right) - \frac{1}{R_2^2}} \quad (\text{check this!})$$

Another example (Ex 3-11 P96)



Assume op-amp is ideal, and  $K$  (op-amp gain) is very large ( $\rightarrow +\infty$ )

Then  $v_2 = 0$ . Using KCL

$$-\frac{e_i}{R_1} - \frac{e_o}{R_2} - sC e_o = 0$$

$$\Rightarrow \frac{e_{o(s)}}{e_{i(s)}} = -\frac{1}{R_1 \left( sC + \frac{1}{R_2} \right)}$$

$$= -\frac{R_2}{R_1} \frac{1}{1 + SCR_2}$$

More examples in table 3-1 pp 103.

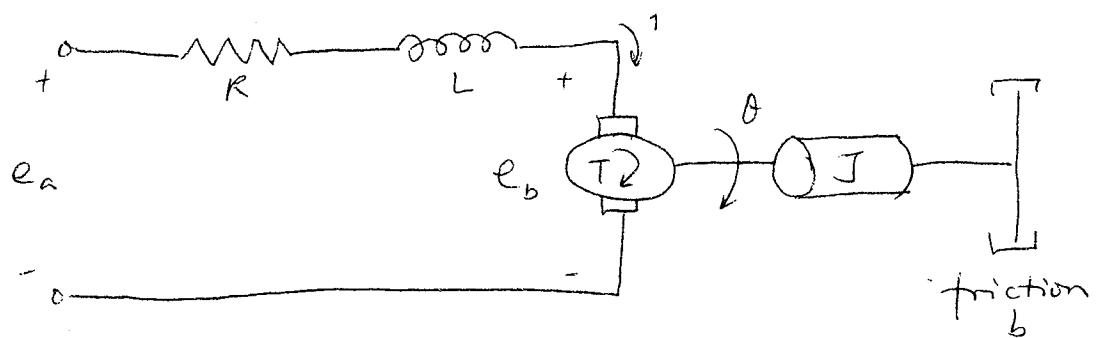
(76)

Putting everything together

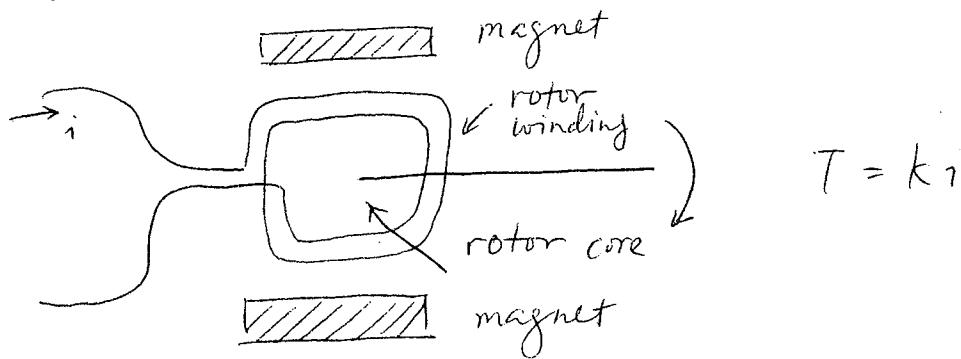
- mechanical
- electrical
- block diagram

An electro-mechanical system (similar to p224 in text)

Armature control of a DC servo-motor



DC motor:



- ① Input voltage  $e_a$  sends a certain current  $i$  through the armature (rotor winding)
- ② The current  $i$  causes a torque  $T$  on the rotor & load.  $T$  is proportional to  $i$

$$T = k_i$$

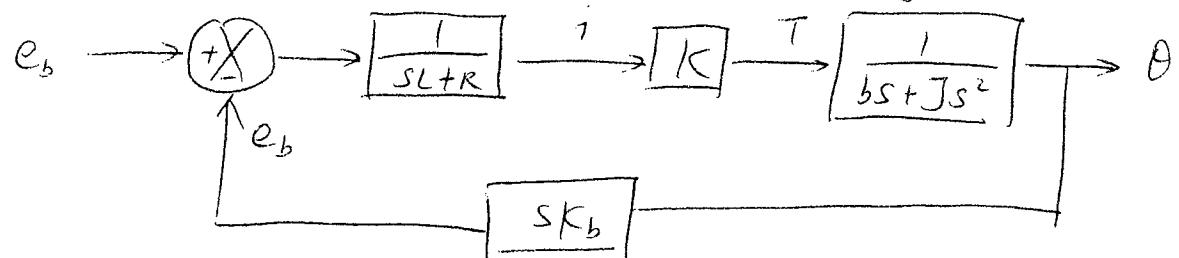
- ③ The rotor + load is modeled as a rotational system with damping
- ④ The rotation of the rotor also generates a "back" voltage  $e_b$  proportional to the angular velocity  $\dot{\theta}$ . This can be viewed as a feedback that impact the armature current in item ①

$$e_b = K_b \dot{\theta}$$

We thus obtain the set of equations as

$$\left\{ \begin{array}{l} \frac{e_a - e_b}{SL + R} = i \\ T = Ki \\ T - b\dot{\theta} = J\ddot{\theta} \Rightarrow \dot{\theta} = \frac{T}{bS + JS^2} \\ e_b = K_b \dot{\theta} \Rightarrow e_b = SK_b \dot{\theta} \end{array} \right.$$

We could model this as a block diagram



$$\frac{\theta(s)}{e_a(s)} = \frac{\frac{1}{SL+R} \cdot K \cdot \frac{1}{JS^2+bS}}{1 + SK_b \cdot \frac{1}{SL+R} K \cdot \frac{1}{JS^2+bS}}$$

$$= \frac{K}{s^2 LJ + s(bL + RJ) + (Rb + K_b K)}$$