Signal Flow Graphs

SFG provide an alternate (and equivalent) way to represent block diagrams. The advantage is that there is a formula called "Mason's rule" that enables us to compute transfer functions between any input & output.

Example:

\[
\begin{align*}
U & \rightarrow Y \\
& \rightarrow G(s) \\
& \rightarrow H(s)
\end{align*}
\]

Basics: represent a block in a block diagram by a branch in SFG.

\[
\begin{align*}
G(s) & \rightarrow \text{block} \\
G(s) & \rightarrow \text{branch}
\end{align*}
\]
In SFGs:

1. **Node**: represents a signal
2. **Transmittance**: "gain" between two nodes (i.e., the transfer function.
3. **Branch**: a directed line segment connecting two nodes.
   Gain between nodes is written below or above the arrow.
4. **Input node**: node with only outgoing branches
5. **Output node**: node with only incoming branches
6. **Mixed node**: has both incoming & outgoing branches.
   A mixed node adds all incoming branches and transmits this to all outgoing branches.

A mixed node can be made into an output node by adding an outgoing branch of gain 1.

**Path**: a traversal of connected branches in the direction of the branch arrows
- \( U \rightarrow U_1 \rightarrow Y \)
- \( U \rightarrow U_1 \rightarrow Y \rightarrow U_1 \)

**Loop or closed path**: a path that ends at the beginning node and does not cross any other
node more than once

\[ u_1 \rightarrow y \rightarrow u_1 \]

(4) Loop gain: the product of the branch transmittance of a loop.

(5) Non-touching loops: multiple loops with no common nodes.

(6) Forward path: path from the input node to output node with no node encountered more than once.

Mason's rule

Based on determinant \( \Delta \) of a graph.

\[ \Delta = 1 - (\text{sum of all individual loop gains}) + (\text{sum of gain products of all combinations of two non-touching loops}) - (\text{sum of gain products of all combinations of three non-touching loops}) \]

Ex. p 111 of text.
Loops

L1: X1 X2 X3 X4 X5 X1
L2: X3 X4 X3
L3: X1 X2 X5 X1
L4: X1 X3 X4 X5 X1

(© is X1 X3 X4 X3 a loop?)

Two non-touching loops

L2 & L3

Gains

- G2 G3 G4 G5 H2
- G4 H1
- G2 G7 H2
- G6 G4 G5 H2

G2 G4 G7 H1 H2

Hint: You can tell whether loops touch each other by looking at the node sequence.

Hence, the determinant is

\[ \Delta = 1 - \left[ -G2 G3 G4 G5 H2 - G4 H1 - G2 G7 H2 - G6 G4 G5 H2 \right] + G2 G4 G7 H1 H2 \]

L2 & L3

Procedures of carrying out Mason's rule

Suppose we want to compute the end-to-end transfer function of a SFG. The steps are:

1. Compute \( \Delta \), the determinant of the SFG.
2. Find all forward paths and the associated forward path gains $P_1, P_2, \ldots, P_k$ (k is the number of forward paths).

3. For each forward path $P_i$, $i = 1, \ldots, k$, find the determinant $\Delta_i$ of the SFG gotten by removing the terms that correspond to loops touching path $P_i$.

$$\Delta_i = \text{cofactor of the forward path } P_i$$

4. Then the required transfer function is

$$H = \frac{1}{\Delta} \sum_{i=1}^{k} P_i \Delta_i$$

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Continuing the earlier example

<table>
<thead>
<tr>
<th>Forward Path</th>
<th>Gains</th>
<th>Cofactor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$G_1 G_2 G_3 G_4 G_5$</td>
<td>1</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$G_1 G_2 G_7$</td>
<td>$1 - (G_4 H_1)$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$G_1 G_6 G_9 G_5$</td>
<td>1</td>
</tr>
</tbody>
</table>

Hint: To compute the cofactor of a path, check whether the path has a common node with each loop, and keep those terms of $\Delta$ that
correspond to non-touching loops.

Hence

\[ H(s) = \frac{G_1 G_2 G_3 G_4 G_5}{1 + G_2 G_3 G_4 G_5 H_2 + G_9 H_1 + G_2 G_7 H_C} \]

\[ + G_6 G_9 G_5 H_2 + G_2 G_4 G_7 H_C \]

Additional examples in A-3-24, p142 of text. You can also try A-3-25, but do not follow the reduction approach.