

## Transfer function: P 55-P 58

The transfer function of a linear-time-invariant system is defined as the ratio of the Laplace transform of the output (i.e., response) to the Laplace transform of the input, under the assumption that all initial conditions are zero.

$$U(s) \rightarrow [H(s)] \rightarrow Y(s)$$

$$\boxed{H(s) = \frac{Y(s)}{U(s)}}$$

Assuming that we know the transfer function  $H(s)$ . Then for any input  $U(s)$

$$Y(s) = H(s) U(s)$$

$$\text{or } y(t) = \int_0^t u(\tau) h(t-\tau) d\tau \quad y(t) = h * u.$$

## Impulse response

If  $U(t) = \delta(t)$ , then

$$y(t) = \int_0^t \delta(\tau) h(t-\tau) d\tau = h(t)$$

$\Rightarrow$  Transfer function is the Laplace transform of the impulse response.

$$H(s) = \mathcal{L}[h(t)]$$

To obtain the transfer function from differential equations :

$$a_0 y + a_1 \dot{y} + \cdots + a_n y^{(n)} = b_0 u + b_1 \dot{u} + \cdots + b_m u^{(m)} \quad (n \geq m)$$

Take Laplace transforms on both sides (assuming zero initial conditions)

$$(a_0 + a_1 s + \cdots + a_n s^n) Y(s) = (b_0 + b_1 s + \cdots + b_m s^m) U(s)$$

$$\Rightarrow H(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} = \frac{b(s)}{a(s)}$$

$b(\cdot)$  : numerator polynomial

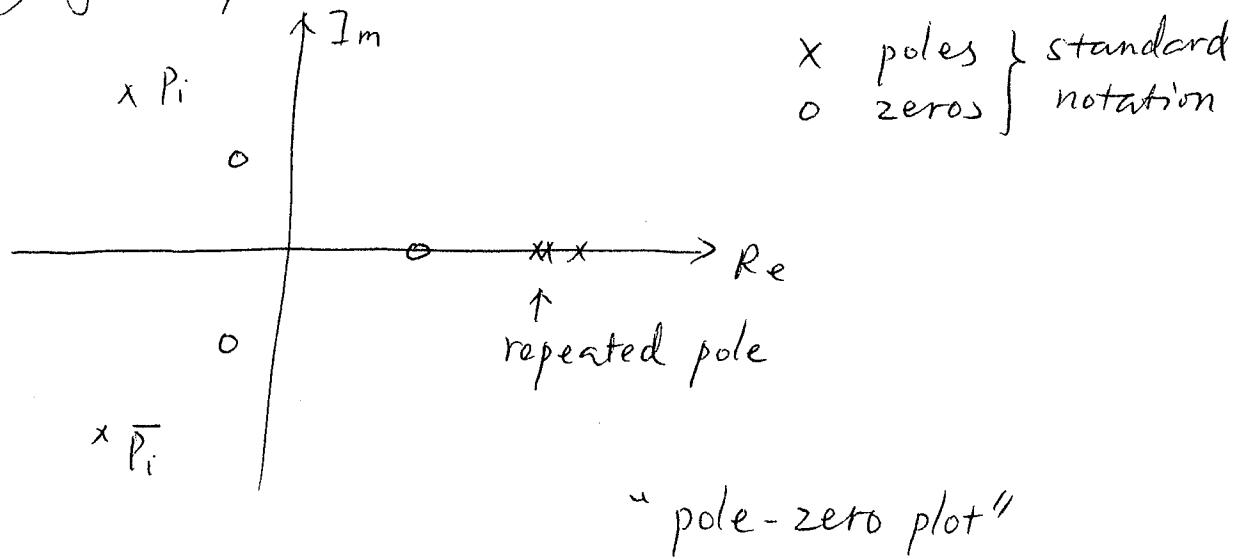
$a(\cdot)$  : denominator polynomial

System is called "nth-order system"

The  $m$  roots of  $b(\cdot)$  are the zeros of  $H$

The  $n$  roots of  $a(\cdot)$  are the poles of  $H$ .

Since  $a_0, \dots, a_n, b_0, \dots, b_m$  are all real numbers, the zeros & poles of  $H(s)$  occur in complex conjugate pairs.



## Some standard forms for transfer functions

- ① Ratio of polynomials

$$H(s) = \frac{b_0 + b_1 s + \dots + b_m s^m}{a_0 + a_1 s + \dots + a_n s^n}$$

- ② Factored form or product form

$$H(s) = K \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

$\nwarrow$  zeros       $\nearrow$  poles  
 $K = \frac{b_m}{a_m}$

Poles & zeros are in general complex.

Since they occur in complex conjugate pairs, we may collect each pair and obtain only real terms.

If  $z = \sigma + j\omega$ ,  $\bar{z} = \sigma - j\omega$ , then

$$(s-z)(s-\bar{z}) = (s-\sigma-j\omega)(s-\sigma+j\omega) \\ = (s-\sigma)^2 + \omega^2 = s^2 - 2\sigma s + (\sigma^2 + \omega^2)$$

Then

$$H(s) = K \frac{(s-z_1)\dots(s-z_\alpha)}{(s-p_1)\dots(s-p_\beta)} \frac{(s^2 - 2\tilde{\sigma}_1 s + (\tilde{\omega}_1^2 + \tilde{\omega}_1^2))\dots(s^2 - 2\tilde{\sigma}_\alpha s + (\tilde{\omega}_\alpha^2 + \tilde{\omega}_\alpha^2))}{(s^2 - 2\tilde{\sigma}_1 s + \tilde{\sigma}_1^2 + \tilde{\omega}_1^2)\dots(s^2 - 2\tilde{\sigma}_\beta s + \tilde{\sigma}_\beta^2 + \tilde{\omega}_\beta^2)}$$

$\nwarrow$   $\alpha$  real terms       $\nearrow$   $\beta$  terms  
 $\uparrow$   $\beta$  real terms       $\uparrow$   $\alpha$  terms

Clearly,

$$\alpha + 2\beta = m, \quad \beta + 2\delta = n$$

This is referred to as "real factored form"

③ DC-normalized form

Suppose 0 is not a pole of  $H(s)$

$$H(0) = \frac{b(0)}{a(0)} = \frac{b_0}{a_0} = K \cdot \frac{\prod_{i=1}^m (-z_i)}{\prod_{i=1}^n (-p_i)} \quad \left. \begin{array}{l} \text{"DC gain"} \\ K_{DC} \end{array} \right\}$$

$$H(s) = K_{DC} \frac{(1 - s/z_1) \cdots (1 - s/z_m)}{(1 - s/p_1) \cdots (1 - s/p_n)}$$

Once again, if there are complex terms, we can collect the conjugate pairs to get the real DC-normalized form

④ Sum form: comes from the partial fraction expansion

Assume all poles are distinct

$$H(s) = K_{HF} + \sum_{i=1}^n \frac{\alpha_i}{s - p_i}$$

(when poles are not distinct, see previous notes or text p35)

$K_{HF} \neq 0$  if  $n=m$ , in which case it is called the "high-freq gain"

$$K_{HF} = \lim_{s \rightarrow +\infty} H(s)$$

$K_{HF} = 0$  if  $n < m$

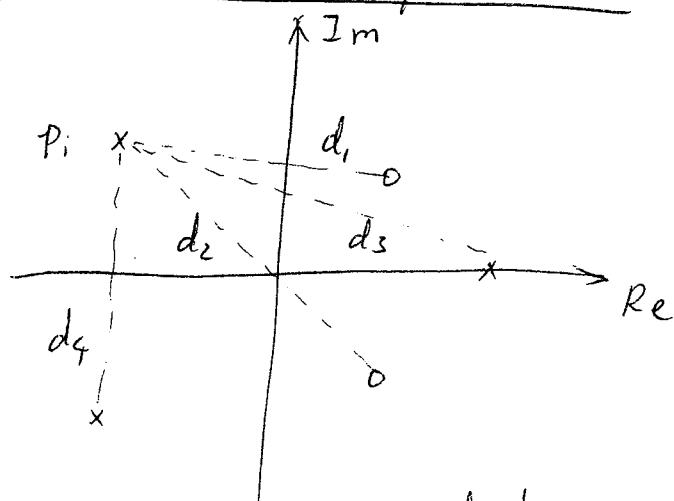
$\alpha_i$  is called the "residue" at the pole  $p_i$

(41)

$$\text{recall } \alpha_i = \lim_{s \rightarrow p_i} H(s)(s-p_i) \\ = K \cdot \frac{\prod_{\substack{j=1 \\ j \neq i}}^m (p_i - z_j)}{\prod_{\substack{j=1 \\ j \neq i}}^n (p_i - p_j)}$$

Thus  $|\alpha_i| = |K| \frac{\prod_{j=1}^m |p_i - z_j|}{\prod_{\substack{j=1 \\ j \neq i}}^n |p_i - p_j|}$  distance between  $(p_i, z_j)$

### Geometric Interpretation



then  $|\alpha_i| = |K| \frac{d_1 d_2}{d_3 d_4}$

As a consequence, if a pole is close to another zero, the residue will be small.

(42)

Example: Derive the various standard forms for

$$H(s) = \frac{3s^2 + 8s + 6}{s^3 - 4s^2 + 6s - 4}$$

Solve for zeros:

$$3s^2 + 8s + 6 = 0$$

$$z_1 = \frac{4+j\sqrt{2}}{3}, \quad z_2 = \frac{4-j\sqrt{2}}{3}$$

Solve for poles

$$s^3 - 4s^2 + 6s - 4 = 0$$

$$p_1 = 2, \quad p_2 = 1+j, \quad p_3 = 1-j$$

Factored form

$$H(s) = 3 \cdot \frac{\left(s - \frac{4+j\sqrt{2}}{3}\right) \left(s - \frac{4-j\sqrt{2}}{3}\right)}{(s-2)(s-(1+j))(s-(1-j))}$$

Real-factored form

$$H(s) = 3 \cdot \frac{s^2 - \frac{8}{3}s + 2}{(s-2)(s^2 - 2s + 2)}$$

DC-normalized form

$$H(s) = -\frac{3}{2} \cdot \frac{\left[1 - \frac{3s}{4+j\sqrt{2}}\right] \left[1 - \frac{3s}{4-j\sqrt{2}}\right]}{\left(1 - \frac{s}{2}\right) \left(1 - \frac{s}{1+j}\right) \left(1 - \frac{s}{1-j}\right)}$$

Sum form

$$H(s) = \frac{\alpha_1}{s-2} + \frac{\alpha_2}{s-(1+j)} + \frac{\alpha_3}{s-(1-j)}$$

$$\alpha_1 = \lim_{s \rightarrow 2} (s-2)H(s) = \lim_{s \rightarrow 2} \frac{3s^2 - 8s + 6}{(s-(1+j))(s-(1-j))} = \frac{2}{2} = 1$$

(43)

$$\alpha_2 = \lim_{s \rightarrow 1+j} (s - (1+j)) H(s) = \lim_{s \rightarrow 1+j} \frac{3s^2 - 8s + 6}{(s-2)(s-(1-j))}$$

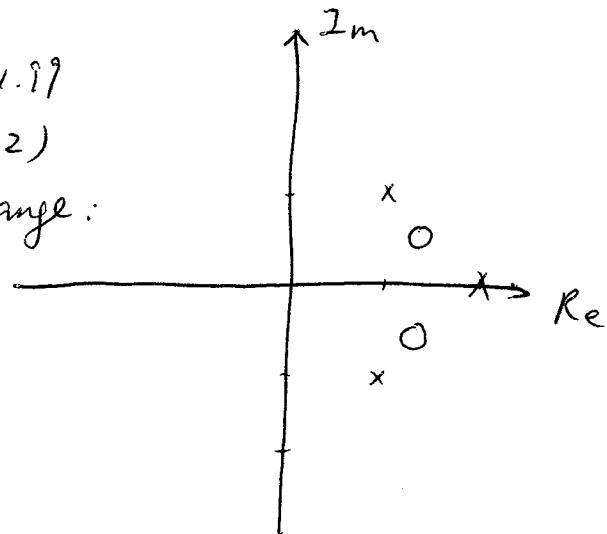
$$= \frac{-2-2j}{-2+2j} = 1$$

$$\alpha_3 = \lim_{s \rightarrow (1-j)} (s - (1-j)) H(s) = 1$$

$$\therefore H(s) = \frac{1}{s-2} + \frac{1}{s-(1+j)} + \frac{1}{s-(1-j)}$$

Let us now add a zero of 1.99  
 (almost cancelling the pole at 2)  
 and see how the residues change:

$$\tilde{H}(s) = (s-1.99) H(s)$$



The sum-form is then

$$\tilde{H}(s) = 3 + \frac{0.01}{s-2} + \frac{-0.99+j}{s-(1+j)} + \frac{-0.99-j}{s-(1-j)}$$

↑ residue is  
High-freq gain. smaller

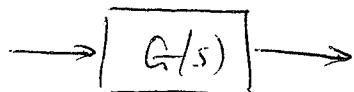
Block diagrams

P58-60

For large control systems, it will be difficult to write down the differential equations for the entire system. How then should we obtain the transfer function?

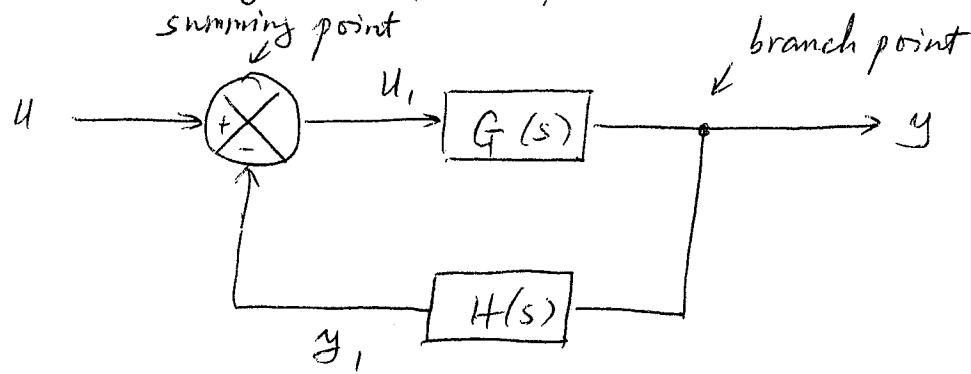
Fortunately, many control systems can be modelled as being made up of several components that do not interact except that the input of one part is the output of another part. Further, the transfer function of each component is easily found. In these cases, a picture called the block diagram is useful in describing & simplifying the relationships between inputs & outputs of interest.

A block:



Blocks (components) interact with each other in such a way that the output of a block is the input of another block.

Ex) Block diagram of a feedback control system



In a block diagram:

- block:  $\rightarrow \boxed{G(s)} \rightarrow$  represents the mathematical operation on the input signal to produce the output signal.

Transfer function is written inside the block.

- arrow: input or output signals (by direction)

- summing point:  $\rightarrow \circlearrowleft \oplus \ominus \rightarrow$  plus and minus

Sign at each arrows head indicates whether that signal is added or subtracted.

- branch point:  $\rightarrow \downarrow \rightarrow$  the same signal goes concurrently to multiple directions.

Suppose we can model a complex system using a block diagram. We can then use the block diagram to derive the transfer function between the input/output of interest.

In general, we can write down the equation for each block, and solve for the relationship between input & output.

$$\text{Ex) } Y(s) = G(s) U_1(s)$$

$$Y_1(s) = H(s) Y(s)$$

$$U_1(s) = U(s) - Y_1(s)$$

Eliminate unknowns  $Y_1(s)$ ,  $U_1(s)$

$$\Rightarrow Y(s) = G(s) (U(s) - Y_1(s)) \\ = G(s) (U(s) - H(s)Y(s))$$

$$\Rightarrow Y(s) \cdot (1 + G(s)H(s)) = G(s)U(s)$$

$$\Rightarrow \boxed{Y(s) = \frac{G(s)}{1 + G(s)H(s)} \cdot U(s)}$$

Hence, the transfer function from  $U \rightarrow Y$  is  $\frac{G}{1+GH}$ .

The block diagram we just saw and the formula  $\frac{G}{1+GH}$  are some of the most frequently occurring items in classical control.

How to remember?

- Feedforward transfer function:  $G(s)$  the ratio of the output to the error signal. (i.e. as if there is no feed back).

Open-loop transfer function:  $G(s)H(s)$  the ratio of the feedback signal to the error signal

(47)

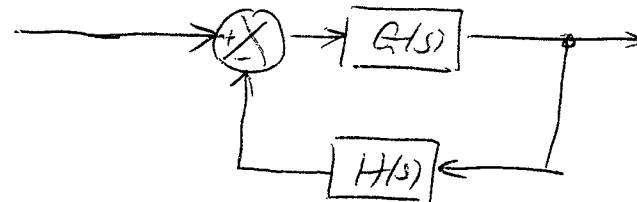
(i.e. there is feedback but as if the feedback is not applied to the input yet).

Closed-loop transfer function:

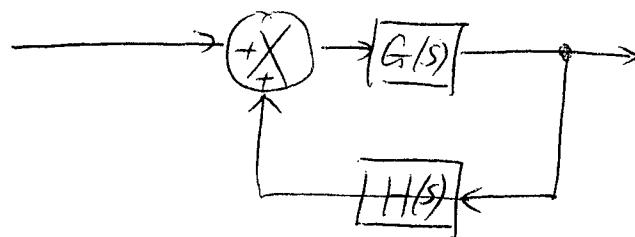
$$\frac{G(s)}{1 + G(s)H(s)}$$

↑                      ↑  
feed forward    open-loop

+ sign for negative feedback



- sign for positive feedback

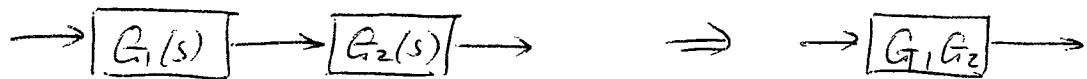


## Block diagram reduction P68-70

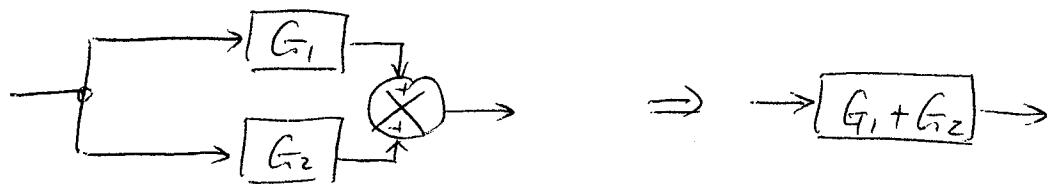
For large systems, the above procedure can be fairly involved. The alternatives are to use block diagram reduction or Mason's rule. We first introduce block diagram reduction.

We can reduce a complicated block diagram via three reduction rules:

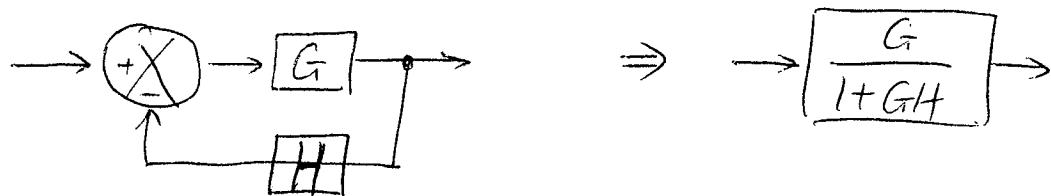
① Cascade ~~rate~~ blocks



② Parallel blocks

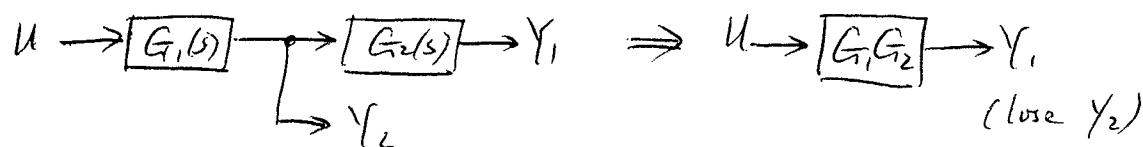


③ Feedback blocks



Key: Keep the input/output relationship the same.

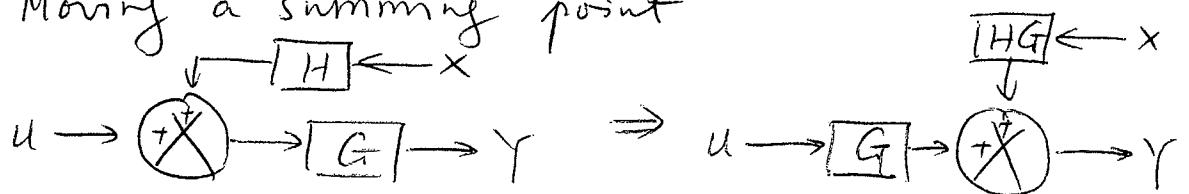
Q) Is this correct? No.



Additionally, we can move the summing point or the branch point ahead or behind a block, so that we can use the three reduction rules.

a) move  
behind  
a block

① Moving a summing point

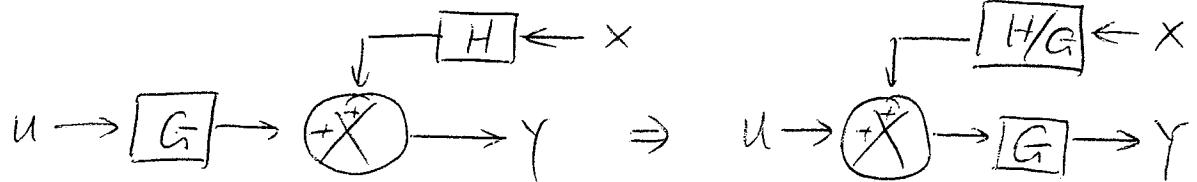


Idea: Again, you need to keep input/output relationships the same

~~Y = G(U + HX)~~

$$Y = G(U + HX) = GU + HGX$$

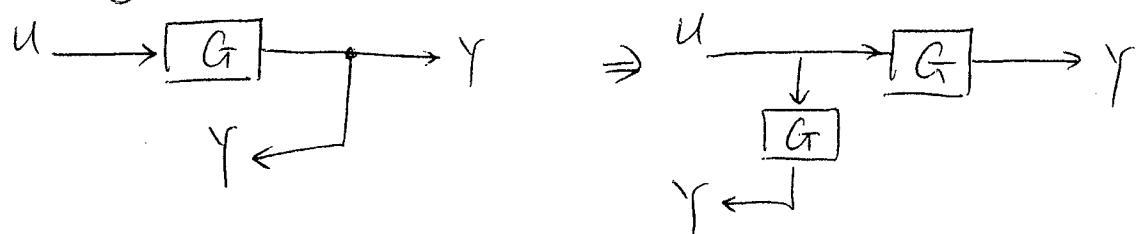
b) move  
ahead of  
a block



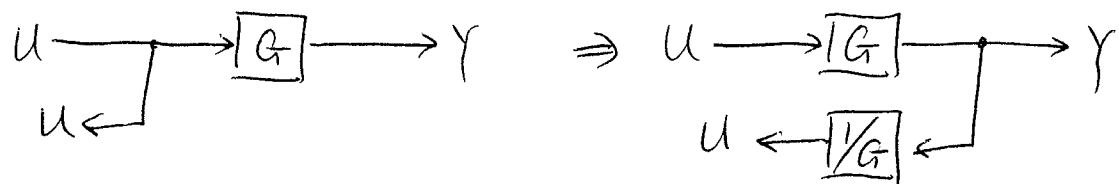
$$Y = GU + HX = G(U + H/GX)$$

② Moving a branch point

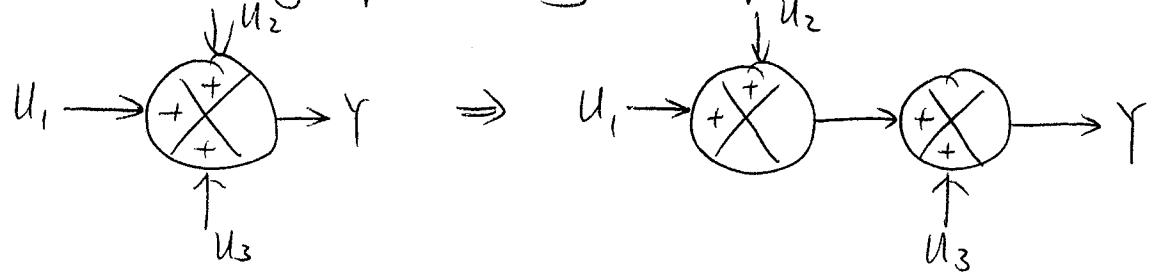
a) move  
ahead of  
a block



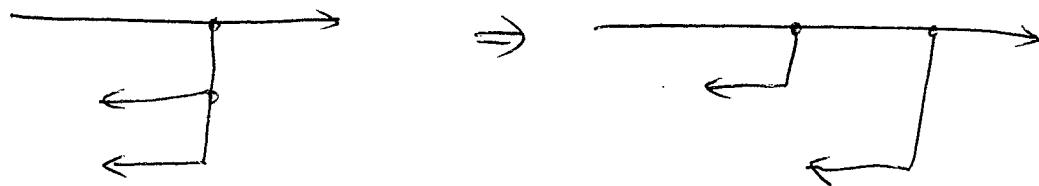
b) move  
behind  
a block



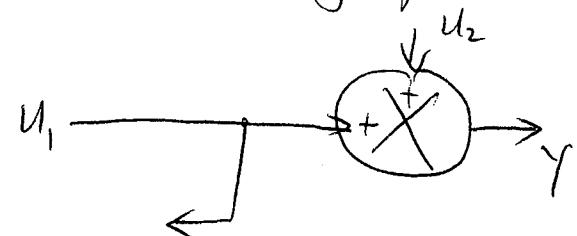
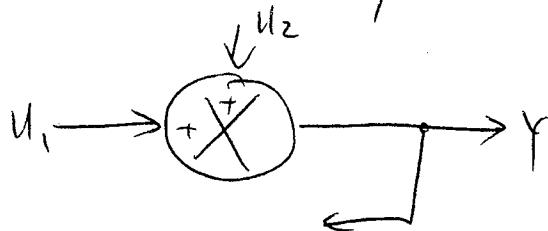
③ A summing point may be split into two



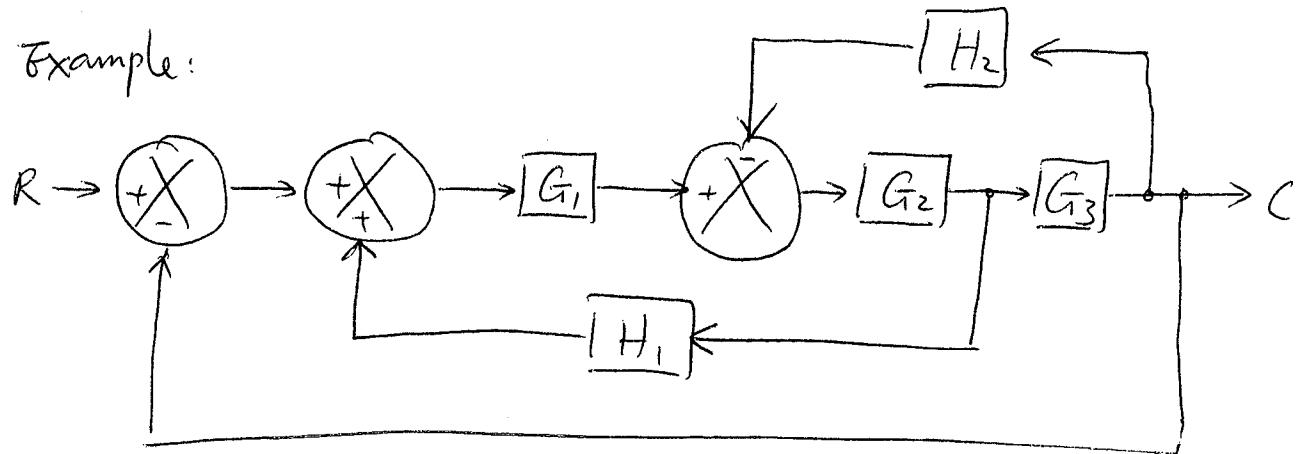
④ A branch point may be split into two



CAUTION : DO NOT MOVE A SUMMING POINT ACROSS A BRANCH POINT, or move a branch point across a summing point



Example:



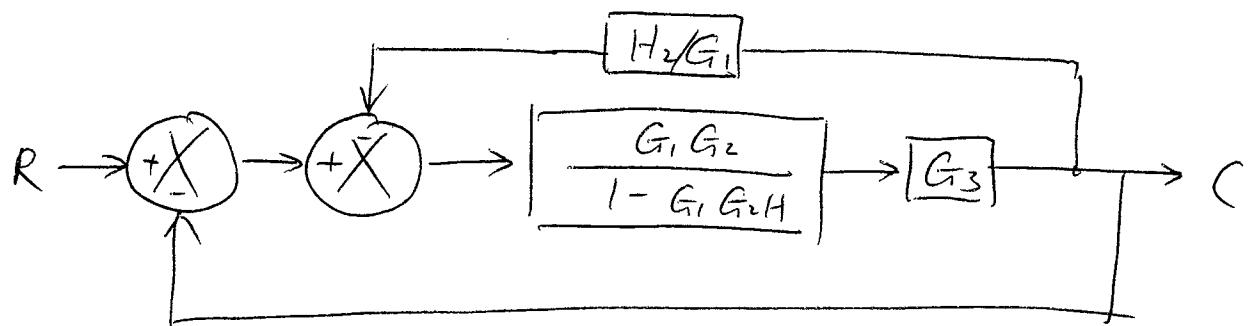
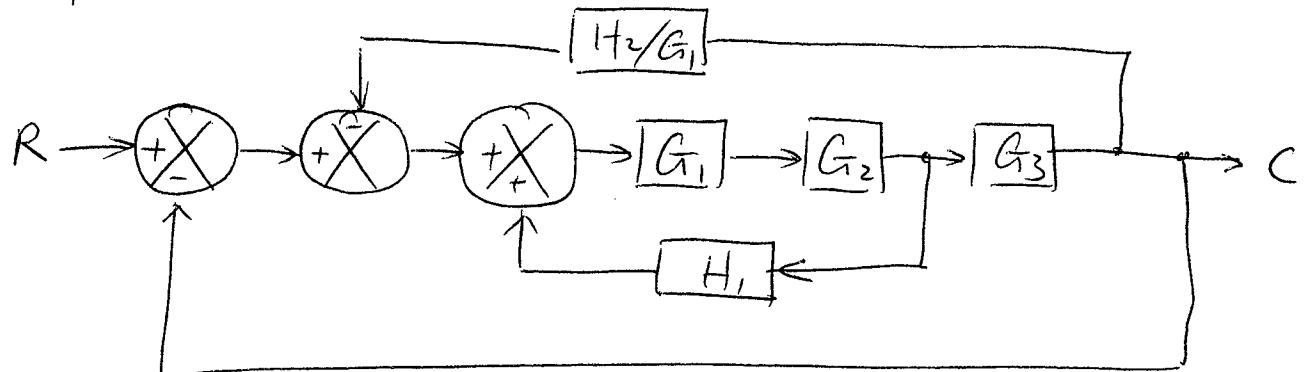
Procedures for block-diagram reduction:

- ① Identify the feedforward paths & the feedback loops (may have parallel paths).
- ② Always start from the most inside loops and/or parallel paths and proceed outwards.
- ③ If possible, reduce the diagram using the three reduction rules.

If multiple loops and/or parallel paths partially overlap with each other, decouple them by moving the branch points and/or summing points according to the 4 moving rules, so that either they do not overlap or one contains the other.

In this example, two loops overlap, involving two summing points and two branch points. There are multiple ways of decoupling them.

However, take caution not to move summing point across branch point.



$$R \rightarrow \text{summing junction} \rightarrow \left[ \frac{\frac{G_1 G_2 G_3}{1 - G_1 G_2 H}}{1 + \frac{H_2}{\alpha_1} \cdot \frac{G_1 G_2 G_3}{1 - G_1 G_2 H}} \right] = \frac{G_1 G_2 G_3}{1 - G_1 G_2 H + G_2 G_3 H} \rightarrow C$$

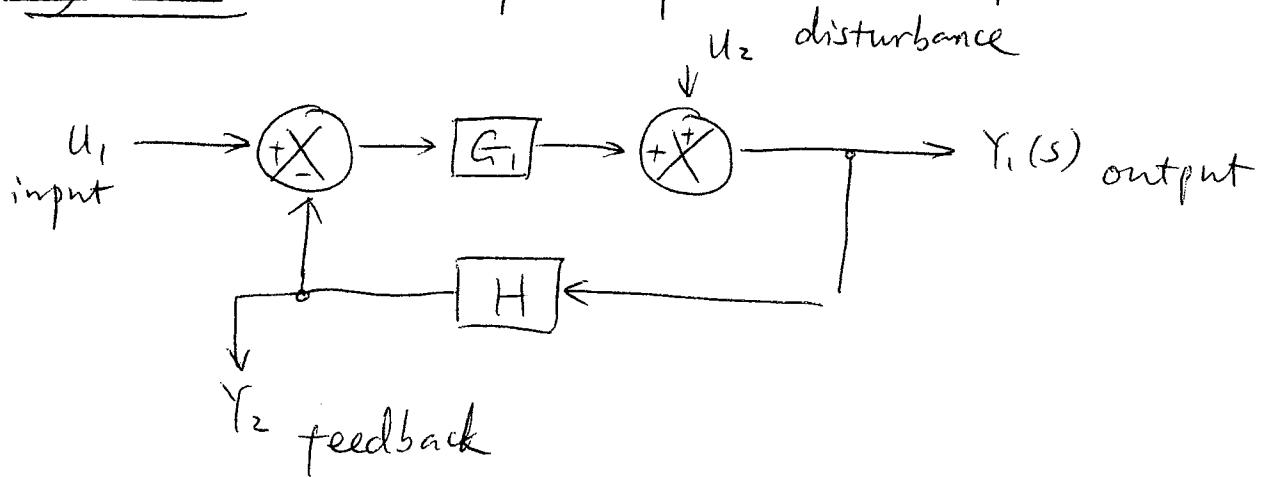
$$R \rightarrow \left[ \frac{\frac{G_1 G_2 G_3}{1 - G_1 G_2 H + G_2 G_3 H}}{1 + \frac{G_1 G_2 G_3}{1 - G_1 G_2 H + G_2 G_3 H}} \right] \rightarrow C$$

$\frac{G_1 G_2 G_3}{1 - G_1 G_2 H + G_2 G_3 H}$  // feedforward path

$1 - G_1 G_2 H + G_2 G_3 H + G_1 G_2 G_3 \leftarrow$  open-loop transfer func.

In general, the numerator of the end-to-end transfer function is the feed-forward transfer function, and the denominator is  $1 \pm$  the open-loop transfer functions. We will see more of this in Mason's rule.

### System with multiple inputs and outputs



Suppose there are  $m$  inputs  $u_1, \dots, u_m$ , and  $n$  outputs  $y_1, \dots, y_n$  to the linear system. Then from linearity, the Laplace transform of the  $k$ -th output  $Y_k(s)$  is given by

$$Y_k(s) = H_{yk}u_1(s) + H_{yk}u_2(s) + \dots + H_{yk}u_m(s)$$

where  $H_{yk}u_j(s)$  is the transfer function from the input  $u_j$  to the  $k$ -th output  $y_k$ , which can be found by letting all other inputs signal  $u_i=0$ ,  $i \neq j$ , and calculate the ratio between  $Y_k(s)$  and  $u_j(s)$ .

(54)

$$H_{y_1 u_1} = \frac{G}{1+GH}$$

$$H_{y_2 u_1} = \frac{GH}{1+GH}$$

$$H_{y_1 u_2} = \frac{1}{1+GH}$$

$$H_{y_2 u_2} = \frac{H}{1+GH}$$