

## The Inverse Laplace Transform

P32-42

(31)

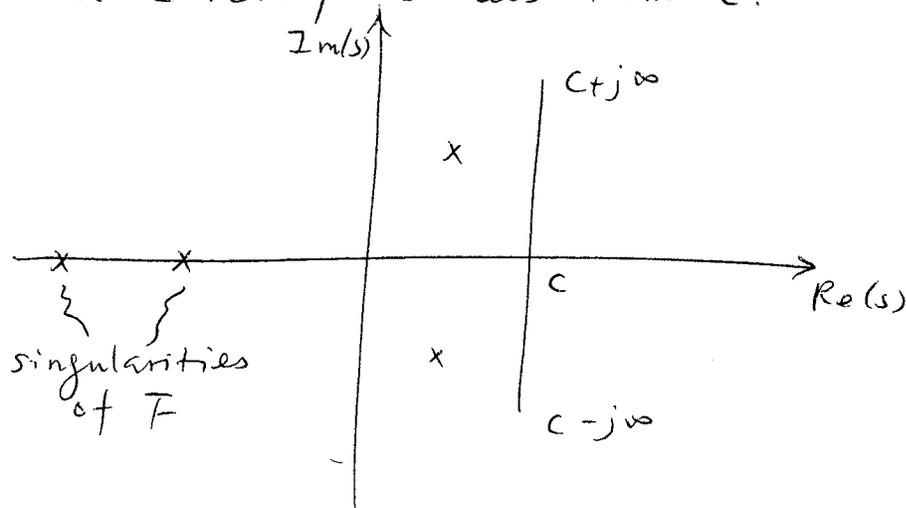
Suppose  $F = \mathcal{L}(f)$ , then  $f$  is called the "inverse Laplace transform" of  $F$ , denoted by  $\mathcal{L}^{-1}(F)$

Ex)  $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1(t)$

### Explicit formula of $\mathcal{L}^{-1}(F)$

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds \quad (t > 0)$$

where  $c$  is chosen such that all singularities of  $F(s)$  have real parts less than  $c$ .



### Partial fraction method (very useful!!!)

Often, we would like to find the inverse Laplace transform of functions of the following form:

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \dots + b_m s^m}{a_0 + a_1 s + \dots + a_n s^n} \quad (n \geq m)$$

Idea: Break  $F(s)$  into a sum of simple terms whose inverse L.T. is obvious

$$F(s) = F_1(s) + F_2(s) + \dots + F_p(s)$$

$$\Rightarrow \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}(F_1) + \mathcal{L}^{-1}(F_2) + \dots + \mathcal{L}^{-1}(F_p).$$

Ex) 
$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

Write partial fraction expansion as

$$\frac{s+3}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

In order to determine  $a_1, a_2$ , multiply both sides by  $(s+1)(s+2)$

$$s+3 = a_1(s+2) + a_2(s+1) = (a_1+a_2)s + (2a_1+a_2)$$

Equating the coefficients of each term

$$\begin{cases} a_1 + a_2 = 1 \\ 2a_1 + a_2 = 3 \end{cases} \Rightarrow \begin{cases} a_1 = 2 \\ a_2 = -1 \end{cases}$$

$$\therefore F(s) = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\Rightarrow \mathcal{L}^{-1}(F) = 2e^{-t} - e^{-2t} \quad t \geq 0.$$

In general, when the roots of <sup>denominator</sup>  $A(s)$  are distinct, i.e.

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1s + \dots + b_ms^m}{a_n(s-\lambda_1)(s-\lambda_2)\dots(s-\lambda_n)} \quad n \geq m$$

We can write the partial-fraction expansion as

$$F(s) = \alpha_0 + \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2} + \dots + \frac{\alpha_n}{s-\lambda_n}$$

Once we find  $\alpha_0, \alpha_1, \dots, \alpha_n$ , we then have

$$\mathcal{L}^{-1}(F) = \alpha_0 \delta(t) + \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} + \dots + \alpha_n e^{\lambda_n t}$$

How to find  $\alpha_0, \alpha_1, \dots, \alpha_n$ ???

- ① Multiply both side of  $F(s)$  by  $A(s)$ , and equate the coefficients for each  $s^i$ . (As in example)
- ② By passing limits. Note

$$(s-\lambda_1)F(s) = \alpha_1 + \left[ \alpha_0(s-\lambda_1) + \frac{\alpha_2}{s-\lambda_2}(s-\lambda_1) + \dots + \frac{\alpha_n}{s-\lambda_n}(s-\lambda_1) \right]$$

Let  $s \rightarrow \lambda_1$ , we get

$$\alpha_1 = \lim_{s \rightarrow \lambda_1} (s-\lambda_1) F(s)$$

Similarly for  $\alpha_2, \dots, \alpha_n$ . Finally

$$\alpha_0 = \lim_{s \rightarrow +\infty} F(s) \quad (\alpha_0 = 0 \text{ if } m < n)$$

Redo the example:

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

$$\Rightarrow a_1 = \lim_{s \rightarrow -1} (s+1)F(s) = \lim_{s \rightarrow -1} \frac{s+3}{s+2} = 2$$

$$a_2 = \lim_{s \rightarrow -2} (s+2)F(s) = \lim_{s \rightarrow -2} \frac{s+3}{s+1} = -1.$$

More general case: The roots of  $A(s)$  are not necessarily distinct.

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1s + \dots + b_ms^m}{a_n (s-\lambda_1)^{t_1} (s-\lambda_2)^{t_2} \dots (s-\lambda_p)^{t_p}}$$

Then the partial fraction expansion will contain linear combinations of

$$1, \frac{1}{s-\lambda_1}, \frac{1}{(s-\lambda_1)^2}, \dots, \frac{1}{(s-\lambda_1)^{t_1}}$$

$$\frac{1}{s-\lambda_2}, \frac{1}{(s-\lambda_2)^2}, \dots, \frac{1}{(s-\lambda_2)^{t_2}}$$

$$\dots$$

$$\frac{1}{s-\lambda_p}, \frac{1}{(s-\lambda_p)^2}, \dots, \frac{1}{(s-\lambda_p)^{t_p}}$$

Important: DO NOT LEAVE OUT TERMS

Example: (text p38)

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

Write the partial-fraction expansion as

$$F(s) = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

then

$$\mathcal{L}(F) = b_1 e^{-t} + b_2 t e^{-t} + b_3 \left( \frac{t^2}{2} e^{-t} \right) \quad t \geq 0$$

To find  $b_1, b_2, b_3$

Method ①: Multiplying both sides by  $(s+1)^3$

$$\begin{aligned} s^2 + 2s + 3 &= b_1(s+1)^2 + b_2(s+1) + b_3 \\ &= b_1 s^2 + (2b_1 + b_2)s + (b_1 + b_2 + b_3) \end{aligned}$$

$$\Rightarrow \begin{cases} b_1 = 1 \\ 2b_1 + b_2 = 2 \\ b_1 + b_2 + b_3 = 3 \end{cases} \Rightarrow \begin{cases} b_1 = 1 \\ b_2 = 0 \\ b_3 = 2 \end{cases}$$

$$\text{Hence } \mathcal{L}^{-1}(F) = e^{-t} + 2 \cdot \frac{t^2}{2} e^{-t} \quad (t \geq 0)$$

What if we leave out a term?

$$F(s) = \frac{b_3}{(s+1)^2}$$

$$\Rightarrow s^2 + 2s + 3 = b_3$$

WRONG!!!

Method (2) : Passing limits . (text p35)

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

$$\Rightarrow s^2 + 2s + 3 = b_1(s+1)^2 + b_2(s+1) + b_3$$

Let  $s \rightarrow -1$  on both sides, get

$$\boxed{2 = b_3}$$

Next, differentiate w.r. to  $s$ :

$$2s + 2 = 2b_1(s+1) + b_2$$

Let  $s \rightarrow -1$

$$\boxed{0 = b_2}$$

Differentiate one more time

$$2 = 2b_1$$

$$\boxed{1 = b_1}$$

Ex)  $F(s) = \frac{2s+12}{s^2+2s+5}$  text P34

We may use terms in PFE such as

$$\frac{1}{s+1+j2}, \quad \frac{1}{s+1-j2}$$

However, often easier to avoid imaginary numbers.

Use

$$\mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

$$\mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{s+\alpha}{(s+\alpha)^2 + \omega^2}$$

Since

$$A(s) = s^2 + 2s + 5 = (s+1)^2 + 2^2$$

We can write

$$F(s) = \frac{a \cdot 2}{(s+1)^2 + 2^2} + \frac{b \cdot (s+1)}{(s+1)^2 + 2^2}$$

Multiply both sides by  $A(s)$

$$2s+12 = 2a + b(s+1) = \cancel{2a} + bs + (a+b)$$

$$\Rightarrow \begin{cases} b = 2 \\ 2a+b = 12 \end{cases} \Rightarrow \begin{cases} a = 5 \\ b = 2 \end{cases}$$

$$\therefore F(s) = \frac{5 \cdot 2}{(s+1)^2 + 2^2} + \frac{2 \cdot (s+1)}{(s+1)^2 + 2^2}$$

$$\mathcal{L}^{-1}(F) = 5 \cdot e^{-t} \sin 2t + 2 \cdot e^{-t} \cos 2t \quad t \geq 0.$$