

## Laplace Transform

The following are perhaps the most frequently used Laplace Transform pairs.

Impulse function	$\delta(t) \Leftrightarrow 1$
Step function	$1(t) \Leftrightarrow \frac{1}{s}$
Ramp function	$t \Leftrightarrow \frac{1}{s^2}$
Exponential	$e^{-at} \Leftrightarrow \frac{1}{s+a}$
Sin/cos	$\sin \omega t \Leftrightarrow \frac{\omega}{s^2 + \omega^2}$ $\cos \omega t \Leftrightarrow \frac{s}{s^2 + \omega^2}$

More on textbook pp 17-18.

Although there are many of them, it should be easy to memorize them once we understand the properties

- linearity
- translation
- exponential weighting
- time-scaling
- differentiation
- integration
- initial value
- final value
- convolution.

## Review of Laplace Transforms

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $f(t) = 0$  for  $t < 0$ .

Then the Laplace transform of  $f$  (if it exists), is denoted by  $F = \mathcal{L}(f)$ , with

$$* F : \mathbb{C} \rightarrow \mathbb{C}$$

$$* F(s) = \int_0^{+\infty} f(t) e^{-st} dt$$

Typically,  $f$  denotes a signal, which is a function of time  $t$ . The Laplace transform maps a function of  $t$  to another function  $F$  that is complex-valued and defined over complex numbers.

$$f(t) \xrightarrow{\begin{array}{|c|} \hline \text{Laplace} \\ \text{Transform} \\ \hline \end{array}} \mathcal{L} \rightarrow F(s)$$

Remarks:

- ① We need  $\int_0^{+\infty} f(t) e^{-st} dt$  to converge for some value of  $s \in \mathbb{C}$  in order for the Laplace transform to exist.

Typically the integral converges when  $\operatorname{Re} s$  is large enough

- ② The smallest real number  $\sigma_c$  such that the integral converges for  $\operatorname{Re} s > \sigma_c$  is called the "abscissa of convergence".

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Ex) Unit step function

$$1(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(s) = \int_0^{+\infty} 1 \cdot e^{-st} dt = -\frac{e^{-st}}{s} \Big|_0^{+\infty} \quad \begin{matrix} \text{the limit is} \\ \text{for } t. \\ s \text{ is fixed} \end{matrix}$$

$$= \frac{1}{s} - \frac{e^{-st}}{s} \Big|_{+\infty}$$

The integral converges if  $\operatorname{Re} s > 0$   
then  $F(s) = \frac{1}{s}$

The abscissa of convergence is 0.

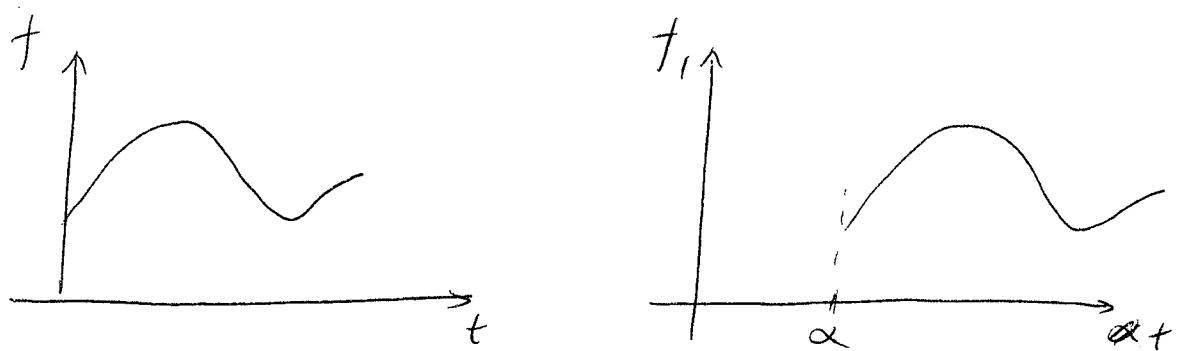
- ③ Although the integral converges only for  $s$  with  $\operatorname{Re} s$  large enough, the function  $F(s)$  is thought to be defined (almost) everywhere in  $\mathbb{C}$ .

### Properties of Laplace Transforms

Linearity :  $\mathcal{L}(\alpha f_1 + \beta f_2) = \alpha \mathcal{L}(f_1) + \beta \mathcal{L}(f_2)$

Proof by direct verification.

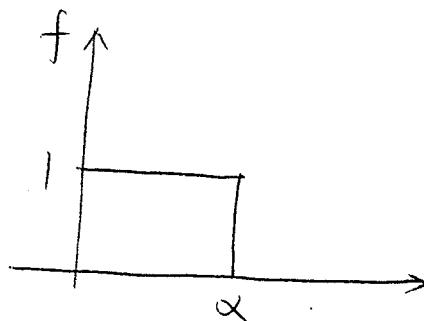
Translation : Let  $F = \mathcal{L}(f)$ . Let  $f_\alpha$  be an  $\alpha$ -shifted version of  $f$ , that is  
 $f_\alpha(t) = f(t-\alpha) 1(t-\alpha)$



Then  $F_1 = \mathcal{L}(f_1)$  is given by

$$\boxed{F_1(s) = e^{-\alpha s} F(s)}$$

Ex)



square function

$$f(t) = 1(t) - 1(t-\alpha)$$

$$\mathcal{L}(f) = \frac{1}{s} - \frac{1}{s} e^{-\alpha s} = \frac{1}{s} (1 - e^{-\alpha s})$$

Exponential weighting : If  $f_1(t) = e^{-\alpha t} f(t)$

then

$$\boxed{F_1(s) = F(s+\alpha)}$$

Ex) (a)  $\mathcal{L}[1(t)] = \frac{1}{s}$

(b) since  $e^{-\alpha t} = e^{-\alpha t} 1(t)$

hence  $\mathcal{L}[e^{-\alpha t}] = \frac{1}{s+\alpha}$

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c) Since  $\sin \omega t = \frac{1}{2j} [e^{j\omega t} - e^{-j\omega t}]$   
 hence  $\mathcal{L}[\sin \omega t] = \frac{1}{2j} \left[ \frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right]$

$$= \frac{1}{2j} \frac{(s+j\omega) - (s-j\omega)}{(s-j\omega)(s+j\omega)} = \frac{1}{2j} \frac{2j\omega}{s^2 + \omega^2}$$

$$= \frac{\omega}{s^2 + \omega^2}$$

d) Similarly

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

Remarks on the "duality":

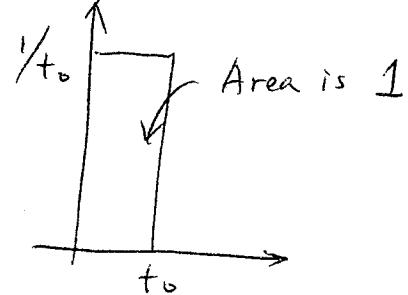
Exponential in  $t \rightarrow$  shift in  $s$   
 shift in  $t \rightarrow$  exponential in  $s$

Time-scaling: If  $f_1(t) = f(\alpha t)$   
 then  $F_1(s) = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$

Ex)  $\mathcal{L}(e^{-t}) = \frac{1}{s+1}$   
 $\mathcal{L}(e^{-\alpha t}) = \frac{1}{\alpha} \left( \frac{1}{\frac{s}{\alpha} + 1} \right) = \frac{1}{s+\alpha}$ .

## Detour from Properties

Impulse function: Consider the following function, commonly known as the "pulse" function



$$P_{t_0}(+) = \begin{cases} \frac{1}{t_0} & 0 \leq t \leq t_0 \\ 0 & \text{otherwise.} \end{cases}$$

As  $t_0 \rightarrow 0$ , the limiting function is called the unit impulse, denoted by  $\delta(+)$   
— also called Dirac delta function.

The unit impulse is zero everywhere except at  $t=0$ , where it is undefined. (You may think of it as being infinite there.) If we integrate from  $-\infty$  to  $+\infty$ , the integral is 1

$$\int_{-\infty}^{+\infty} \delta(+) dt = 1.$$

The Laplace transform of the unit impulse

$$\mathcal{L}(\delta(+)) = 1$$

(can be shown as the limit of the Laplace transform of pulse function as  $t_0 \rightarrow 0$ . see text p20)

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Closer look: whenever we deal with functions that have an impulse at 0, we need to distinguish between  $0^-$  &  $0^+$

$0^-$  is "just before zero"

$0^+$  is "just after zero"

Similarly, we can define two versions of Laplace transform.

$$\mathcal{L}_+(f(t)) = \int_{0^+}^{+\infty} f(t) e^{-st} dt$$

$$\mathcal{L}_-(f(t)) = \int_{0^-}^{+\infty} f(t) e^{-st} dt$$

$$= \int_{0^+}^{+\infty} f(t) e^{-st} dt + \int_{0^-}^{0^+} f(t) e^{-st} dt$$

$\approx \int_{0^-}^{0^+} f(t) dt$

When we write  $\mathcal{L}(\delta(t)) = 1$ , we implicitly integrating from  $0^-$  <sup>are</sup>

$$\mathcal{L}_-(\delta(t)) = \int_{0^-}^{+\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

What if I pick  $\mathcal{L}_+$  (integrating from  $0^+$ )

$$\mathcal{L}_+(\delta(t)) = \int_{0^+}^{+\infty} \delta(t) e^{-st} dt = 0 \quad (\text{not useful})$$

If  $f(t)$  does not possess an impulse function at  $t=0$ , then  $\mathcal{L}_+(f(t)) = \mathcal{L}_-(f(t)) = \mathcal{L}(f(t))$

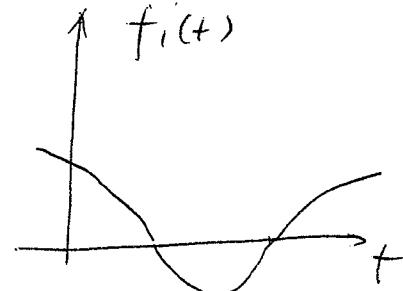
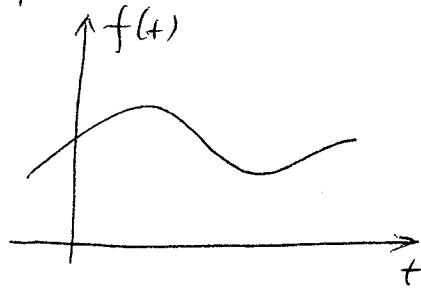
If it does, we need to make a choice. Often implicitly we use  $\mathcal{L}_-$ .

## Back to properties of Laplace Transform

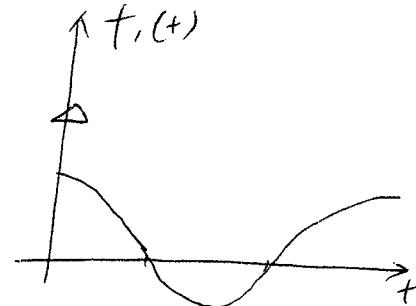
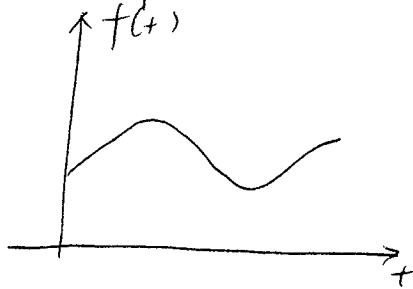
Differentiation: If  $f_1(t) = \frac{d}{dt} f(t)$   
then

$$\mathcal{L}(f_1(t)) = sF(s) - f(0)$$

This is the form that will be used most often in this class. It is unambiguous when  $f(0^-) = f(0^+)$ , i.e., when the function does not jump at time 0.



If the function jumps at time 0

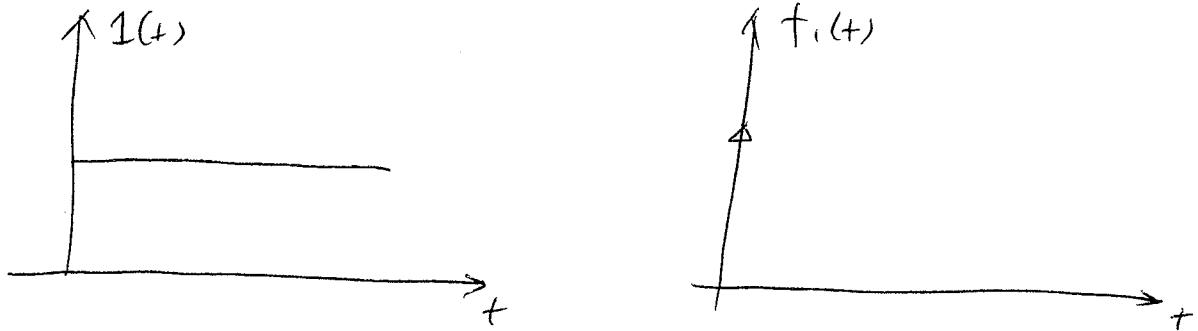


Its derivative then has an impulse function at time 0. Then depending on whether or not we want to include the impulse function into consideration, the differentiation properties change to:

$$\text{or } \mathcal{L}_-(f_1(t)) = sF(s) - f(0^-)$$

$$\text{or } \mathcal{L}_+(f_1(t)) = s\bar{F}(s) - f(0^+)$$

Example:  $f(t) = 1(t)$  jumps at time 0  
 $f_1(t) = \frac{d}{dt}f(t) = \delta(t)$



If we want to consider the impulse function  
in  $f_1(t)$ , use  
 $\mathcal{L}_-(\delta(t)) = s \cdot \frac{1}{s} - f(0^-) = 1$

If we don't, use  
 $\mathcal{L}_+(\delta(t)) = s \cdot \frac{1}{s} - f(0^+) = 0$

BOTH ARE CORRECT!!! Again, often we pick  $\mathcal{L}_-$   
since we want to incorporate the impulse.

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$$\underline{\text{Integration}} \quad : \quad \text{If} \quad f_1(t) = \int_0^t f(\tau) d\tau \\ \text{then} \quad F_1(s) = \frac{F(s)}{s}$$

$$\begin{aligned} \text{Ex}) \quad & \mathcal{L}[\delta(t)] = 1 \\ \Rightarrow & \mathcal{L}[1(t)] = \frac{1}{s} \\ & \mathcal{L}[t] = \frac{1}{s^2} \\ & \mathcal{L}\left[\frac{t^2}{2}\right] = \frac{1}{s^3} \end{aligned}$$

### Summary

$$\begin{aligned} \mathcal{L}(\alpha f_1 + \beta f_2) &= \alpha \cdot \mathcal{L}(f_1) + \beta \mathcal{L}(f_2) \\ \mathcal{L}(f(t-\alpha)1(t-\alpha)) &= e^{-\alpha s} \mathcal{L}(f) \\ \mathcal{L}(e^{-\alpha t} f(t)) &= F(s+\alpha) \\ \mathcal{L}(f(\alpha t)) &= \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) \\ \mathcal{L}\left(\frac{d}{dt} f(t)\right) &= s F(s) - f(0) \\ \mathcal{L}\left(\int_0^t f(u) du\right) &= \frac{F(s)}{s} \end{aligned}$$

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Often for the same  $f(t)$ , you may find multiple ways to use these properties. Of course they should produce the same Laplace transform. This would be a good way of cross-checking.

Example:  $f(t) = \cos(t - \frac{\pi}{4})$

It appears that there are at least 3 ways to check it

### ① Linearity

$$f(t) = \cos t \cdot \cos \frac{\pi}{4} + \sin t \cdot \sin \frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2} \cos t + \frac{\sqrt{2}}{2} \sin t$$

$$\text{Since } \mathcal{L}[\cos t] = \frac{s}{s^2+1}, \quad \mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

∴

$$\mathcal{L}[f(t)] = \frac{\sqrt{2}}{2} \frac{s}{s^2+1} + \frac{\sqrt{2}}{2} \frac{1}{s^2+1}$$

### ② Time-shift

One may view  $\cos(t - \frac{\pi}{4})$  as a time-shifted version of  $\cos t$  by  $\frac{\pi}{4}$ .

$$\text{Since } \mathcal{L}[\cos t] = \frac{s}{s^2+1}$$

we get

$$\mathcal{L}(\cos(t - \frac{\pi}{4})) = e^{-\frac{\pi}{4}s} \frac{s}{s^2+1}$$

Which one is correct ???

**IMPORTANT:** When using time-shift property, Need to fill  $(0, \infty]$  by zero!

$$\mathcal{L}(f(t-\alpha) \mathbf{1}(t-\alpha)) = e^{-\alpha s} \mathcal{L}(f(t))$$

↑

### ③ Differentiation

One note that

$$\cos(t - \frac{\pi}{4}) = \frac{d}{dt} \sin(t - \frac{\pi}{4})$$

Assume we know

$$\mathcal{L}(\sin(t - \frac{\pi}{4})) = \frac{\sqrt{2}}{2} \frac{1}{s^2 + 1} - \frac{\sqrt{2}}{2} \frac{s}{s^2 + 1}$$

We would get

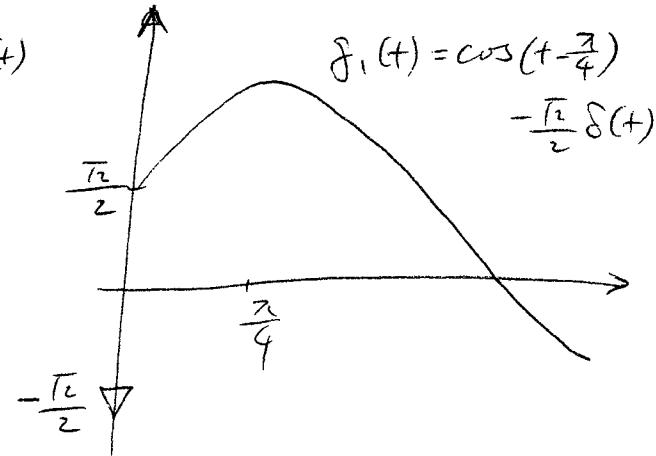
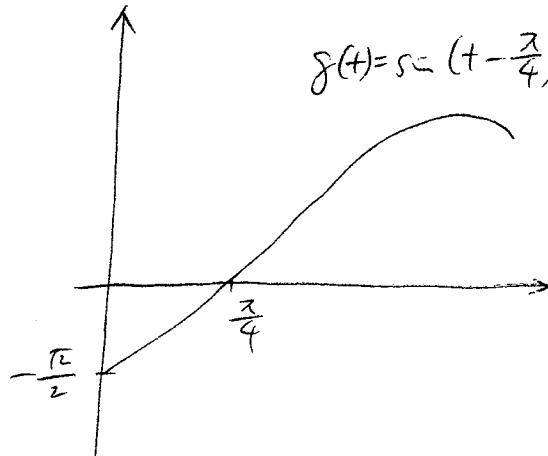
$$\mathcal{L}(\cos(t - \frac{\pi}{4})) = s \left( \frac{\sqrt{2}}{2} \frac{1}{s^2 + 1} - \frac{\sqrt{2}}{2} \frac{s}{s^2 + 1} \right) - g(0)$$

which  $g(0) ???$  Assuming we that  $g(0^-) = 0$ , we get

$$\mathcal{L}(\cos(t - \frac{\pi}{4})) = s \cdot \left( \frac{\sqrt{2}}{2} \frac{1}{s^2 + 1} - \frac{\sqrt{2}}{2} \frac{s}{s^2 + 1} \right)$$

which one is correct ???

**IMPORTANT:** when using differentiation rule, take special note of the delta function!!!



$$\begin{aligned}\mathcal{L}_-(f_1(t)) &= s \cdot \mathcal{L}(f_1(+)) - f(0^-) \\ &= s \left( \frac{\sqrt{2}}{2} \frac{1}{s^2 + 1} - \frac{\sqrt{2}}{2} \frac{s}{s^2 + 1} \right)\end{aligned}$$

$$\text{But } \mathcal{L}_-(f_1(+)) = \mathcal{L}(\cos(t - \frac{\pi}{4}) - \frac{\sqrt{2}}{2} \delta(t))$$

$$\Rightarrow \mathcal{L}(\cos(t - \frac{\pi}{4})) = s \left( \frac{\sqrt{2}}{2} \frac{1}{s^2 + 1} - \frac{\sqrt{2}}{2} \frac{s}{s^2 + 1} \right) + \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{2} \frac{s}{s^2 + 1} + \frac{\sqrt{2}}{2} \frac{1}{s^2 + 1}$$

correct

## EVEN MORE useful properties of LT:

### ① Initial value theorem

$$f(0^+) = \lim_{s \rightarrow +\infty} sF(s)$$

### ② Final value theorem

If all poles of  $F(s)$  lie on the left-half plane of  $\mathbb{C}$ , (this implies that  $\lim_{t \rightarrow \infty} f(t)$  exists) then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Remarks on duality again:

Need value of  $f$  at 0  $\rightarrow$  look at  $s = +\infty$

Need value of  $f$  at  $+\infty$   $\rightarrow$  look at  $s = 0$

### ③ Convolution

$$\mathcal{L}(f_1) \cdot \mathcal{L}(f_2) = \mathcal{L}(f_1 * f_2)$$

where

$$(f_1 * f_2)(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

Convolution in time  $\rightarrow$  multiplication of LT

VERY IMPORTANT!!! Basis for transfer functions.

An important application of LT is to solve linear time-invariant differential equations

Example: ① Let us solve the differential equation  
 $\ddot{y} + 3\dot{y} + 2y = 2u + u$

assuming that  $y(0) = \dot{y}(0) = 0$ , and  $u(0) = 0$

Take Laplace Transforms. Note that

$$\mathcal{L}(y(t)) = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}(\dot{y}(t)) = s[sY(s) - y(0)] - \dot{y}(0) = s^2Y(s)$$

$$\Rightarrow s^2Y(s) + 3sY(s) + 2Y(s) = 2sU(s) + U(s)$$

$$\Rightarrow Y(s) = \frac{2s+1}{s^2+3s+2} U(s)$$

IMPORTANT: If  $\dot{y}(0)$ ,  $y(0)$  or  $u(0)$  are not zero, we ~~need~~ need to have them in the equations.

We can check that

$$\mathcal{L}[3e^{-2t} - e^{-t}] = \frac{2s+1}{s^2+3s+2}$$

Hence

$$y(t) = \int_0^t [3e^{-2(t-\tau)} - e^{-(t-\tau)}] u(\tau) d\tau$$

② If  $u(t)$  is the unit step function, let us find the final value of  $y(t)$ .

Since  $\mathcal{L}[1(t)] = \frac{1}{s}$

$$Y(s) = \frac{2s+1}{s^2+3s+2} \cdot \frac{1}{s}$$

We can verify that all poles are on the LHP of  $\mathbb{C}$ .

$$\lim_{t \rightarrow +\infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \frac{1}{2}.$$