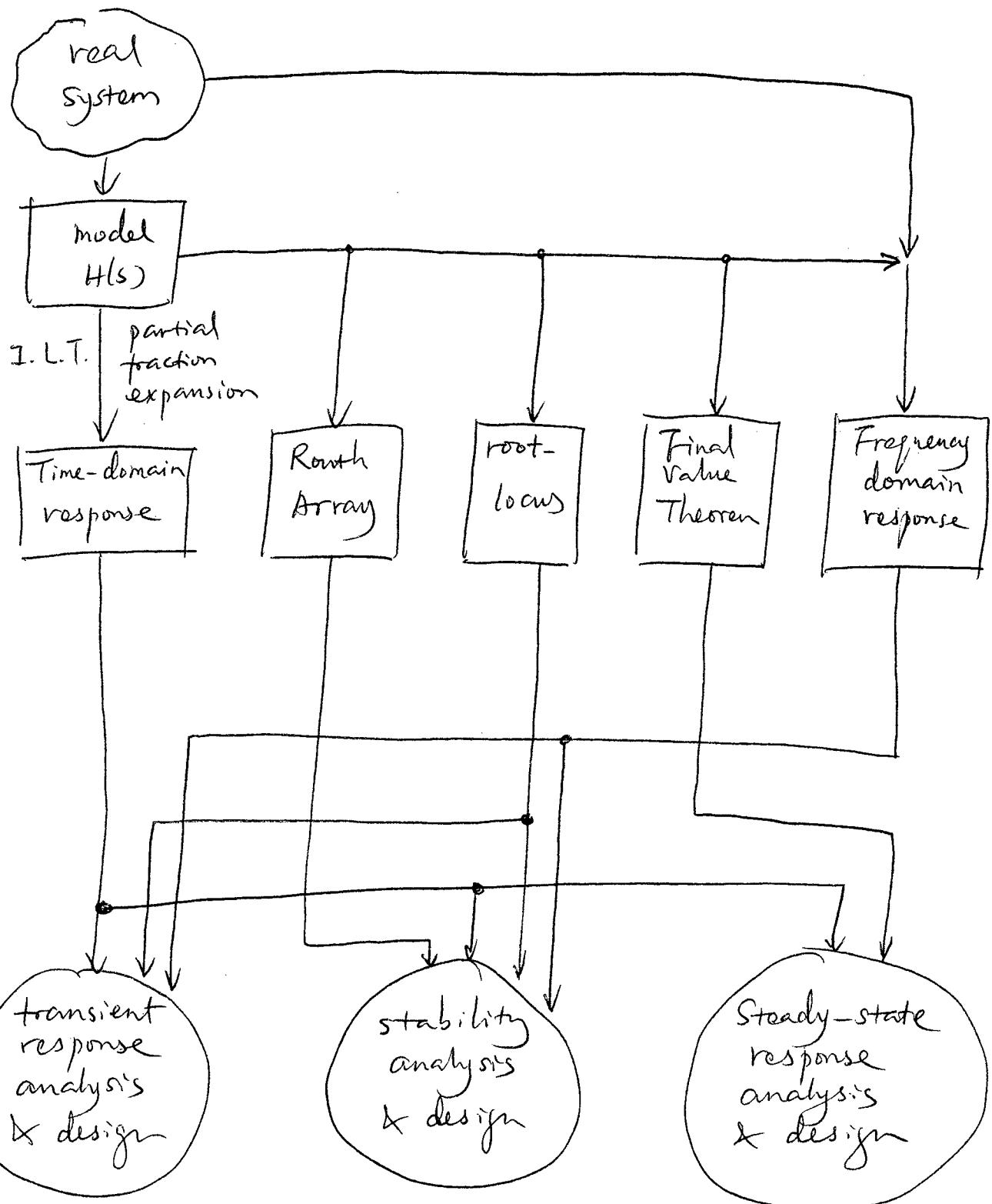


①

What we have learned:

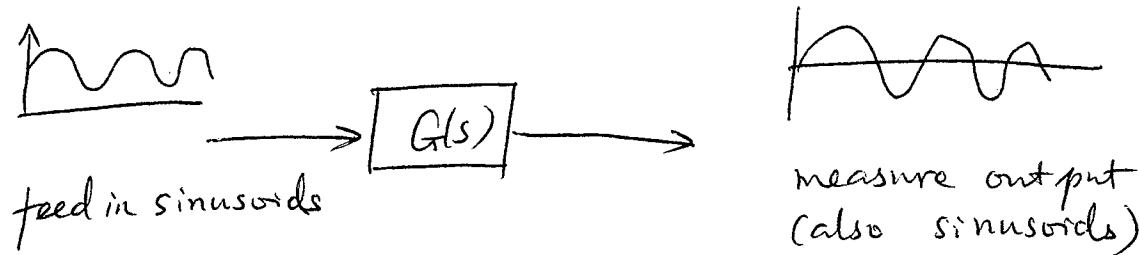


(2)

Frequency-domain methods can not only work with known transfer function, but also work with systems whose transfer function is not available!

Frequency Response Analysis & Design P492-497

Basic Idea:



Frequency response: the steady-state response of a system to a sinusoidal input.

If the system is LTI (linear and time-invariant), the output at steady-state is also a sinusoid, but the magnitude & phase are usually different from the input.

With frequency-response methods, we vary the frequency of the input signal over a certain range and study the resulting response. We can also design the closed-loop controllers based on such responses.

(3)

Relationship between frequency-response and the transfer function $G(s)$

We can show that, if the input is $e^{j\omega t}$, then the output at steady-state is

$$G(j\omega) e^{j\omega t} = |G(j\omega)| e^{j(\omega t + \angle G(j\omega))}$$

Hence, we can construct frequency-response from $G(j\omega)$, $-\infty < \omega < +\infty$.

$|G(j\omega)|$: the amplitude ratio of the output sinusoid to the input sinusoid.

$\angle G(j\omega)$: phase shift.

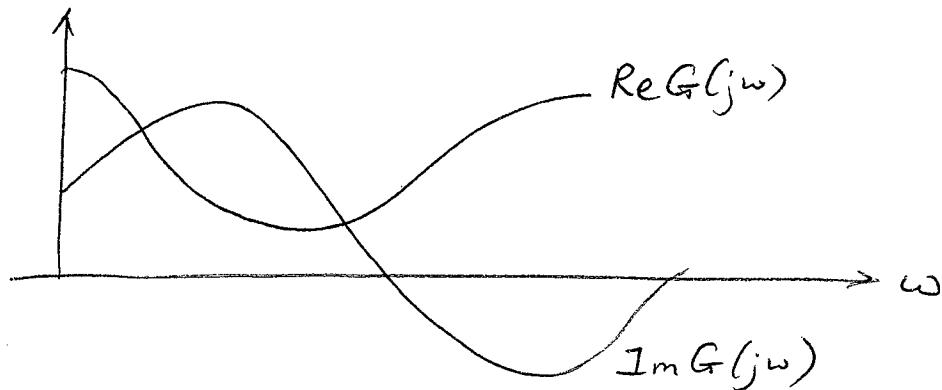
Why do we use frequency-response methods?

- * Methods studied so far rely on knowing $G(s)$. Often $G(s)$ is not available (not modelled or not identified). On the other hand, $G(j\omega)$ is easy to measure empirically using high-precision signal generators & measurement techniques.
- * Frequency-response methods also applicable to non-rational transfer functions. (e.g. $G(s) = e^{-sT}$).

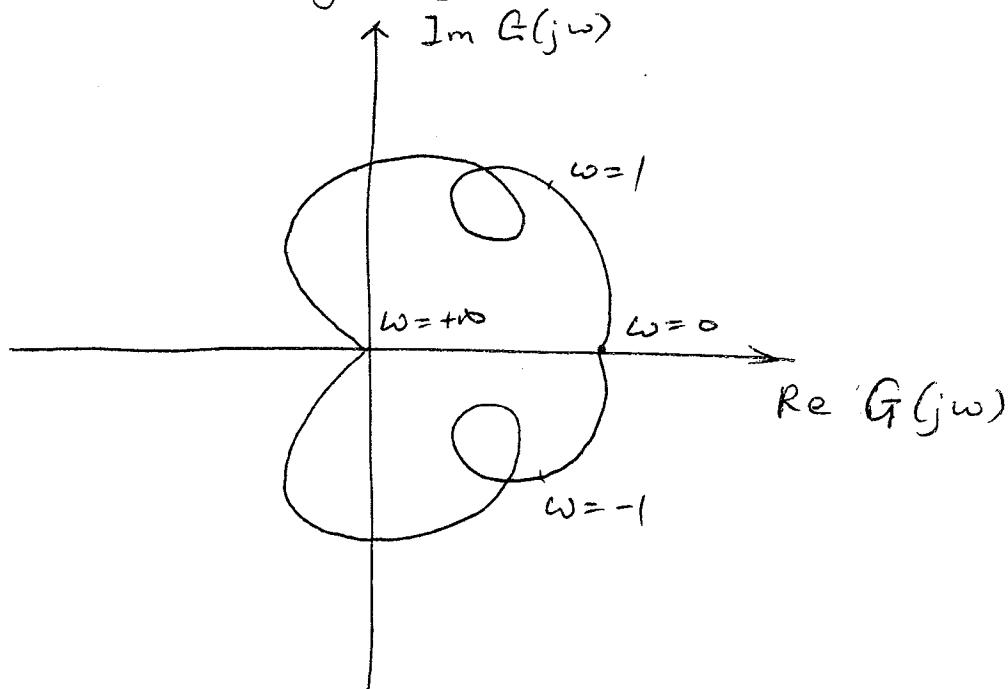
(4)

How to represent frequency responses

Several possibilities

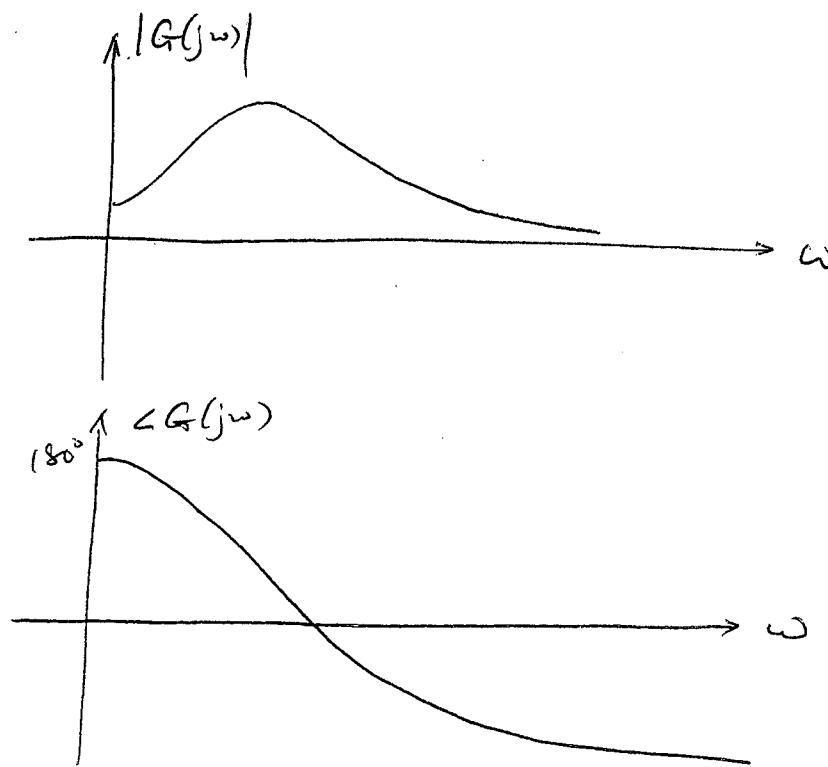


Real & Imaginary parts as ω varies



Polar plot
(Nyquist plot)

(5)



Magnitude & Phase Plot
(Bode plot).

Bode-plot 9497-515

(frequency ω)

- The x-axis is always drawn in log-scale
- The standard representation of the logarithmic magnitude of $G(j\omega)$ is

$$20 \log |G(j\omega)|$$

The unit is decibel (dB).

Each decade is 20 dB

$$K=1$$

$$K=10$$

$$K=100$$

$$20 \log_{10} K = 0 \text{ dB}$$

$$20 \log_{10} K = 20 \text{ dB}$$

$$20 \log_{10} K = 40 \text{ dB}$$

(6)

$$K=0.1$$

$$20 \log_{10} K = -20 \text{ dB}$$

Each octave is 6 dB

$$K=1$$

$$20 \log_{10} K = 0 \text{ dB}$$

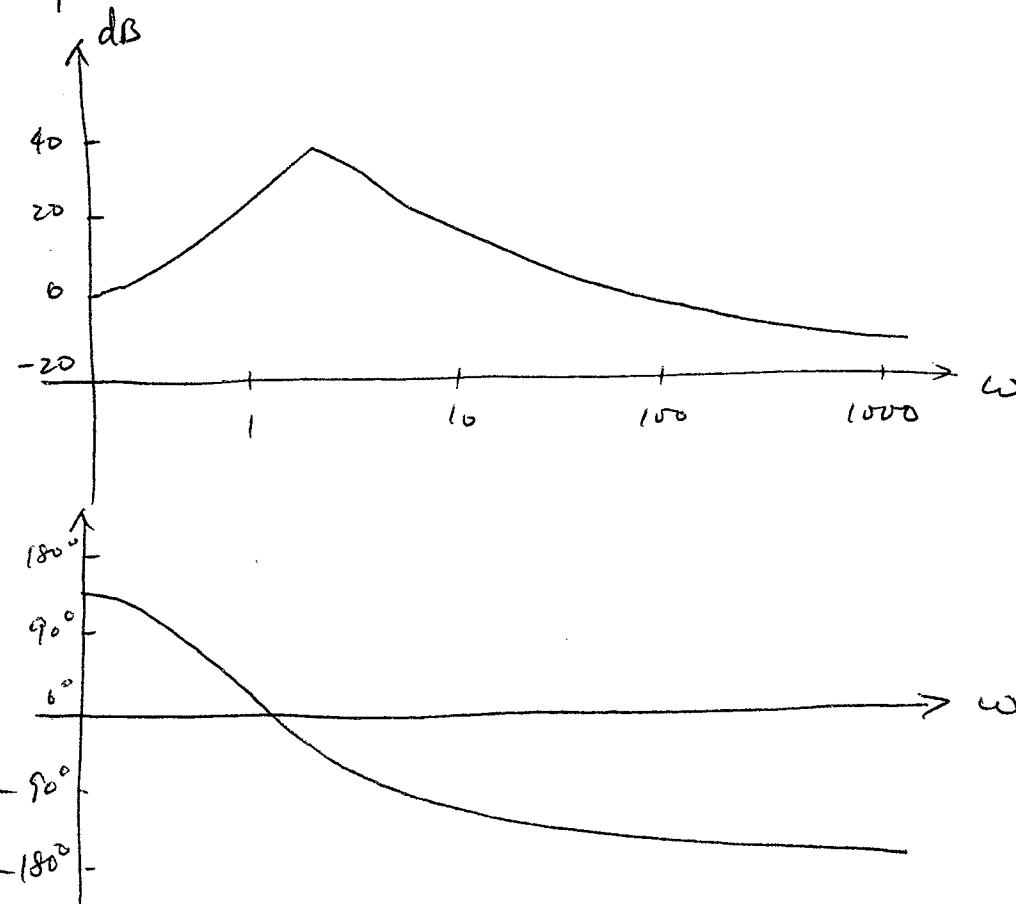
$$K=2$$

$$20 \log_{10} K = 6 \text{ dB}$$

$$K=4$$

$$20 \log_{10} K = 12 \text{ dB}$$

- The phase is drawn on normal scale.



Advantage of working with the Bode plots

- * When a transfer function is expressed in a product form, then Bode plot can be obtained by merely "adding" the Bode — plots of each factor.

⑦

If $G(s) = G_1(s) G_2(s)$
then

$$\begin{aligned}20 \log_{10} |G(j\omega)| &= 20 \log_{10} |G_1(j\omega)| + 20 \log_{10} |G_2(j\omega)| \\ \angle G(j\omega) &= \angle G_1(j\omega) + \angle G_2(j\omega).\end{aligned}$$

Thus, Bode plots for complicated transfer functions can be quickly drawn using a number of simple rules

- * A wide range of frequencies can be displayed on the log-scale.
- * Easy to measure, useful for analysis & design

Bode plots for standard terms

* $k(j\omega)$, $\frac{k}{j\omega}$, $k(j\omega)^n$

* $\frac{1}{1+j\omega T}$, $(1+j\omega T)$

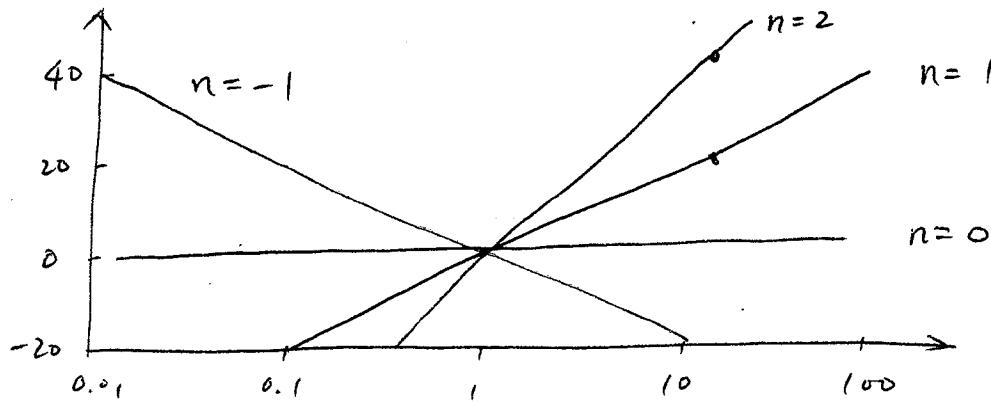
* $\frac{1}{1+2f\frac{j\omega}{\omega_n} + (\frac{j\omega}{\omega_n})^2}$, $(1+2f\frac{j\omega}{\omega_n} + (\frac{j\omega}{\omega_n})^2)$

8

① $K(j\omega)^n$ — integral or derivative $KS^n, \frac{K}{\omega^n}$

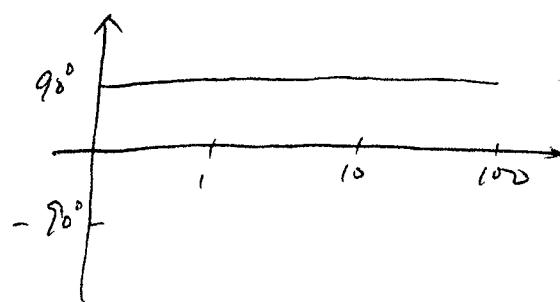
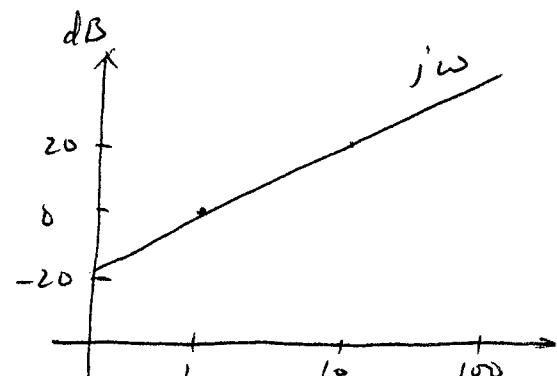
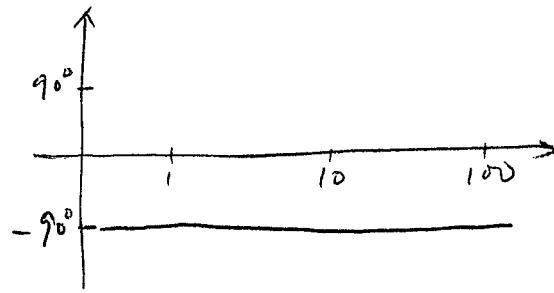
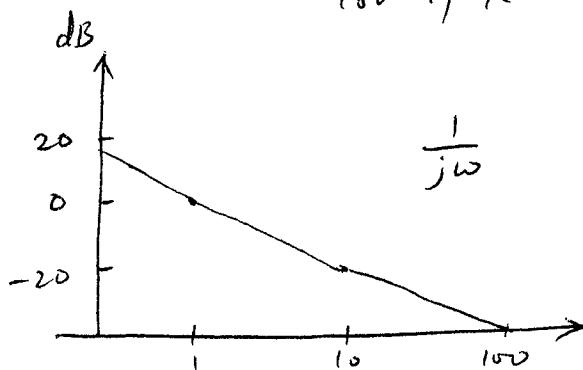
$$20 \log_{10} |K(j\omega)^n| = \underbrace{20 \log_{10} |K|}_{\text{constant}} + \underbrace{n \cdot 20 \log_{10} |\omega|}_{\text{linear with slope } n}$$

20 n dB every decade



Bode-plot when $K=1$

$$\begin{aligned}\angle K(j\omega)^n &= \angle K + \angle (j\omega)^n \\ &= \underbrace{\angle K}_{0^\circ \text{ if } K > 0} + n \cdot 90^\circ \\ &\quad 180^\circ \text{ if } K < 0\end{aligned}$$



(9)

② First-order factor $1+j\omega T$, $\frac{1}{1+j\omega T}$

Ex) first-order system $\frac{\frac{1}{T}}{s+\frac{1}{T}}$

Substitute s by $s=j\omega$

$$G(j\omega) = \frac{\frac{1}{T}}{j\omega + \frac{1}{T}} = \frac{1}{1+j\omega T}$$

For $\omega T \ll 1$, $\frac{1}{1+j\omega T} \approx 1$

magnitude $20 \log_{10} \left| \frac{1}{1+j\omega T} \right| \approx 0$

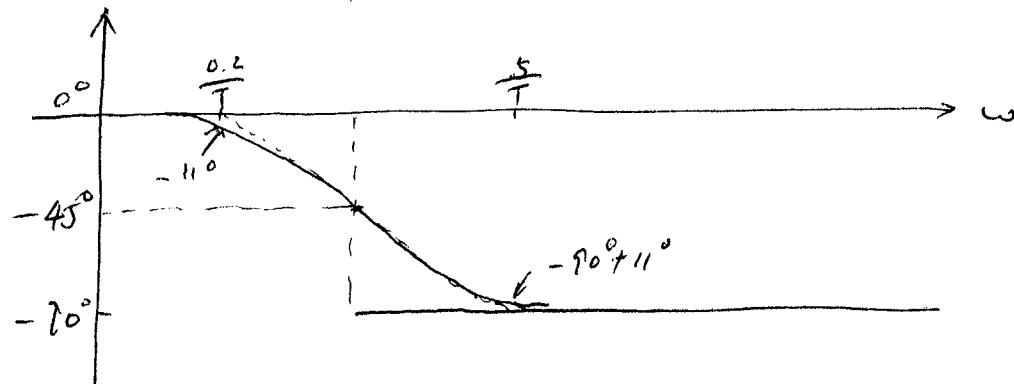
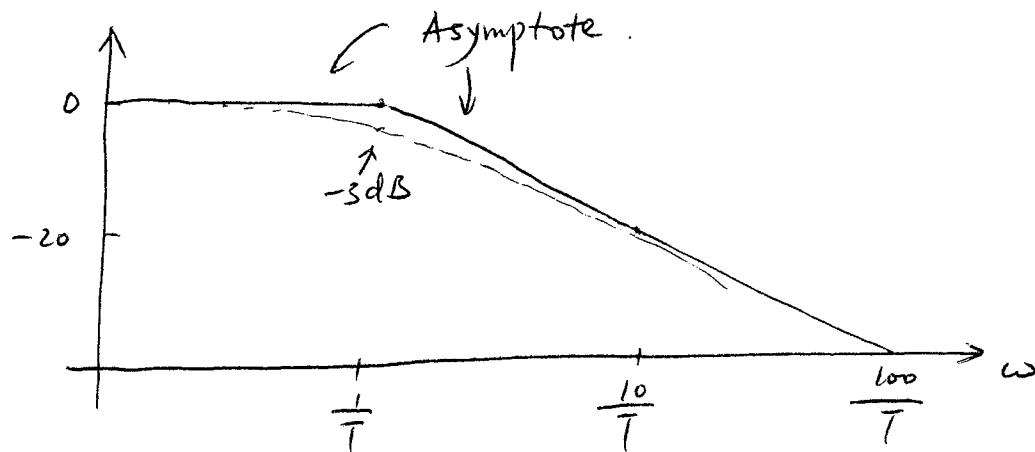
angle 0°

For $\omega T \gg 1$, $\frac{1}{1+j\omega T} \approx \frac{1}{j\omega T}$

magnitude $20 \log_{10} \left| \frac{1}{1+j\omega T} \right| \approx -20 \log_{10} |\omega T|$

-20 dB every decade

angle -90°



(10)

Thus, as $\omega \rightarrow 0$, the magnitude curve asymptotically approaches the 0dB line, and the phase approaches 0° .

As $\omega \rightarrow +\infty$, the magnitude curve asymptotically approaches the $\frac{1}{j\omega T}$ curve, and the angle approaches -90° .

These limiting curves are called asymptotes.

KEY: The asymptotes are of the form $(j\omega T)^n$.

The point where two asymptotes intersect is

$$\omega T = 1$$

$$\Rightarrow \omega = \frac{1}{T}$$

$\omega = \frac{1}{T}$ is called the corner frequency.

We may make additional adjustment of the magnitude and phase at the corner frequency.

When $\omega = \frac{1}{T}$,

$$20 \log_{10} \left| \frac{1}{1+j\omega T} \right| = 20 \log_{10} \frac{1}{\sqrt{2}} = -3 \text{ dB}$$

$$\angle \frac{1}{1+j\omega T} = \angle \frac{1}{1+j} = -45^\circ$$

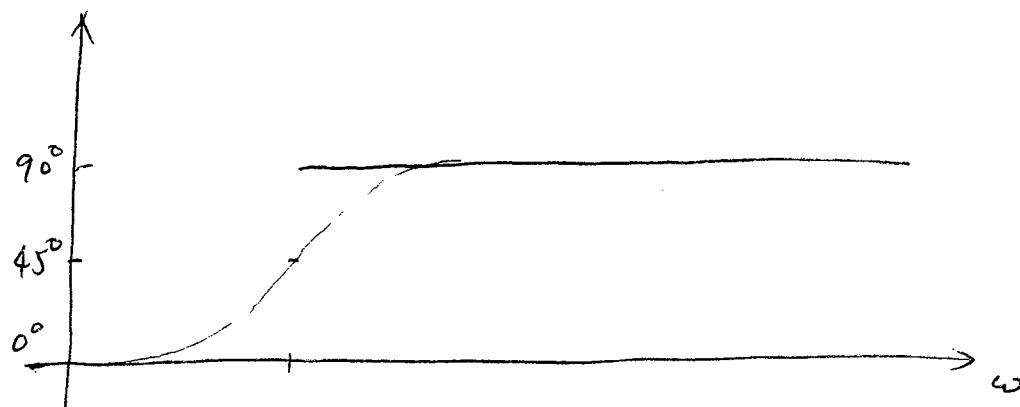
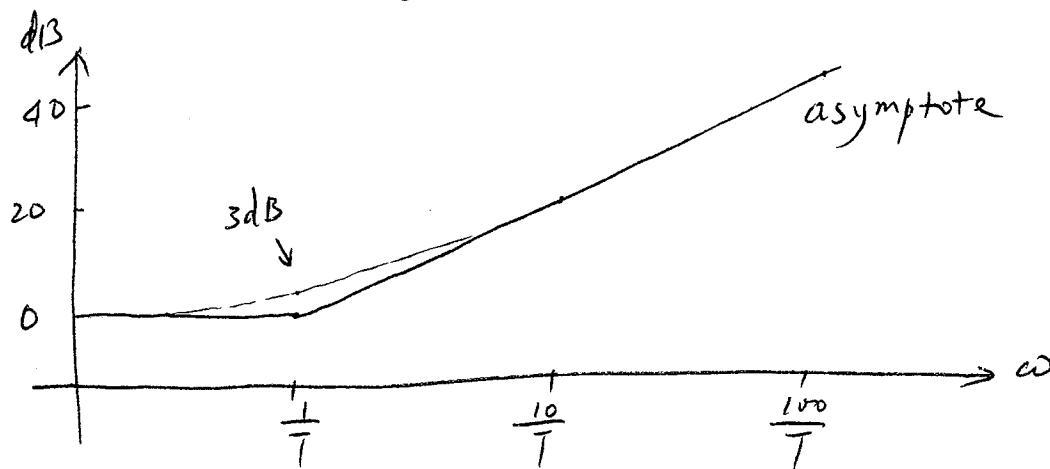
Further, on the phase plot, the tangent at the corner-frequency intersects the two asymptotes at $\frac{0.2}{T} \neq \frac{5^\circ}{T}$.

(11)

Similarly, for $G(j\omega) = 1 + j\omega T$

For $\omega T \ll 1$, $1 + j\omega T \approx 1$

$\omega T \gg 1$, $1 + j\omega T \approx j\omega T$



We can also make adjustments at the corner frequency $\omega = \frac{1}{T}$

$$\Rightarrow 1 + j\omega T = 1 + j = \sqrt{2} < 45^\circ$$

Note: Both the magnitude-plot & the phase-plot are negative of those of $\frac{1}{1+j\omega T}$.

In general, the Bode-plot of $\frac{1}{G(s)}$ is the negative of the Bode-plot of $G(s)$

$$20 \log_{10} \left| \frac{1}{G(j\omega)} \right| = -20 \log_{10} |G(j\omega)|$$

$$\angle \frac{1}{G(j\omega)} = -\angle G(j\omega).$$

(12)

As we just see, one of the big advantage of the Bode-plot is that we can sketch it easily using knowledge of the asymptotes (which are always $(j\omega T)^n$). Then we can make small adjustment if needed at the corner frequency -

⑤ Second-order factors

$$\frac{1}{1 + 2f \left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2}$$

Ex) Second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2f\omega_n s + \omega_n^2}$$

$$\begin{aligned} G(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2f\omega_n(j\omega) + \omega_n^2} \\ &= \frac{1}{1 + 2f \frac{j\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2} \end{aligned}$$

Use the idea of asymptotes again:

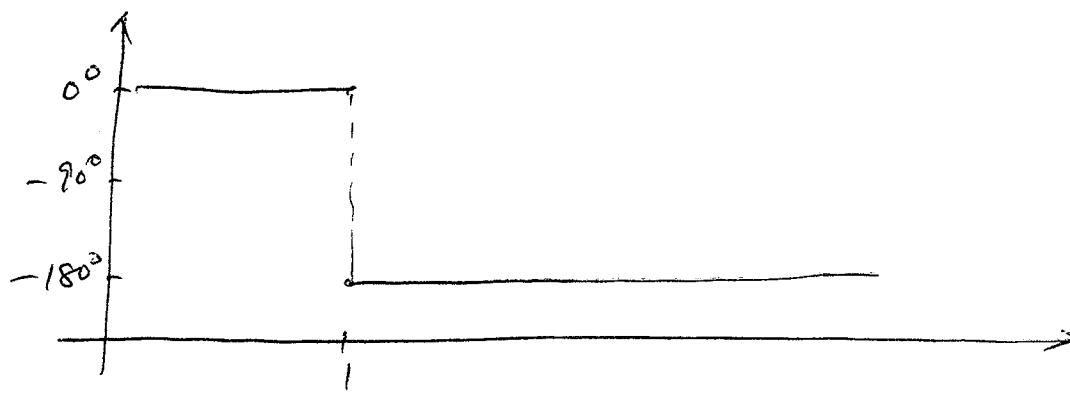
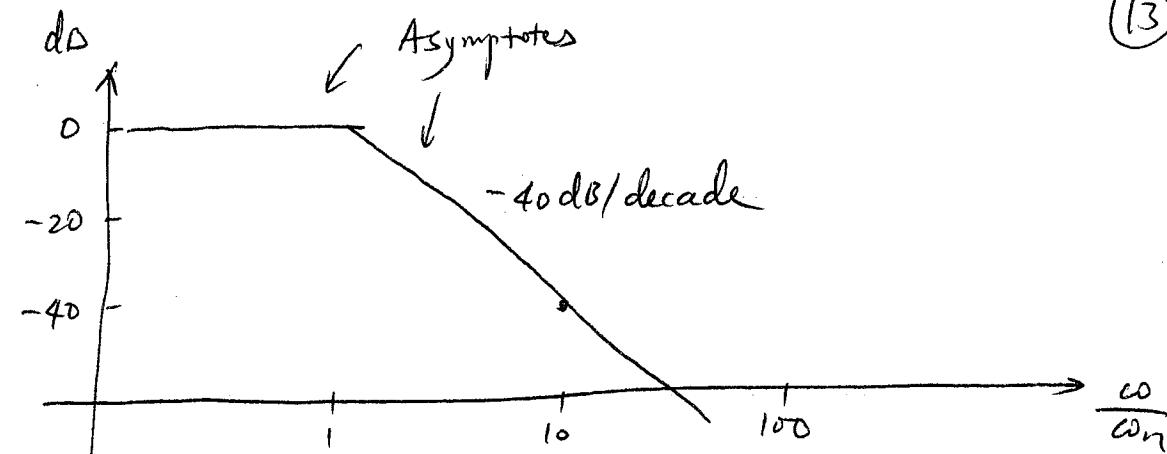
For $\omega \ll \omega_n$, $G(j\omega) \approx 1$

For $\omega \gg \omega_n$, $G(j\omega) \approx \left(\frac{j\omega}{\omega_n}\right)^{-2}$

The corner frequency is ω_n . At lower than ω_n , the magnitude plot is flat. At higher than ω_n , the magnitude decreases 40dB per decade.

The angle changes from 0° to -180°

(13)



Note: The asymptotes are independent of ζ
 The asymptote of the magnitude plot is at 0dB
 at the corner frequency.

Let us adjust the Bode-plot at the corner-frequency.

At $\omega = \omega_n$

$$|G(j\omega)| = \left| \frac{1}{1 + 2\zeta j + j^2} \right| = \frac{1}{2\zeta}$$

$$\angle G(j\omega) = \angle \frac{1}{2\zeta j} = -90^\circ$$

When $\zeta < 0.5$, the magnitude plot will be above 0dB
 at the corner-frequency. As $\zeta \rightarrow 0$, the
 magnitude at ω_n will increase to infinity,
 a sign that the system is more and more oscillatory.

We can use MATLAB to plot the Bode-plot for different ξ . See Fig 8-9, p 505 in text.

It turns out that whenever $\xi \leq 0.707$, $|G(j\omega)|$ will have a peak value at some frequency close to ω_n .

We call this peak frequency as the "resonant freq." As $\xi \rightarrow 0$, the magnitude of the resonant peak goes to infinity.

When $\xi > 0.707$, there is no resonant peak. The magnitude-plot will decrease monotonically to zero. (Recall that step response exhibits little oscillation when $\xi > 0.707$).

Plotting the Bode-plot for general transfer functions

Step 1 : factor out $G(j\omega)$.

Normalize the constant of each factor to be 1.

$$\text{Ex) } G(s) = \frac{10(s+3)}{s(s+2)(s^2+s+2)}$$

$$\begin{aligned} G(j\omega) &= \frac{10(j\omega+3)}{j\omega(j\omega+2)(j\omega^2+j\omega+2)} \\ &= \frac{10 \times 3}{2 \times 2} \frac{(1 + \frac{j\omega}{3})}{j\omega(1 + \frac{j\omega}{2})(1 + \frac{j\omega}{2} + (\frac{j\omega}{2})^2)} \\ &\quad \parallel 7.5 \end{aligned}$$

Step 2 : draw the Bode-plot for each factor
 $7.5, 1 + \frac{j\omega}{3}, \frac{1}{j\omega}, 1 + \frac{j\omega}{2}, \frac{1}{1 + \frac{j\omega}{2} + (\frac{j\omega}{2})^2}$

Add them together to obtain the overall Bode-plot.

Two Approaches for step 2 :

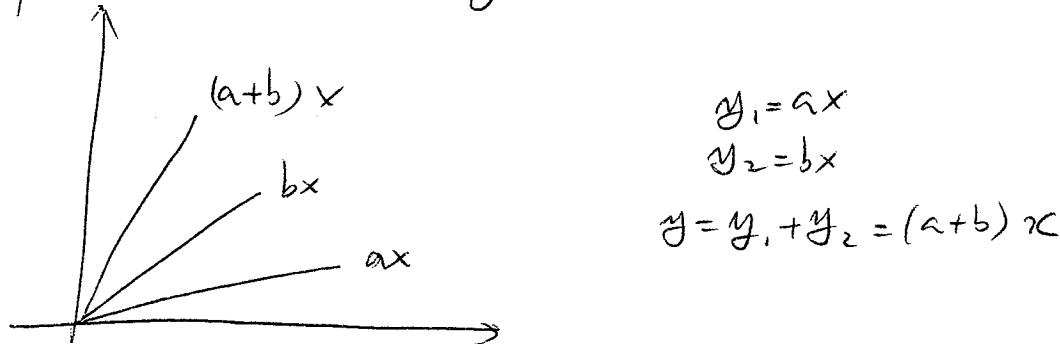
Approach 1 : P 507-509

(16)

Approach 2: Focus on asymptotes

Key: understand how to add asymptotes together

- ① On magnitude-plot, asymptotes are segments of straight lines that change slope at corner frequencies.
- ② When two straight lines add up, the sum is also a straight line and the slope of the sum straight line is the sum of the slopes of individual straight lines



① + ② \Rightarrow The sum of the asymptotes are also segments of straight lines that change slope at each corner frequency.

Approach 2:

- ① Sort the factors by their corner frequency

$$\text{Ex) } H(s) = \frac{1}{s} \cdot \frac{s+1}{s+10}$$

$$\begin{aligned} H(j\omega) &= \frac{1}{j\omega} \cdot \frac{j\omega+1}{j\omega+10} \\ &= \frac{1}{10} \cdot \frac{1}{j\omega} \cdot \frac{1+j\omega}{1+\frac{j\omega}{10}} \end{aligned}$$

Factors are

$$\frac{1}{10}, \frac{1}{j\omega}, 1+j\omega, \frac{1}{1+\frac{j\omega}{10}}$$

corner freq.	N/A	N/A	1	10.
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- ② Start from very small frequency ω . Write down the slope and angle of the asymptote
- ③ As ω increases, whenever ω crosses a corner-frequency, add the slope of the corresponding factor-term to the slope of the magnitude plot. Similarly, add the angle of the corresponding factor-term to the phase plot.

Worksheet for Bode-plot

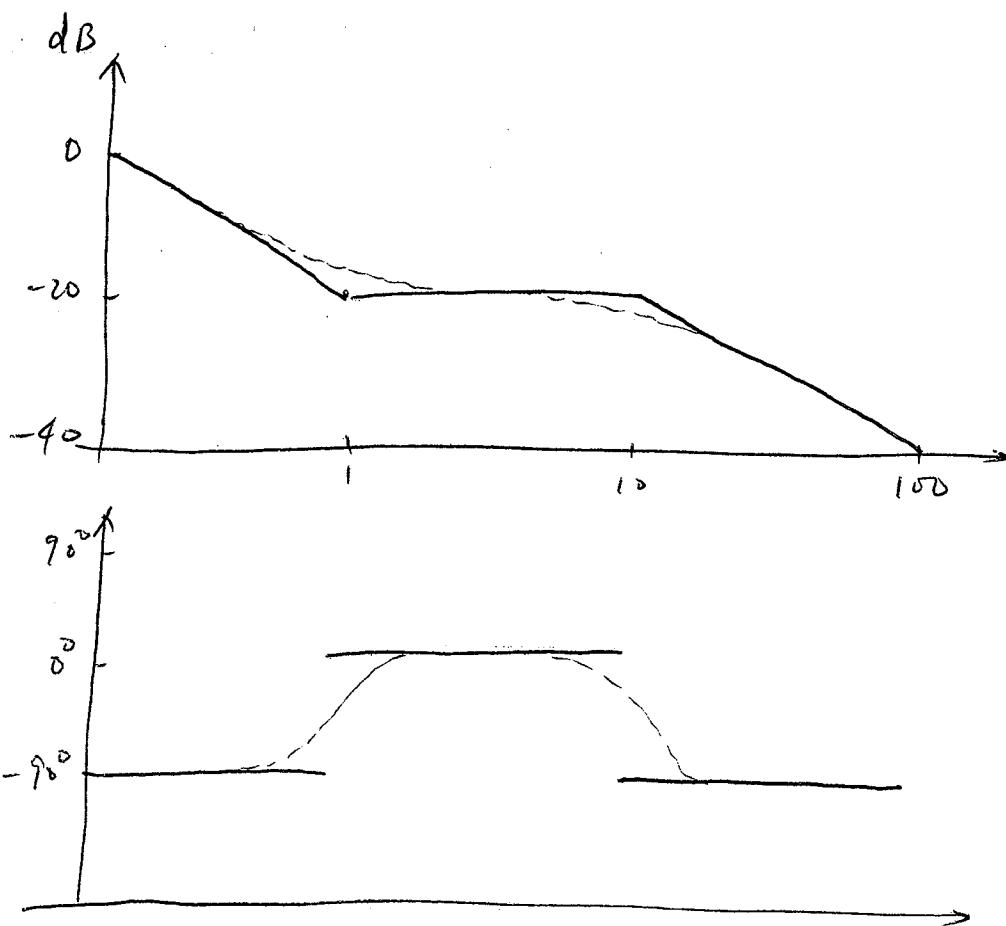
Factors are

	$\frac{1}{\omega}$	$\frac{1}{j\omega}$	$1+j\omega$	$\frac{1}{1+j\frac{\omega}{\omega_c}}$	sum
corner freq	N/A	N/A	1	ω_c	
$\omega < 1$	slope angle	0dB/dec 0°	-20dB/dec -90°	0 0	-20dB/dec -90°
$1 < \omega < \omega_c$	slope angle	0dB/dec 0°	-20dB/dec -90°	20dB/dec 90°	0dB/dec 0°
$\omega > \omega_c$	slope angle	0dB/dec 0°	-20dB/dec -90°	20dB/dec 90°	-20dB/dec -90°

↑
simply copy from the
row above

As ω increases, whenever ω crosses
a corner-freq., add the slope/angle
of the corresponding factor term

(19)



- ④ make adjustments at the corner frequency if necessary.

20

$$\text{Ex) } G(s) = \frac{10(s+3)}{s(s+2)(s^2+s+2)} \quad 1508$$

$$\text{Step 1. } G(j\omega) = 7.5 \frac{1 + \frac{j\omega}{3}}{j\omega(1 + j\frac{\omega}{2})(1 + j\frac{\omega}{2} + (\frac{j\omega}{T_2})^2)}$$

Step 2. Add Bode-plots

Factors are

	7.5	$\frac{1}{j\omega}$	$\frac{1}{1 + j\frac{\omega}{2} + (\frac{j\omega}{T_2})^2}$	$\frac{1}{1 + j\frac{\omega}{2}}$	$1 + j\frac{\omega}{3}$	Sum
corner freq.	N/A	N/A	T_2	2	3	

$\omega < T_2$

slope	0 dB/dec	-20 dB/dec	0	0	0	-20 dB/dec
angle	0	-90°	0	0	0	-90°

$T_2 < \omega < 2$

slope	↓	↓	-40 dB/dec	0	0	-60 dB/dec
angle			-180°	0	0	-270°

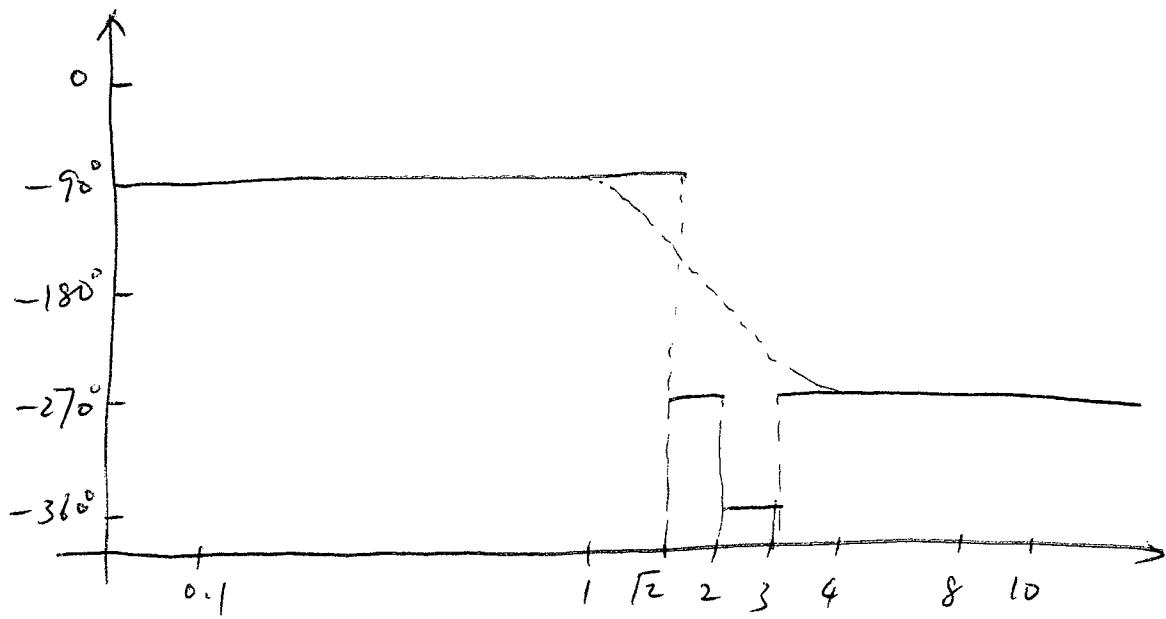
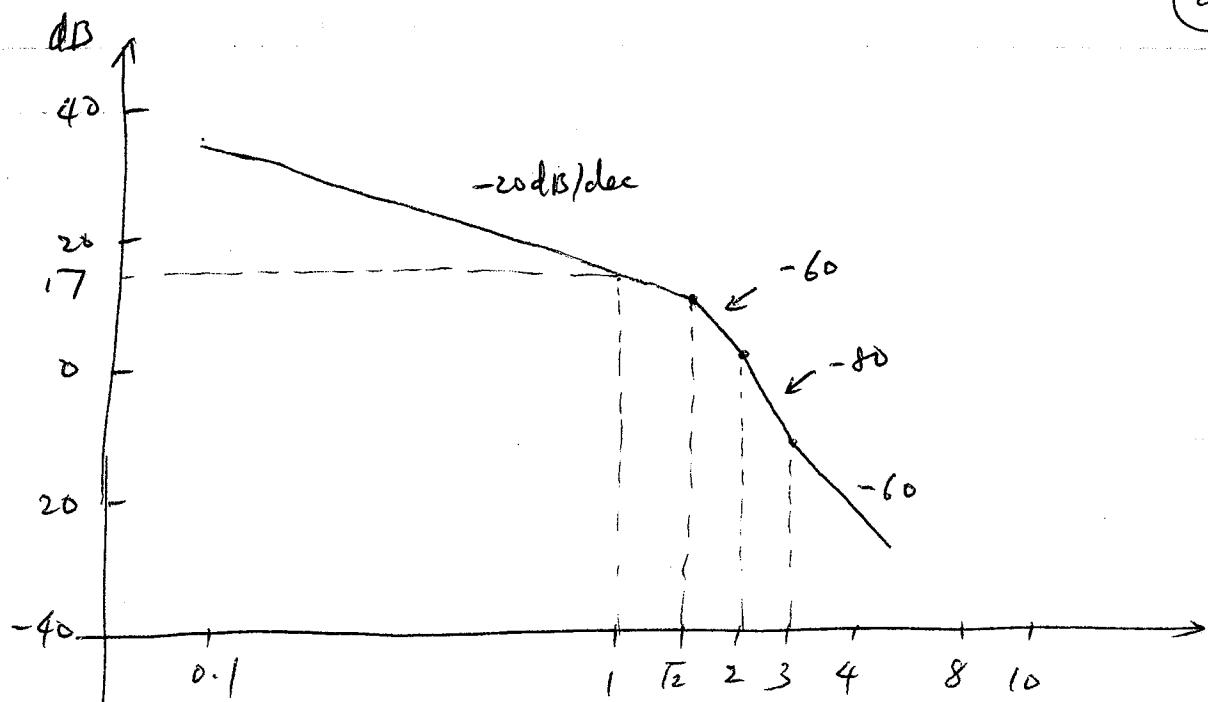
$2 < \omega < 3$

slope	↓	↓	↓	-20 dB/dec	0	-80 dB/dec
angle				-90°	0	-360°

$\omega > 3$

slope	↓	↓	↓	↓	20 dB/dec	-60 dB/dec
angle					90°	-270°

(21)



What can we tell from the Bode-plot?

- ① Steady-state error constants p 513 - 515

Recall that a system is of Type-N if it has N poles at the origin

$$G(s) = \frac{K (T_1 s + 1)(T_2 s + 1) \cdots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)}$$

$N=0$: Type-0, track step-input, $K_p = K$

$N=1$: Type-1, track ramp-input, $K_v = K$

$N=2$: Type-2, track acceleration-input, $K_a = K$.

On the Bode-plot of $G(j\omega)$, when $\omega \rightarrow 0$, the asymptotes become

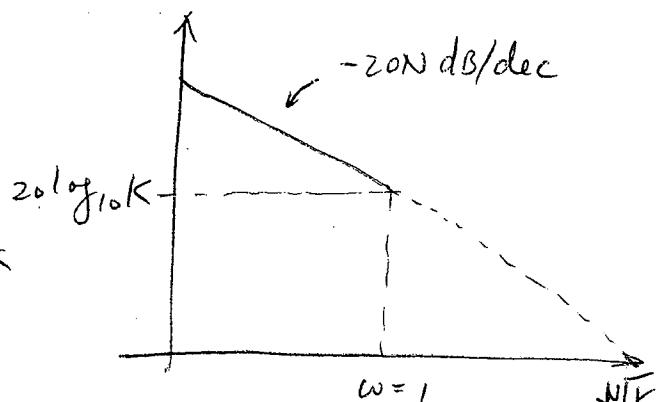
$$\frac{K}{(j\omega)^N}$$

slope: $-20N \text{ dB/dec}$

intersection with $\omega=1$: $20 \log_{10} K$

intersection with 0dB line:

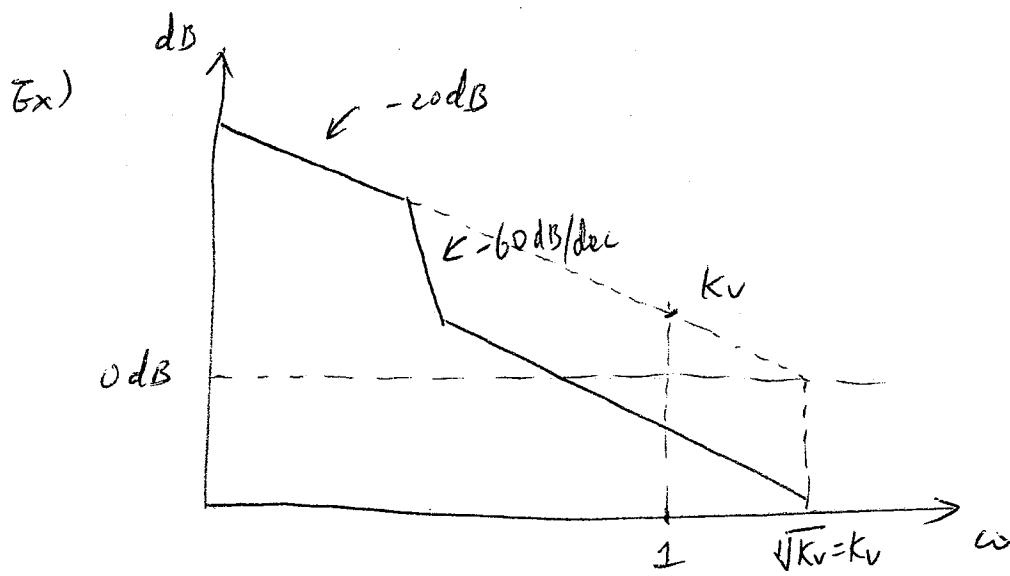
$$\begin{aligned} \frac{K}{\omega^N} &= 1 \\ \Rightarrow \omega &= \sqrt[N]{K} \end{aligned}$$



We can therefore determine the Type & steady-state error-constant from the low-frequency range of the Bode-plot.

(23)

Type	Magnitude plot at low freq	SS error constant
0	0 dB/dec	$K_p = \text{intersection with } \omega=1$
1	-20 dB/dec	K_v
2	-40 dB/dec	K_a



Type-1 system.

K_v = intersection (of extension) with $\omega=1$
 = intersection with 0dB line

- ② Let q be the degree of the denominator , $q = N + P$
 m be the degree of the numerator

As $\omega \rightarrow +\infty$, the asymptotes become

$$\frac{K'}{(j\omega)^{q-m}}$$

$$\Rightarrow -20(q-m) \text{ dB/dec}$$

How about the phase-plot?

$$G(s) = \frac{K(T_1 s + 1)(T_2 s + 1) \dots (T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1) \dots (T_p s + 1)}$$

$$\text{As } \omega \rightarrow 0, \quad \angle G(j\omega) = \angle \frac{K}{(j\omega)^N} = -90^\circ N + 2K$$

As $\omega \rightarrow +\infty$, Assuming all poles & zeros are on LHP

$$\angle G(j\omega) = \angle \frac{K}{(j\omega)^{p-m}} = -90^\circ(p-m) + 2K$$

Therefore, as ω from 0 to $+\infty$, the phase change is
 $-90^\circ(p-m) - (-90^\circ N)$.

Systems whose transfer function has neither poles nor zeros on RHP is called minimum-phase systems

$$\text{Ex)} \quad \frac{1+j\omega T_1}{1+j\omega T}$$

Non-minimum phase systems have either poles or zeros in RHP.

$$\text{Ex)} \quad \frac{1-j\omega T_1}{1+j\omega T}$$

Minimum-phase systems

- has the minimum amount of phase change as ω varies from 0 to $+\infty$, among all systems have the same magnitude-plot.

Note: If $+ \rightarrow -$, magnitude plot does not change phase-plot will.

- the transfer function can be uniquely determined from the magnitude-plot alone.
- typically has better (faster) transient response.

Ex) We may use MATLAB to compare minimum-phase systems and non-minimum-phase systems

$$\text{num} = [1 \ 1]$$

$$\text{den} = [1 \ 1 \ 1]$$

bode (num, den)

step (num, den)

$$\text{num2} = [1 \ -1]$$

bode (num2, den)

step (num2, den)