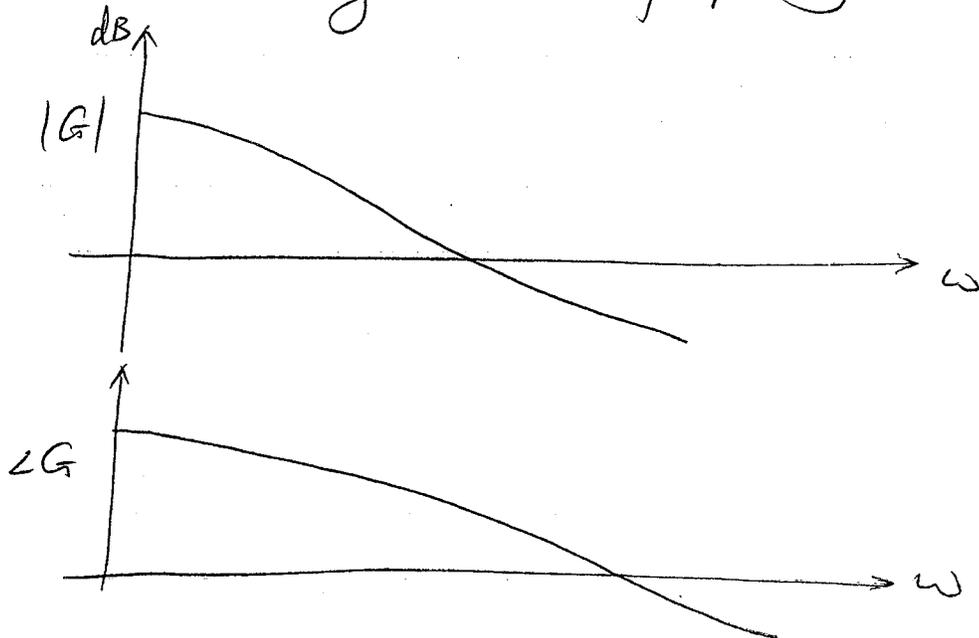


Gain-Margin & Phase Margin P562-565

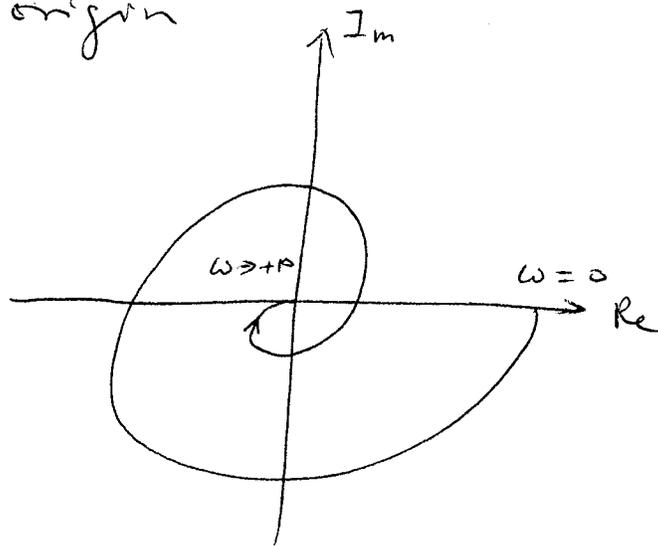
Many "typical" plants are stable and have frequency responses with both magnitude and phase decreasing as the frequency increases



The corresponding Nyquist-plot will "spirals into" the origin

Since the plant is stable
 $\Rightarrow P = 0$

According to the Nyquist stability criterion, the closed-loop system is stable if and only if the Nyquist plot does not encircle the point $-1/k$



(2)

Assume that the Nyquist-plot first intersects the negative-real axis at $-\frac{1}{K_{max}}$. Then the system is stable for

$$0 \leq K < K_{max}$$

If K_{max} is large, there is a larger "margin" to vary the gain K while still keeping the closed-loop system stable. That is why we refer to K_{max} as the gain margin.

Gain Margin

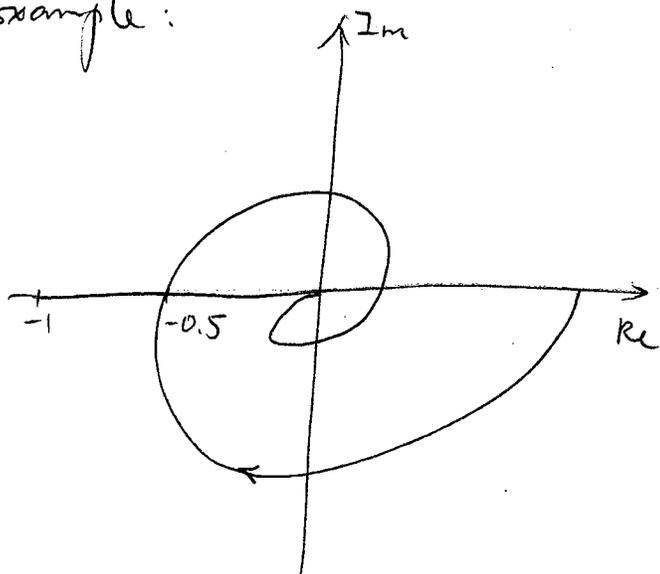
For such a "typical" plant, the gain margin is defined as follows:

- ① Find the frequency $\tilde{\omega}$ for which $\angle G(j\tilde{\omega})$ first crosses 180°
- ② Find $|G(j\tilde{\omega})|$. Then $\frac{1}{|G(j\tilde{\omega})|}$ is called the gain margin (GM).

For proportional-feedback control systems, the GM is the value by which the gain K can be increased from $K=1$ before instability occurs.

3

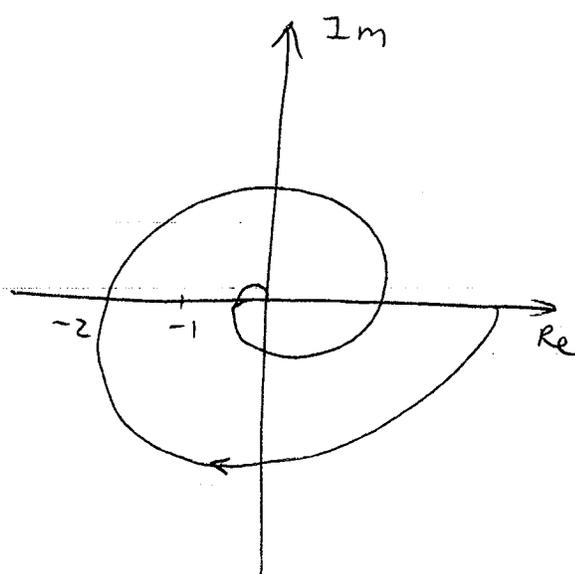
Example:



$$GM = \frac{1}{0.5} = 2$$

System stable with $K=1$

$$\Leftrightarrow GM > 1$$



$$GM = \frac{1}{2} = 0.5$$

System unstable with $K=1$

$$\Leftrightarrow GM < 1$$

Phase Margin

For a "typical" plant, the phase margin is defined as follows:

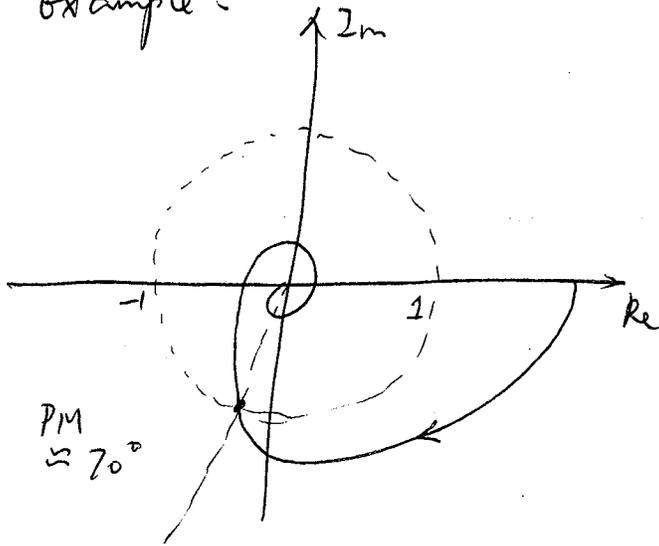
- ① Find the frequency $\tilde{\omega}$ for which $|G(j\omega)|$ first crosses the unit circle $|G(j\tilde{\omega})|=1$.
- ② Find $\angle G(j\tilde{\omega})$, with value in $(-360^\circ, 0^\circ]$.

Then the phase margin

$$PM = 180^\circ + \angle G(j\tilde{\omega}).$$

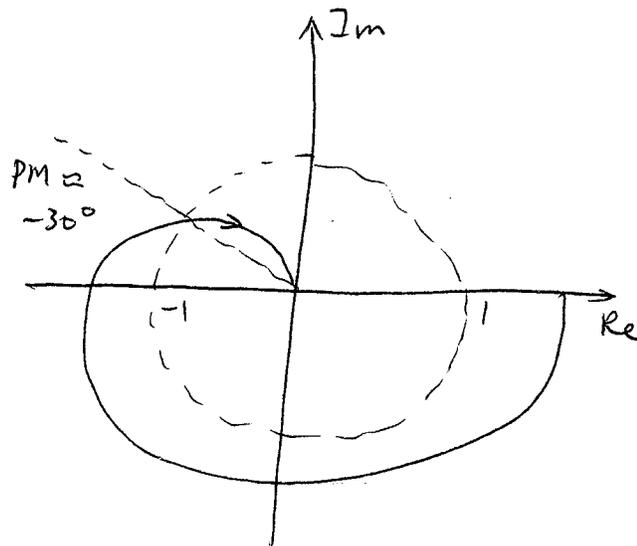
(4)

Example:



System stable with $K=1$

$$\Leftrightarrow PM > 0$$



System unstable with

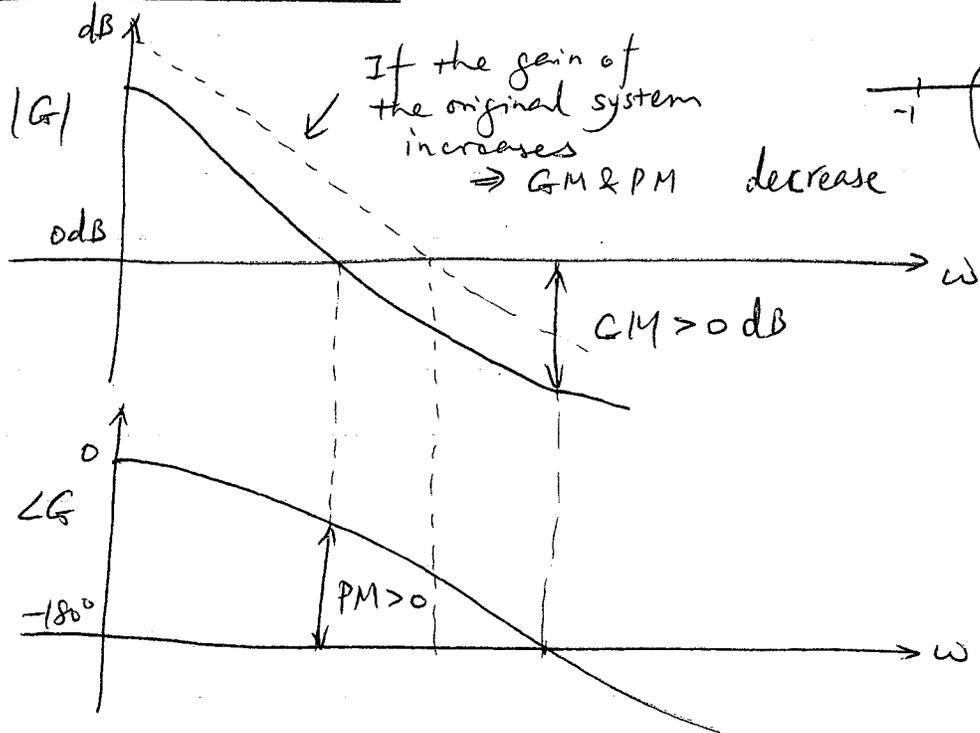
$$K=1 \Leftrightarrow PM < 0$$

Summary: Both the gain margin & phase margin characterize the "tolerance" before the system becomes unstable

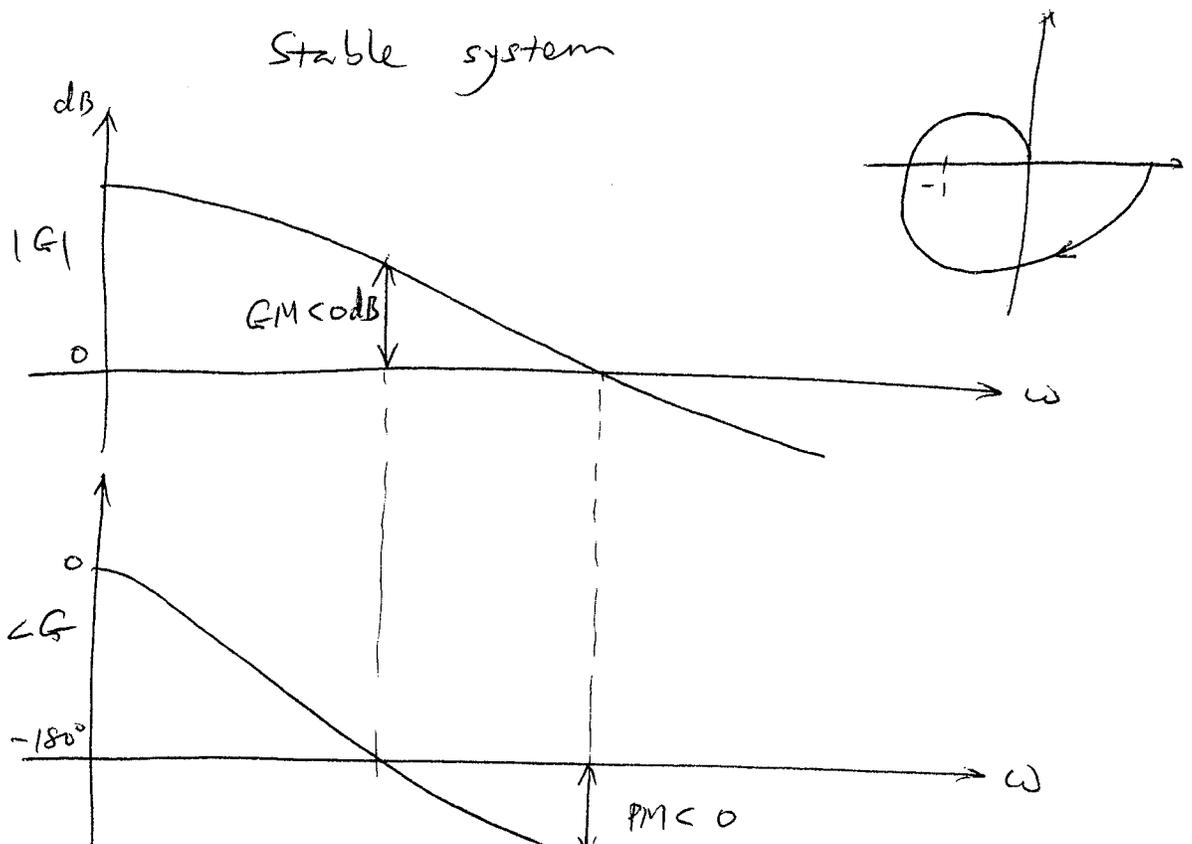
Gain margin: tolerance for additional gain
Phase margin: tolerance for additional phase-lag.

For satisfactory transient-performance, the phase margin should be between 30° to 60° ; the gain margin should be greater than 6dB (2 times).

Reading Gain-margin and Phase-margin from Bode-plots



Stable system



unstable system.

⑥

What if the gain of the original system is increased? The magnitude-plot moves up (or, equivalently, the "0dB" line moves down
 \Rightarrow Both GM & PM become smaller
system closer to instability.

Later in the design process, when we make changes to the system, we are often asked the following question:

- What will be the phase margin if I choose a particular gain K ?
- or, conversely, what will be the corresponding gain K if I want the phase-margin to be a particular value ϕ ?

① Find the phase margin with a given gain K .

- Draw a horizontal line corresponding to $|G(j\omega)| = \frac{1}{K}$. (This is the "0dB" line when the gain is K .)
- Find the intersection of this line with the Bode plot (magnitude plot)
let the corresponding freq be $\tilde{\omega}$
- $PM = 180^\circ + \angle G(j\tilde{\omega})$

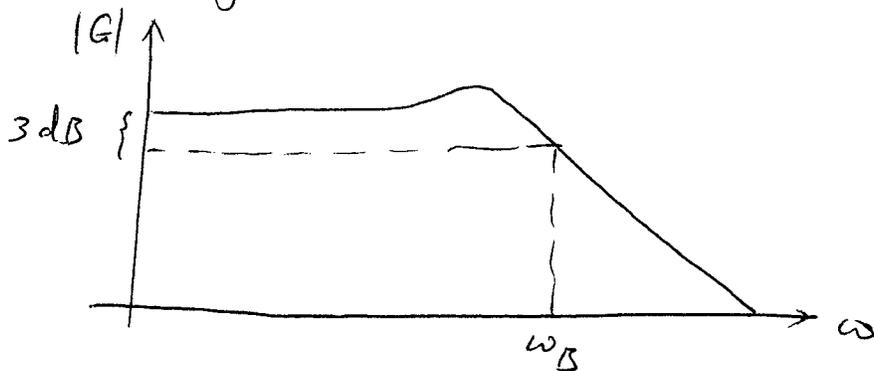
- ② Find the gain that corresponds to a given phase margin ϕ
- Find the frequency $\tilde{\omega}$ with the desired phase-margin ϕ (i.e. $\angle G(j\tilde{\omega}) = \phi - 180^\circ$)
 - Read $|G(j\tilde{\omega})|$ from the magnitude-plot then

$$K = \frac{1}{|G(j\tilde{\omega})|}$$

(This is the gain to make $\tilde{\omega}$ the intersection point with the new "0dB" line)

Bandwidth

For "low-pass" transfer function (i.e. TF that becomes small as $\omega \rightarrow +\infty$), we define the bandwidth as the lowest frequency when the magnitude is 3dB smaller than the DC-magnitude.



* For signals varying much slower than $\cos \omega_B t$, system acts approximately as a gain of $|G(0)|$

8

* Faster-varying signals are attenuated.

Frequency-domain Design

Recall that in time-domain design (e.g. the root-locus method), the key to achieve the desired transient response is to place the dominant closed-loop pole $- \zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}$ at the desired location

Desired Transient Response

Fast rise
Quick settling
Small overshoot
Stability

Close-loop poles

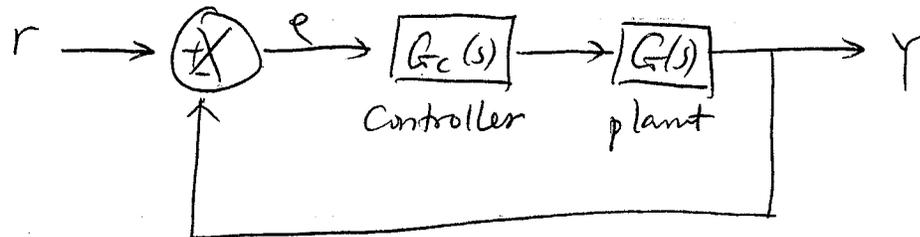
} large ω_n
 $\zeta \in (0.5, 0.7)$
pole on RHP

Also, adjust the DC-gain to achieve the desired steady-state performance.

In frequency-domain design, the key is to "shape" the frequency-response of the open-loop transfer function.

The relationship between design objectives and the frequency-domain specs

Consider a standard closed-loop system



$$L(s) \triangleq G_c(s) G(s)$$

loop-gain (open-loop transfer function).

$$\frac{Y(s)}{R(s)} = \frac{L(s)}{1 + L(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + L(s)}$$

For frequencies where the loop gain $L(s)$ is large

$$Y(s) \approx R(s), \quad E(s) \approx 0$$

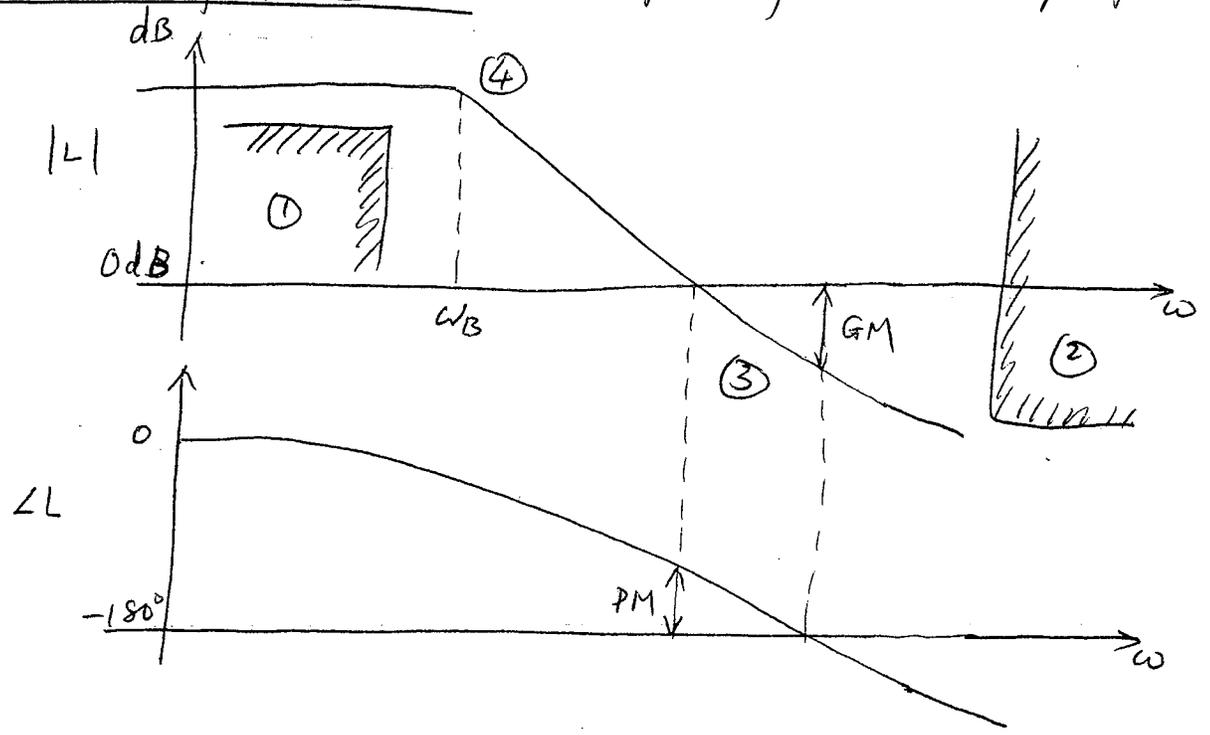
Large loop-gain \Rightarrow Good tracking

For frequencies where the loop-gain $L(s)$ is small

$$Y(s) \approx 0, \quad E(s) \approx R(s)$$

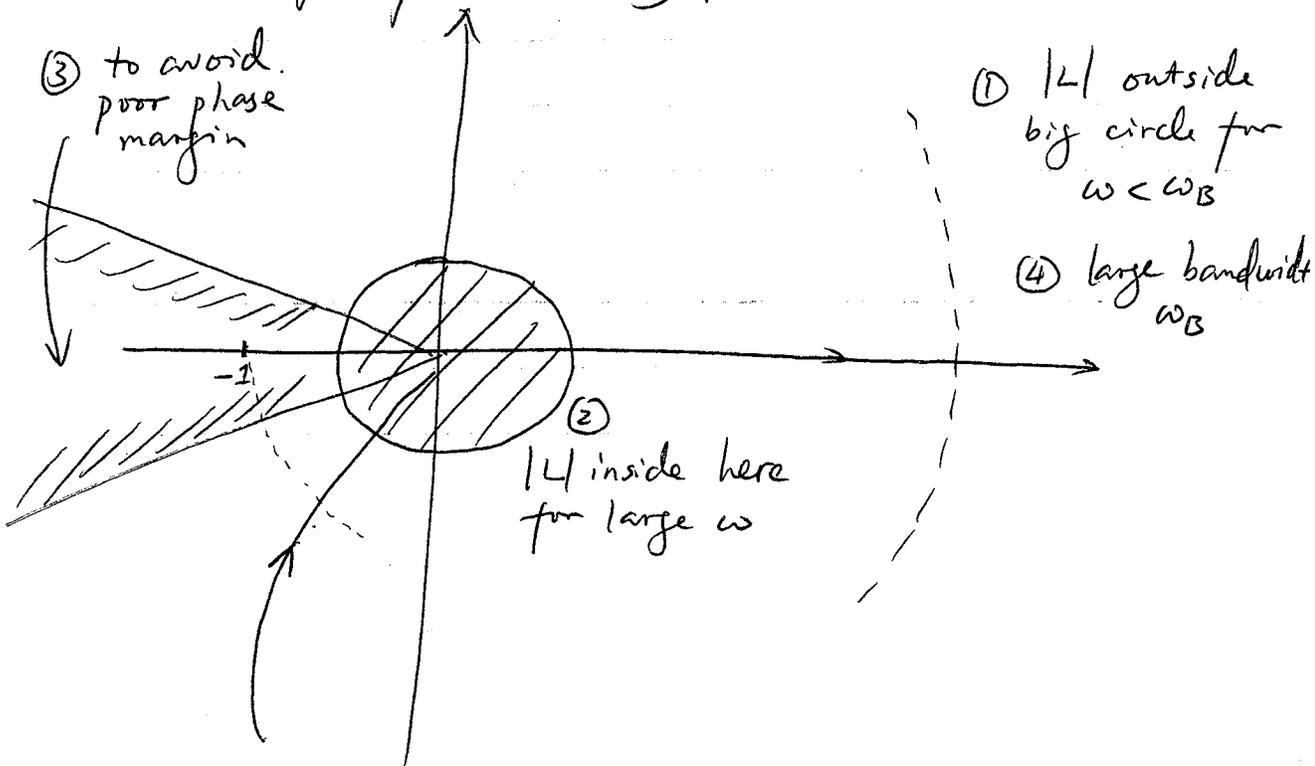
Small loop-gain \Rightarrow Good rejection

Desired frequency-domain specs for better perf.



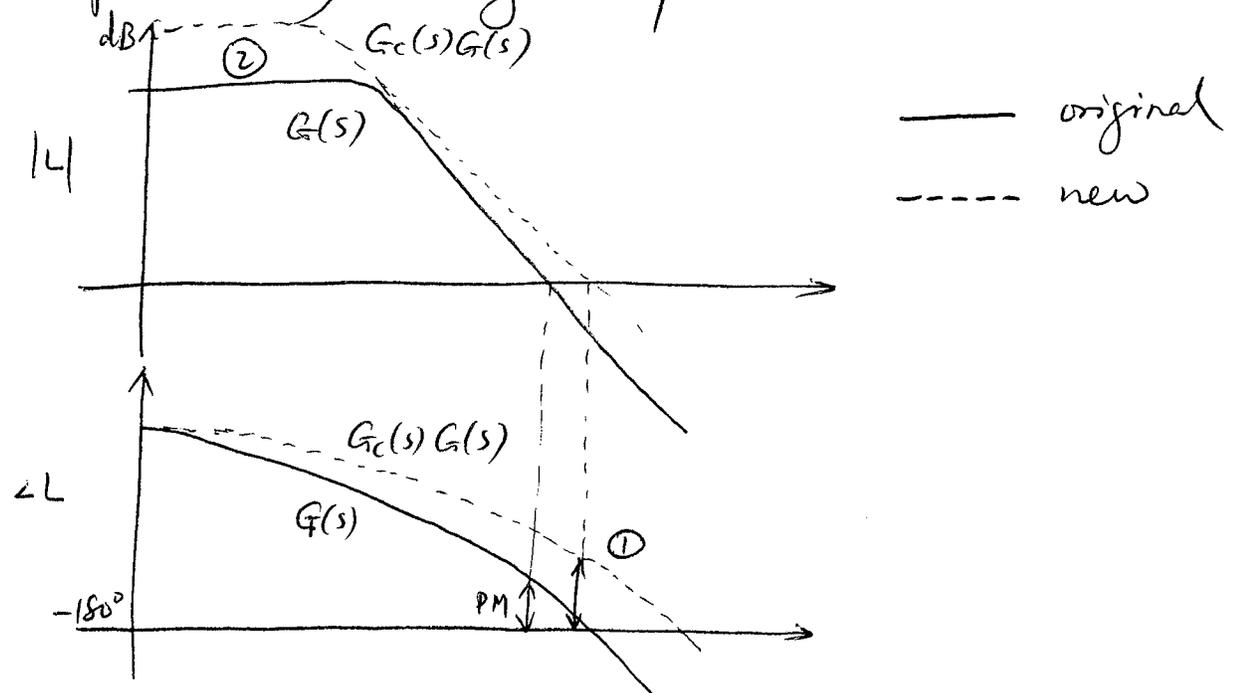
- ① Large loop-gain at low-frequency
 \Rightarrow good tracking, small steady-state error
- ② small loop-gain at high frequency
 \Rightarrow rejection of high frequency noise (disturbance)
- ③ High phase margin } \Rightarrow robustness
 High gain margin } stability
 small overshoot
- ④ High bandwidth ω_B
 \Rightarrow larger tracking bandwidth
 fast rise, quick settling

Desired shape for the Nyquist-plot



Basic idea behind frequency-domain design

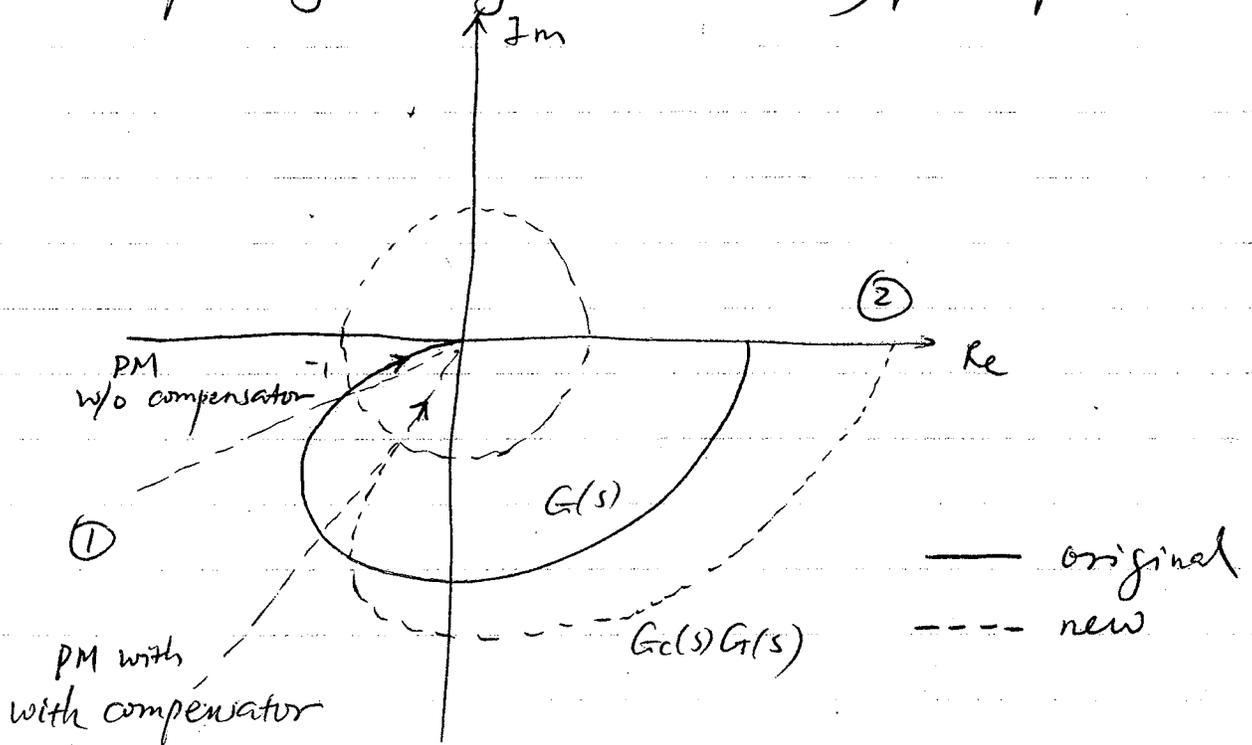
"Shape" L by adding compensator $G_c(s)$



① Increase PM

② Increase low-freq gain

Corresponding changes on the Nyquist plot



Basic Compensators

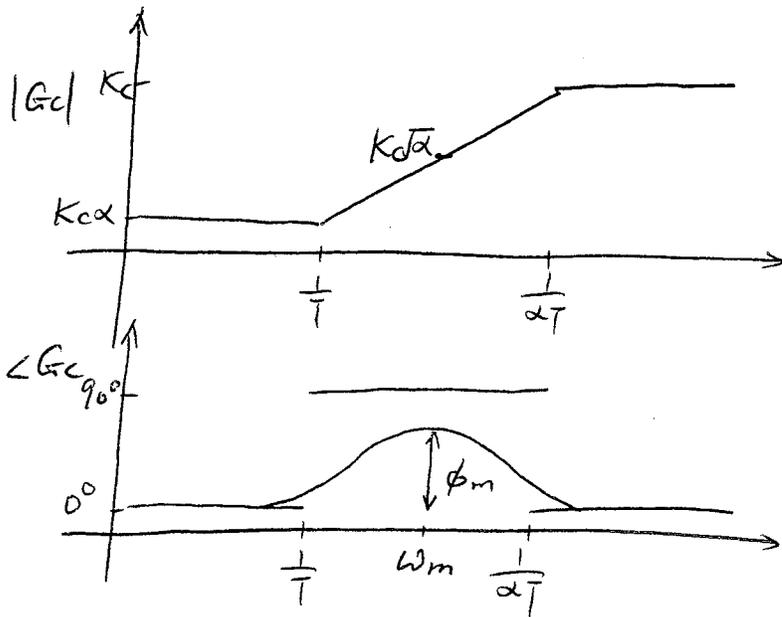
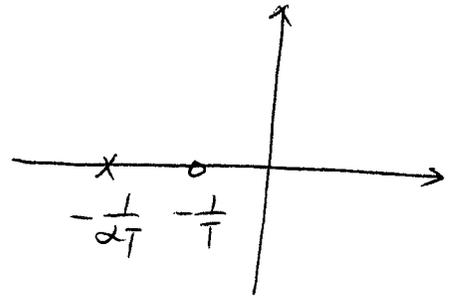
Lead-compensator P621

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

$$0 < \alpha < 1$$

The frequency-response is

$$G_c(j\omega) = K_c \alpha \frac{1 + j\omega T}{1 + j\alpha\omega T}$$



Lead compensator

- * is a high-pass filter
- * adds a phase lead

ϕ_m : maximum phase lead

$$\sin \phi_m = \frac{1 - \alpha}{1 + \alpha}$$

$$\alpha \downarrow \quad \phi_m \uparrow$$

However, in practice α cannot be too small.

$$\alpha > 0.05 \Rightarrow \phi_m < 65^\circ$$

ω_m : frequency for maximum phase-lead

$$\omega_m = \frac{1}{\sqrt{\alpha} T} \quad \text{geometric mean of the two corner freq.}$$

The gain at ω_m is

$$K_c \alpha \left| \frac{1 + j\omega T}{1 + j\alpha\omega T} \right|_{\omega = \frac{1}{\sqrt{\alpha} T}} = K_c \sqrt{\alpha}$$

- geometric mean of the low- and high-frequency gain
- mid point in the magnitude-plot

Use of Lead-compensator:

- * Improve phase-margin by adding phase
- * Move the 0dB cross-over freq to the right

Advantage:

- * Increase phase-margin
- * Increase Bandwidth.

Disadvantage:

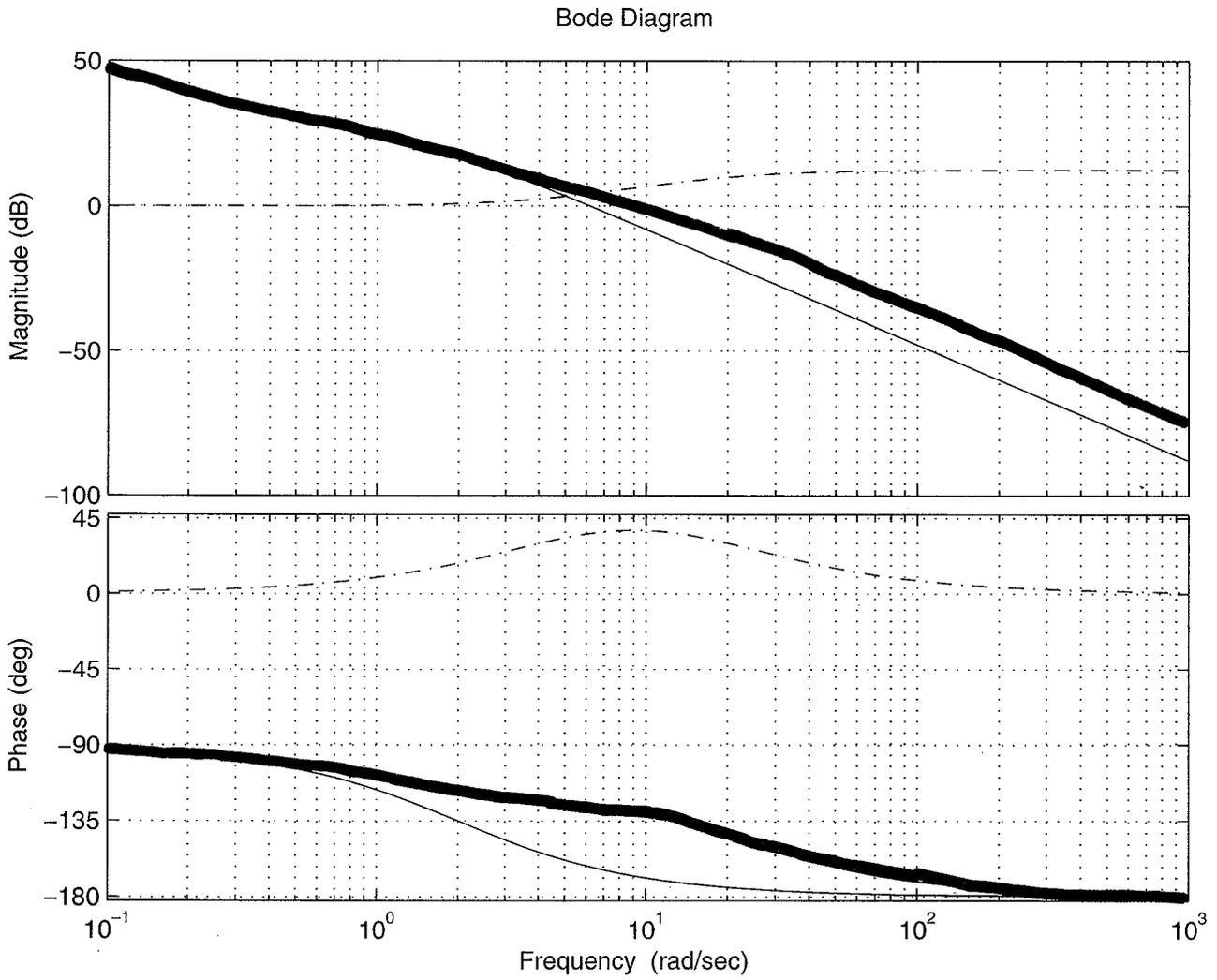
- * Increase high-frequency gain (susceptible to noise)
- * Require additional gain K_c to offset the attenuation α at low-frequency

$$\underline{\underline{K_c \alpha}} \cdot \frac{1+j\omega T}{1+j\omega \alpha T}$$

(Large $K_c \rightarrow$ Larger force, larger cost)

Curve-shaping with a lead-compensator

(18)

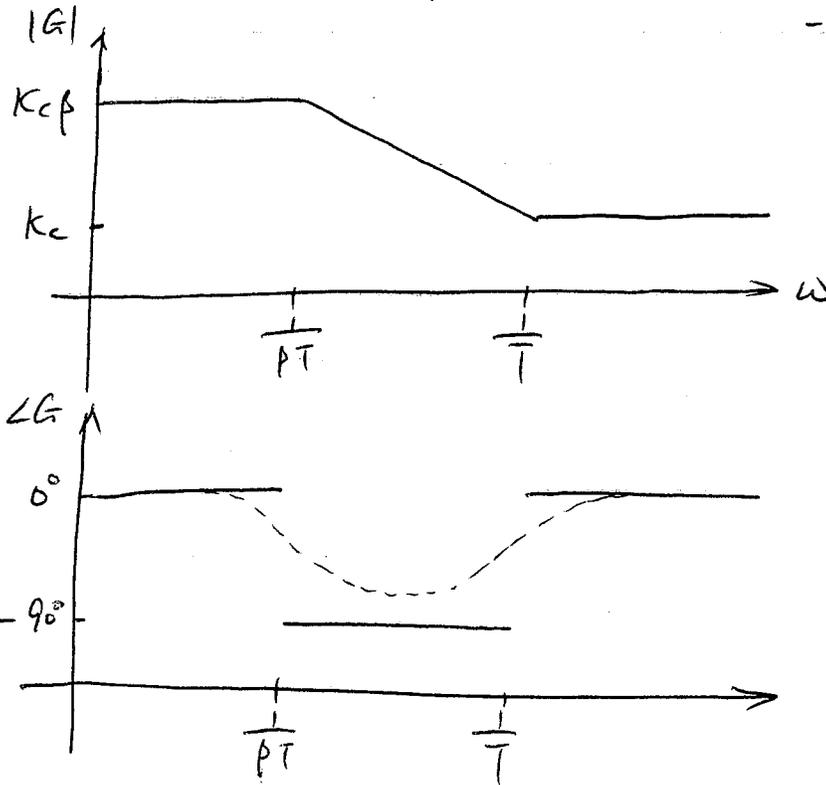
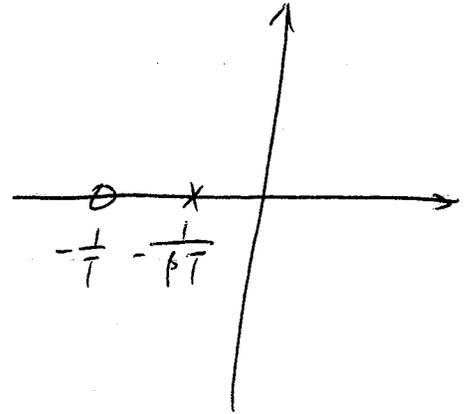


Lag-Compensator P631

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \cdot \frac{1 + sT}{1 + s\beta T}, \beta > 1$$

The frequency-response is

$$G_c(j\omega) = K_c \beta \cdot \frac{1 + j\omega T}{1 + j\omega \beta T}$$



Lag-compensator

- * is a low-pass filter
- * add phase-lag (may be detrimental)

Use of Lag-compensator:

- * Increase the phase-margin by moving the 0dB cross-over freq to the left
- * Use the attenuation at high freq to move the 0dB cross-over freq to the left
- * As in root-locus method, we would like the corner-freq of the lag-compensator to be much smaller than the 0dB cross-over freq, in order to avoid the negative effects of the phase-lag.

Advantage:

- * Reduce high-freq gain (rejecting noise)
- * Require less-gain at low freq (due to the increased gain at low freq.)

Smaller K_c to reach the same steady-state response.

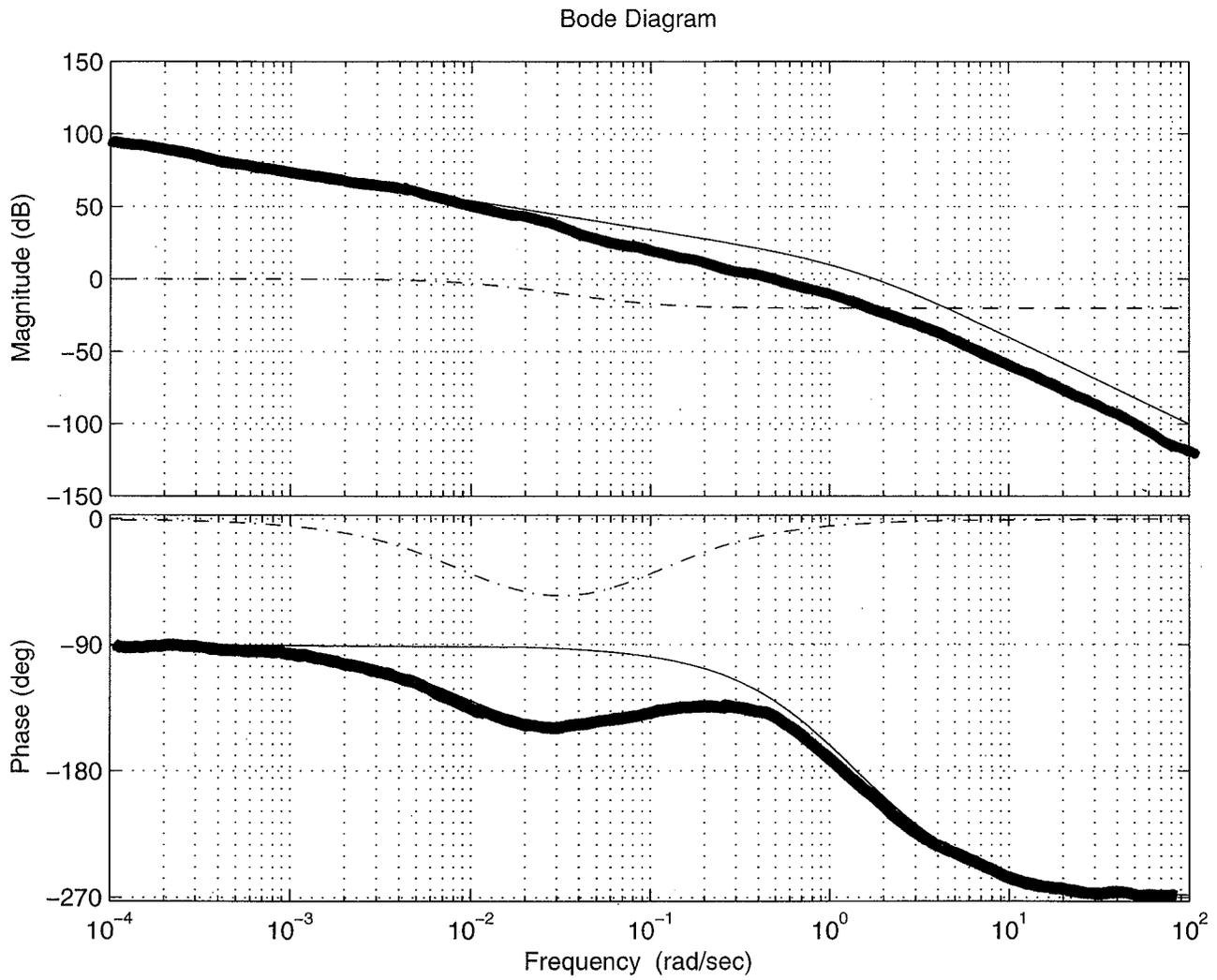
(Smaller $K_c \Rightarrow$ Smaller force/cost)

Disadvantage:

- * Decrease bandwidth \Rightarrow slower response
- * Long tail
(that settles very slowly)

Curve-shaping with a lag-compensator

(21)



Summary:

- * Lead-compensator improves stability by adding phase. It moves the cross-over freq to the right.
- * Lag-compensator improves stability by attenuation. It moves the cross-over freq to the left.

Some system can be solved with either lead-compensator or Lag-compensator.

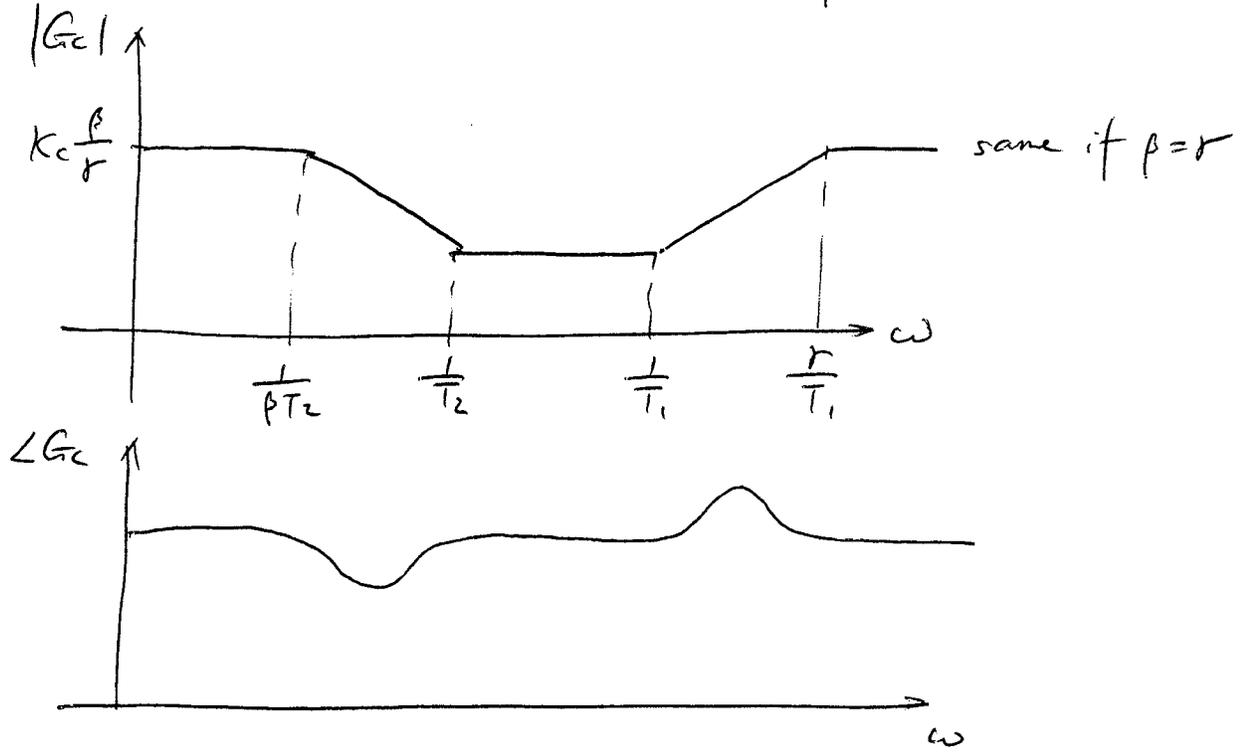
Rules of Thumb:

- * Lead-compensator will typically result into larger bandwidth
- * If the original phase-margin is too small, the required phase-lead can be large ($> 65^\circ$). Then lead-compensator may be ineffective. Need lag-compensator or lag-lead compensator

Lag-lead compensator p639

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)$$

$$= \underbrace{K_c \frac{\beta}{\gamma}}_K \cdot \underbrace{\frac{1 + sT_1}{1 + s\frac{T_1}{\gamma}}}_{\text{Lead } \gamma > 1} \cdot \underbrace{\frac{1 + sT_2}{1 + s\beta T_2}}_{\text{Lag } \beta > 1}$$



- * use the lead-part to add phase
- * use the lag-part to reduce gain at the desired cross-over freq.