Gain-Margin & Phase Margin  P562-565

Many "typical" plants are stable and have frequency responses with both magnitude and phase decreasing as the frequency increases.

The corresponding Nyquist-plot will "spirals into" the origin.

Since the plant is stable
\[ \Rightarrow p = 0 \]
According to the Nyquist stability criterion, the closed-loop system is stable if and only if the Nyquist plot does not encircle the point \(-1\).
Assume that the Nyquist plot first intersects the negative-real axis at $-\frac{1}{K_{\text{max}}}$. Then the system is stable for $0 \leq K < K_{\text{max}}$

If $K_{\text{max}}$ is large, there is a larger "margin" to vary the gain $K$ while still keeping the closed-loop system stable. That is why we refer to $K_{\text{max}}$ as the gain margin.

**Gain Margin**

For such a "typical" plant, the gain margin is defined as follows:
1. Find the frequency $\omega$ for which $\angle G(j\omega)$ first crosses $180^\circ$.
2. Find $|G(j\omega)|$. Then $\frac{1}{|G(j\omega)|}$ is called the gain margin (GM).

For proportional-feedback control systems, the GM is the value by which the gain $K$ can be increased from $K=1$ before instability occurs.
Example:

\[ GM = \frac{1}{0.5} = 2 \]

System stable with \( K = 1 \)
\[ \Leftrightarrow GM > 1 \]

\[ GM = \frac{1}{2} = 0.5 \]

System unstable with \( K = 1 \)
\[ \Leftrightarrow GM < 1 \]

**Phase Margin**

For a "typical" plant, the phase margin is defined as follows:

1. Find the frequency \( \omega \) for which \( |G(j\omega)| \) first crosses the unit circle \( |G(j\omega)| = 1 \).
2. Find \( \angle G(j\omega) \), with value in \((-360^\circ, 0^\circ]\). Then the phase margin
   \[ PM = 180^\circ + \angle G(j\omega) \].
Example:

System stable with $K=1$  \[ \implies \text{PM} > 0 \]

System unstable with $K=1$  \[ \implies \text{PM} < 0 \]

Summary: Both the gain margin & phase margin characterize the "tolerance" before the system becomes unstable.

Gain margin: tolerance for additional gain
Phase margin: tolerance for additional phase-lag.

For satisfactory transient performance, the phase margin should be between $30^\circ$ to $60^\circ$, the gain margin should be greater than $6$ dB (2 times).
Reaching Gain-margm and Phase-margin from Bode-plots

If the gain of the original system increases, GM & PM decrease.

Stable system

Unstable system.
What if the gain of the original system is increased? The magnitude plot moves up (or equivalently, the “0dB” line moves down)
⇒ Both GM & PM become smaller system closer to instability.

Later in the design process, when we make changes to the system, we are often asked the following question:
- What will be the phase margin if I choose a particular gain $K$?
- Or, conversely, what will be the corresponding gain $K$ if I want the phase margin to be a particular value $\phi$?

1. Find the phase margin with a given gain $K$.
   - Draw a horizontal line corresponding to $|G(j\omega)| = \frac{1}{K}$. (This is the “0dB” line when the gain is $K$.)
   - Find the intersection of this line with the Bode plot (magnitude plot)
     let the corresponding freq be $\omega$
   - $PM = 180^\circ + \angle G(j\omega)$
6 Find the gain that corresponds to a given phase margin \( \phi 
\)

- Find the frequency \( \omega \) with the desired phase margin \( \phi \) (i.e. \( \angle G(j\omega) = \phi - 180^\circ \))

- Read \( |G(j\omega)| \) from the magnitude plot

then

\[
K = \frac{1}{|G(j\omega)|}
\]

(This is the gain to make \( \omega \) the intersection point with the new "0 dB" line)

**Bandwidth**

For "low-pass" transfer function (i.e. \( TF \) that becomes small as \( \omega \to \pm \infty \)), we define the bandwidth as the lowest frequency when the magnitude is 3dB smaller than the DC magnitude.

\[
|G|
\]

3dB

\[
\omega_b
\]

*For signals varying much slower than \( \cos(\omega t) \), system acts approximately as a gain of \( |G(0)| \)
Faster-varying signals are attenuated.
Frequency-domain Design

Recall that in time-domain design (e.g. the root-locus method), the key to achieving the desired transient response is to place the dominant closed-loop pole in \( j \omega \) at the desired location.

<table>
<thead>
<tr>
<th>Desired Transient Response</th>
<th>Close-loop poles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fast rise</td>
<td>( \omega ) large</td>
</tr>
<tr>
<td>Quick settling</td>
<td>( \omega \in (0.5, 0.7) )</td>
</tr>
<tr>
<td>Small overshoot</td>
<td>pole on RHP</td>
</tr>
<tr>
<td>Stability</td>
<td></td>
</tr>
</tbody>
</table>

Also, adjust the DC-gain to achieve the desired steady-state performance.

In frequency-domain design, the key is to "shape" the frequency-response of the open-loop transfer function.
The relationship between design objectives and the frequency-domain specs

Consider a standard closed-loop system

\[ r \rightarrow \overset{e}{\bigtriangledown} \rightarrow G_c(s) \rightarrow G(p) \rightarrow y \]

\[ L(s) \triangleq G_c(s) G(p) \]

loop-gain (open-loop transfer function)

\[ \frac{Y(s)}{R(s)} = \frac{L(s)}{1 + L(s)} \]

\[ \frac{E(s)}{R(s)} = \frac{1}{1 + L(s)} \]

For frequencies where the loop gain \( L(s) \) is large

\[ Y(s) \approx R(s), \quad E(s) \approx 0 \]

Large loop-gain \( \Rightarrow \) Good tracking

For frequencies where the loop-gain \( L(s) \) is small

\[ Y(s) \approx 0, \quad E(s) \approx R(s) \]
Small loop gain $\Rightarrow$ Good rejection

Desired frequency-domain specs for better perf.

1. Large loop gain at low frequency $\Rightarrow$ good tracking, small steady-state error
2. Small loop gain at high frequency $\Rightarrow$ rejection of high frequency noise (disturbance)
3. High phase margin $\Rightarrow$ robustness, stability
   High gain margin $\Rightarrow$ small overshoot
4. High bandwidth $\mathrm{WB}$ $\Rightarrow$ larger tracking bandwidth, fast rise, quick settling
Desired shape for the Nyquist plot

1. \|L\| outside big circle for \( \omega < \omega_B \)

2. \|L\| inside here for large \( \omega \)

3. To avoid poor phase margin

Basic idea behind frequency-domain design

"Shape" L by adding compensator \( G_c(s) \)

1. \[ \text{original} \]

2. \[ \text{new} \]
0. Increase PM

2. Increase low-freq gain. Correspondingly changes on the Nyquist plot
Basic Compensators

Lead-compensator P621

\[ G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \quad \quad 0 < \alpha < 1 \]

The frequency response is

\[ G_c(j\omega) = K_c \alpha \frac{1 + j\omega T}{1 + j\omega \alpha T} \]

Lead compensator

* is a high-pass filter
* adds a phase lead

\[ \phi_m : \text{maximum phase lead} \]

\[ \varepsilon - \phi_m = \frac{1 - \alpha}{1 + \alpha} \]

\[ \alpha \downarrow \phi_m \uparrow \]
However, in practice $\alpha$ cannot be too small.

$\alpha > 0.05 \implies \phi_m < 65^\circ$

$\omega_m$: frequency for maximum phase-lead

$\omega_m = \frac{1}{\sqrt{\alpha T}}$

geometric mean of the two corner freq.

The gain at $\omega_m$ is

$K_c \alpha \left| \frac{1 + j\omega T}{1 + j\alpha \omega T} \right| _{\omega = \frac{1}{\sqrt{\alpha T}}} = K_c \alpha$

- geometric mean of the low- and high-frequency gain
- mid point in the magnitude-plot

Use of lead-compensator:
* Improve phase-margin by adding phase
* Move the 0dB cross-over freq to the right

Advantage:
* Increase phase-margin
* Increase Bandwidth
Disadvantages:
* Increase high-frequency gain (susceptible to noise)
* Require additional gain $K_c$ to offset the attenuation $\alpha$ at low-frequency

$$K_c \alpha = \frac{1 + j\omega T}{1 + j\omega T}$$

(Large $K_c$ $\Rightarrow$ larger force, larger cost)
Curve-shaping with a lead-compensator

Bode Diagram

Magnitude (dB)

Phase (deg)

Frequency (rad/sec)

\[10^{-1}, 10^0, 10^1, 10^2, 10^3\]
Lag Compensator

\[ G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \cdot \frac{1 + sT}{1 + s\beta T}, \beta > 1 \]

The frequency response is

\[ G_c(j\omega) = K_c \beta \cdot \frac{1 + j\omega T}{1 + j\beta \omega T} \]

Lag-Compensator

* Is a low-pass filter
* Add phase-lag (may be detrimental)
Use of Lag-compensator:
* Increase the phase-margin by moving the 0dB crossover freq to the left
* Use the attenuation at high freq to move the 0dB crossover freq to the left
* As in root-locus method, we would like the corner freq of the lag-compensator to be much smaller than the 0dB crossover freq, in order to avoid the negative effects of the phase-lag.

Advantage:
* Reduce high-freq gain (rejecting noise)
* Require less gain at low freq (due to the increased gain at low freq.)
  Smaller Kc to reach the same steady-state response.
  (Smaller Kc \Rightarrow Smaller force/cost)

Disadvantage:
* Decrease bandwidth \Rightarrow Slower response
* Long tail (that settles very slowly)
Curve-shaping with a lag-compensator
Summary:

* Lead-compensator improves stability by adding phase. It moves the cross-over freq to the right.
* Lag-compensator improves stability by attenuation. It moves the cross-over freq to the left.

Some system can be solved with either lead-compensator or lag-compensator.

Rules of Thumb:

* Lead-compensator will typically result into larger bandwidth.

* If the original phase-margin is too small, the required phase-lead can be large (>65°). Then lead-compensator may be ineffective. Need lag-compensator or lag-lead compensator.
Lag-lead compensator \( G_c(s) \)

\[
G_c(s) = K_c \left( \frac{s + \frac{1}{T_1}}{s + \frac{1}{T_1}} \right) \left( \frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)
\]

\[
= K_c \frac{\beta}{\delta} \frac{1 + sT_1}{1 + sT_1/\delta} \frac{1 + sT_2}{1 + s\beta T_2}
\]

\( K_c \) \hspace{1cm} \text{Lead} \hspace{1cm} \delta > 1 \hspace{1cm} \text{Lag} \hspace{1cm} \beta > 1

\[|G_c|\]

\[\angle G_c\]

\[\frac{K_c \beta}{\delta}\]

\[\frac{1}{\beta T_2} \quad \frac{1}{T_2} \quad \frac{1}{T_1} \quad \frac{r}{T_1} \rightarrow \omega\]

\[\text{same if } \beta = r\]

* use the lead-part to add phase
* use the lag-part to reduce gain at the desired cross-over freq.