ECE-382: Midterm #2

March 29, 2006
7:00PM—8:00PM

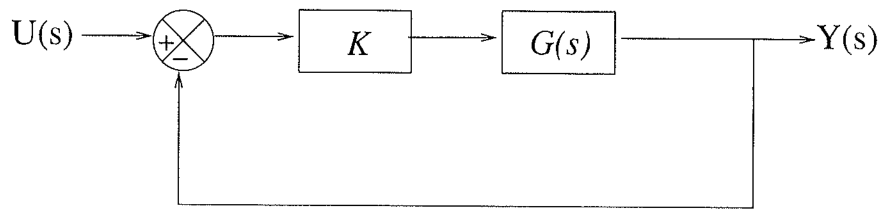
- **Do not write your answers at the back of any page. Answers written at the back of page will NOT be graded.**
- This is a closed book exam. You are only allowed to bring a pen (or pencil), a one-sided crib sheet, and an eraser to the exam. No calculator is allowed.
- Some common sin/cos/tan function values and common square root values are provided at the end of the exam booklet.
- Write your name and PUID at the space provided below.
- There are four problems in the exam. The total points are 150. You have one hour to complete the exam.
- **Show all intermediate steps to get full credits.**

Solution.

Your Name

10-digit PUID

Problem 1 (50 points)



Consider the unit negative feedback system shown above with

$$G(s) = \frac{s+2}{s^2+2s+2}.$$

Answer the following questions (use the blank figure on the next page for marking your answers on the complex plane).

(a) (5 points) Find the open-loop zeros and poles and plot them on the complex plane.

(b) (5 points) Find the segment (or segments) of the real axis that belong to the root locus of $G(s)$. Mark them on the complex plane.

(c) (5 points) For the root locus, find the asymptotes (if any) and the point (if any) that the asymptotes intersect the real axis. Mark them on the complex plane.

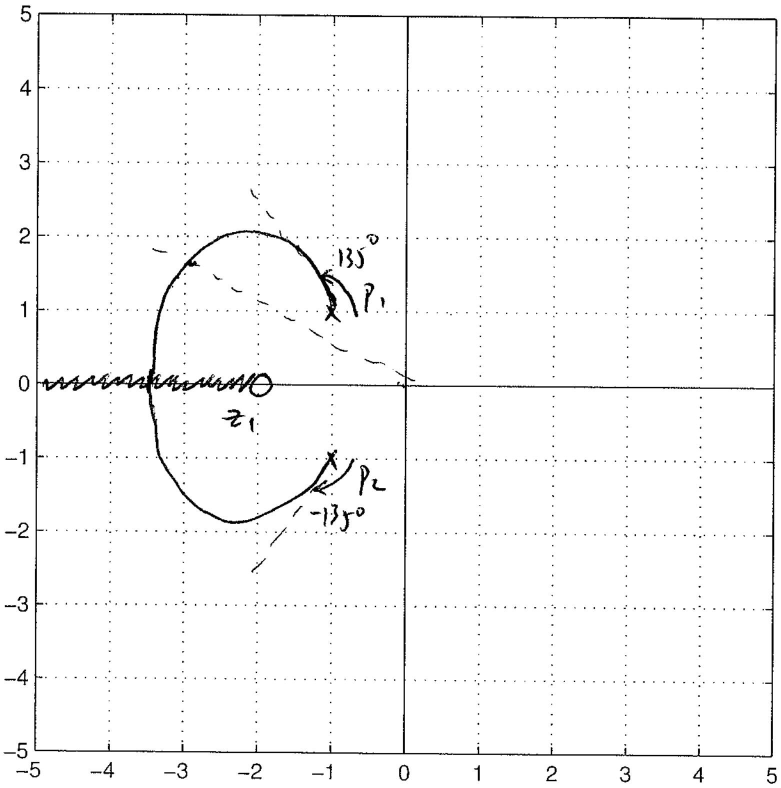
You are given the following information: the root locus DOES NOT cross the imaginary axis.

(d) (10 points) Find the break-in/break-away points (if any) of the root locus, and the corresponding value of K. Mark these points on the complex plane.

(e) (10 points) For the root locus, find the angle of departure from the open loop poles, and the angle of arrival at the open loop zeros.

(f) (10 points) Sketch the complete root-locus on the complex plane.

(g) (5 points) Is it possible to choose $K > 0$ such that the closed-loop system has dominant poles with damping ratio $\zeta = \sqrt{3}/2$. Justify your answer. If the answer is yes, mark the corresponding location for the closed-loop poles on the complex plane, and write down the approximate values of the closed-loop poles.



Solution:

a) The open-loop zero is at -2

the open-loop poles are

$$\frac{-2 \pm \sqrt{2^2-8}}{2} = -1 \pm j$$

b) on the left of the zero at -2

c) There is only one asymptote at $180°$.

It overlaps with the negative real-axis

(Note: By now we can already tell the shape of the root-locus. The two branches start from the poles, break in at some point left of (-2) on the real-axis, end at -2 and $-\infty$. The exact break-in point and the angle of departure are computed in step (d) & (e).)

d) To compute the break-in point,

Write

$$K = -\frac{s^2+2s+2}{s+2} = -s - \frac{2}{s+2} \qquad (*)$$

Let $\frac{dK}{ds} = -1 + \frac{2}{(s+2)^2} = 0$

We thus get

$$(s+2)^2 = 2$$

$$s = -2 \pm \sqrt{2}$$

Only $-2-\sqrt{2}$ is on the real-segments of the root-locus.

Hence, the break-in point is at $-2-\sqrt{2} = -3.414$

The corresponding value of K is given by $(*)$

$$K = -s - \frac{2}{s+2}\bigg|_{s=-2-\sqrt{2}} = -(-2-\sqrt{2}) - \frac{2}{-2-\sqrt{2}+2}$$

$$= 2+\sqrt{2}+\sqrt{2} = 2+2\sqrt{2}$$

e) To compute the angle of departure of the upper pole p_1, use the angle condition

$$\angle(p_1 - z_1) - (\angle(p_1 - p_2) + \theta) = 180^\circ$$

$$45^\circ - 90^\circ - \theta = 180^\circ$$

$$\therefore \theta = -45^\circ - 180^\circ = -225 = 135^\circ$$

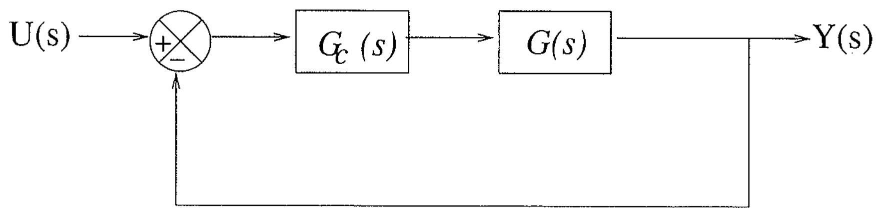
By symmetry, the angle of departure from the lower pole p_2 is -135°.

The angle of arrival to the zero (-2) is 180°.

f) see figure.

g) When $\zeta = \frac{\sqrt{3}}{2}$,
it corresponds to a half-line from the origin with angle $30°$ with the negative real axis.
Clearly, this half-line will intersect the root-locus. Therefore, there exists some value of $K > 0$ such that the closed-loop system has dominant poles with $\zeta = \frac{\sqrt{3}}{2}$.
The corresponding closed-loop poles are approximately $-2.5 \pm 1.5j$ (from figure).

Problem 2 (50 points)



Consider the system shown above with

$$G(s) = \frac{s+2}{s(s+1)^2}.$$

Answer the following questions (use the blank figure on the next page for marking your answers on the complex plane).

(a) (10 points) Suppose $G_c(s) = K$ with $K > 0$. Determine whether or not it is possible to choose a proper K such that the dominant poles of the closed-loop transfer function $Y(s)/U(s)$ are $p_d = -1 \pm \sqrt{3}j$. Justify your answer.

(b) (5 points) Find the angle of deficiency so that the dominant poles of the closed-loop transfer function $Y(s)/U(s)$ are $p_d = -1 \pm \sqrt{3}j$.

(c) (15 points) Design a lead compensator of the form

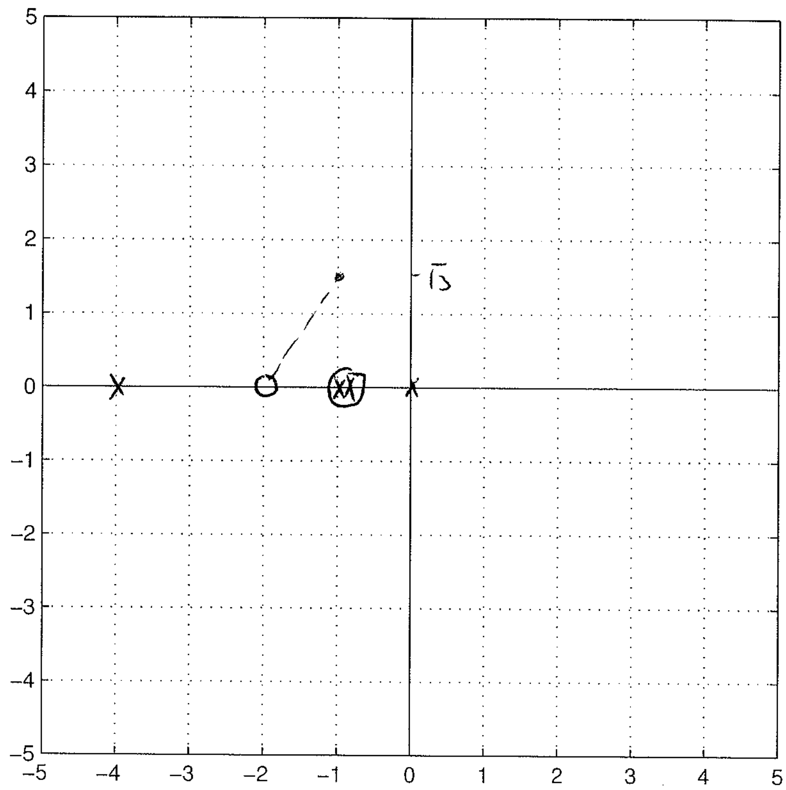
$$G_c(s) = K\frac{s+1}{s+\alpha},$$

for some constant α so that the closed-loop transfer function of the compensated system can have dominant poles at $p_d = -1 \pm \sqrt{3}j$.

(d) (5 points) Using the lead compensator obtained above, what is the steady-state velocity error constant K_v of the closed-loop system when tracking a unit ramp input?

(e) (10 points) Design a lag-lead compensator $G_c(s)$ so that the closed-loop system has dominant poles at approximately $p_d = -1 \pm \sqrt{3}j$, and that the steady-state velocity error constant of the closed-loop system when tracking a unit ramp input is 30.

(f) (5 points) Write down the overall open-loop transfer function $G_c(s)G(s)$ of the compensated system using the lag-lead compensator in part (e).



desired closed-loop pole: $-1+\sqrt{3}j$

open loop zeros	angle		distance
-2	60°		2
new -1		$+90^\circ$	$\sqrt{3}$
	60°	150°	
open loop poles			
0	120°		2
-1	90°		$\sqrt{3}$
-1	90°		$\sqrt{3}$
new -4		$+30^\circ$	$2\sqrt{3}$
	300°	330°	
Total angle	-240°	-180° ✓ ok.	

a) From figure, the angle of $G(s)$ at the desired closed-loop pole $-1+\sqrt{3}j$ is

$$60^\circ - (120^\circ + 90^\circ + 90^\circ) = -240^\circ = 120^\circ \neq 180^\circ$$

$\therefore$ No $K>0$ can achieve the closed-loop pole.

b) The angle deficiency is

$$180^\circ - 120^\circ = 60^\circ$$

c) The angle contributed by the new zero at (-1) is 90°.

Since $90^\circ - 30^\circ = 60^\circ \leftarrow$ the angle deficiency, the angle contributed by the new pole at $(-\alpha)$ should be 30°. Use trigonometry,

$$\frac{\sqrt{3}}{-1-(-\alpha)} = \tan 30^\circ = \frac{\sqrt{3}}{3}$$

$$\therefore \quad \alpha = 4$$

The gain K can be determined from the magnitude condition

$$K = \frac{\prod \text{distance to all poles}}{\prod \text{distance to all zeros}}$$

$$= \frac{2 \times \sqrt{3} \times \sqrt{3} \times 2\sqrt{3}}{2 \times \sqrt{3}} = 6.$$

d) The static velocity error constant is

$$K_v = \lim_{s \to 0} s \cdot 6 \cdot \frac{s+1}{s+4} \cdot \frac{s+2}{s(s+1)^2} = \frac{6 \times 2}{4} = 3$$

e) We can reuse the lead-compensator from step (c) since the closed-loop pole is already $-1 \pm \sqrt{3}j$.

In order to improve K_v to 30, we need a lag-compensator with DC-gain

$$\frac{30}{3} = 10$$

The lag-part can be

$$\frac{s+0.1}{s+0.01}$$

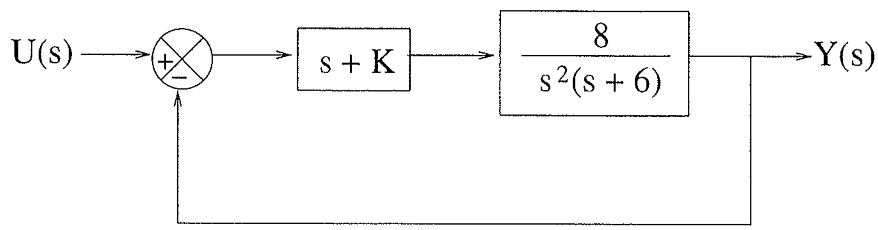
The overall lag-lead compensator is

$$6 \cdot \frac{s+1}{s+4} \frac{s+0.1}{s+0.01}$$

f) The open-loop transfer function is

$$G_c(s) = G(s) = 6 \cdot \frac{s+1}{s+4} \frac{s+0.1}{s+0.01} \cdot \frac{s+2}{s(s+1)^2}$$

Problem 3 (40 points)



Consider the feedback control system shown above, where $K > 0$ is a parameter that one can adjust freely. Answer the following questions (use the blank figure on the next page for marking your answers on the complex plane).

(a) (10 points) Write down the equation for determining the closed-loop poles of the system. In order to carry out root-locus analysis, rearrange the equation such that it is in the standard form

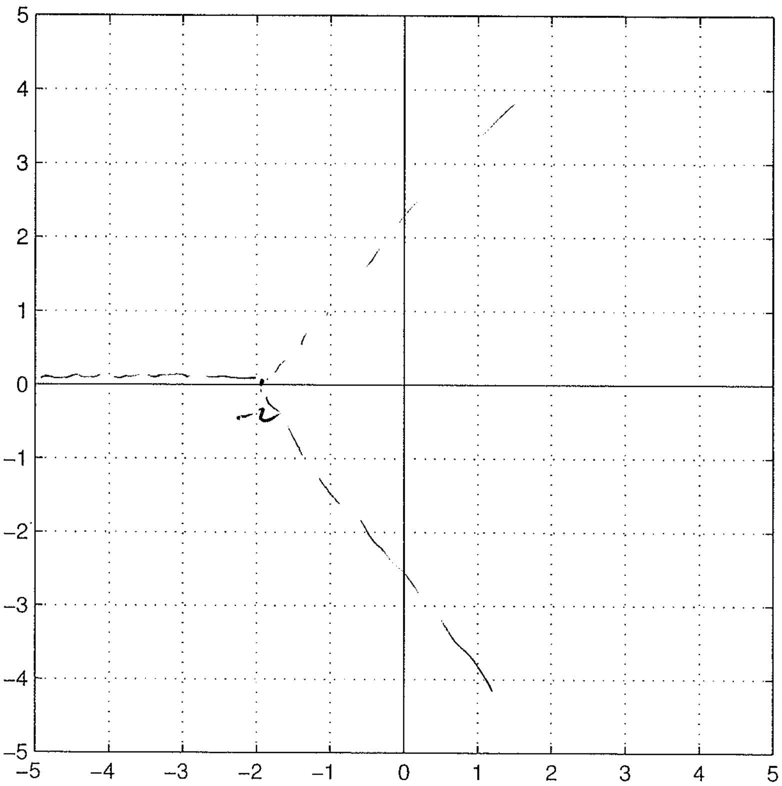
$$1 + KG(s) = 0.$$

Find $G(s)$.

(b) (10 points) For the root-locus, find the asymptotes (if any) and the points (if any) where the asymptotes intersect the real axis. Mark the asymptotes on the complex plane.

(c) (10 points) Find the points (if any) where the root locus cross the imaginary axis, and the corresponding value of K.

(d) (10 points) Determine the values of K such that the closed-loop system is stable.



Solution:

a) The closed-loop poles are solutions of the following equation

$$1 + (s+K)\frac{8}{s^2(s+6)} = 0$$

$$\Rightarrow \quad s^2(s+6) + 8(s+K) = 0$$

$$\Rightarrow \quad s^3 + 6s^2 + 8s + 8K = 0$$

$$\Rightarrow \quad 1 + K \cdot \frac{8}{s^3+6s^2+8s} = 0$$

$$\therefore \quad G(s) = \frac{8}{s^3+6s^2+8s}$$

b) The open-loop poles of $G(s)$ are

$0, \; -2, \; -4$

There is no open-loop zero.

$\therefore$ # of asymptotes is 3

The angles of the asymptotes are

$$\frac{180^\circ}{3} + l \cdot \frac{360^\circ}{3} = 60^\circ + l \cdot 120^\circ$$

$$= 60^\circ, \; 180^\circ, \; -60^\circ$$

The asymptotes intersect with the real-axis at

$$\frac{0 + (-2) + (-4)}{3} = -2$$

See figure

c) To find the points where the root-locus cross the imaginary axis, construct the Routh array for

$$s^3 + 6s^2 + 8s + 8K = 0$$

s^3	1	8
s^2	6	$8K$
s^1	$\frac{48-8K}{6}$	
s^0	$8K$	

$\Leftarrow$ when $K=6$, this row will be entirely zero

The auxillary polynomial is thus $6s^2 + 8 \times 6 = 0$

$$\Rightarrow \quad s = \pm 2\sqrt{2}\, j$$

d) In order to ensure stability, the Routh array should have positive elements in the first column, hence

$$\frac{48-8K}{6} > 0 \quad \Rightarrow \quad K < 6$$

$$8K > 0$$

$$\therefore \quad 0 < K < 6$$

Problem 4 (10 points)

Given the following equation

$$s^3 - s + 1 = 0,$$

use the Routh criteria to find the number of roots with positive real parts, negative real parts, and zero real parts (i.e., purely imaginary roots), respectively. Summarize your final answers in the spaces below. (**Note: You are required to provide the details of the Routh array to get full credits!**.)

(a) Number of roots with positive real parts: 2

(b) Number of roots with negative real parts: 1

(c) Number of roots with zero real parts: 0

Construct the Routh array for

$$s^3 - 0 \cdot s^2 - s + 1 = 0$$

$\varepsilon \to 0^+$	$\varepsilon - 0^-$			
+	+	s^3	1	-1
+	−	s^2	$\cancel{0}\ \varepsilon$	1
−	+	s^1	$\dfrac{-\varepsilon - 1}{\varepsilon}$	
+	+	s^0	1	

$\therefore \quad N^+ = 2$

$\quad\quad N^- = 2$

$\therefore$ # of RHP roots $= \max(N^+, N^-) = 2$

(Note this includes purely imaginary roots)

# of purely imaginary roots $= |N^+ - N^-| = 0$

Since the total # of roots is 3, the number of roots with negative real parts is $3 - 2 = 1$.