ECE 5314: Power System Operation & Control

Lecture 7: Power Flow Problem

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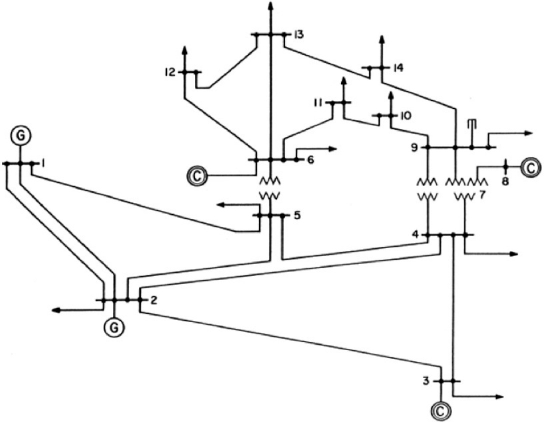
R2 A. Gomez-Exposito, A. J. Conejo, C. Canizares, *Electric Energy Systems: Analysis and Operation*, Chapter 3.

R1 A. J. Wood, B. F. Wollenberg, and G. B. Sheble, Power Generation, Operation, and Control, Wiley, 2014, Chapter 6.

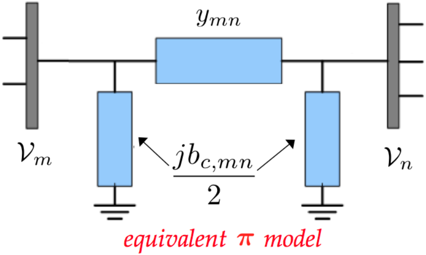
Power transmission network as an electric circuit

- N nodes (generator/load buses) and L edges (lines, transformers)
- AC voltages and currents as phasors (at nominal frequency)

  $\mathcal{V} = Ve^{j\theta} = V_r + jV_i$
- From scalar to multivariate Ohm's law: $\mathcal{V} = Z\mathcal{I} \ \rightarrow \ \mathbf{v} = \mathbf{Zi}$



Transmission lines

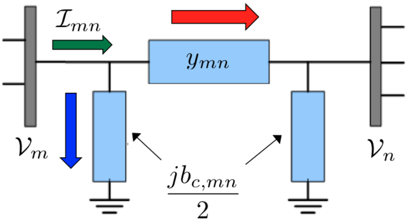


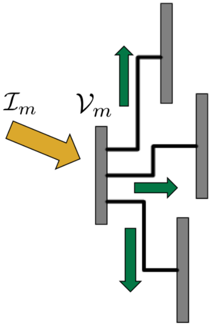
- Line series impedance: $z_{mn} = r_{mn} + jx_{mn}$ $(x_{mn} > 0)$
- Line series admittance: $y_{mn} = \frac{1}{z_{mn}} = g_{mn} - jb_{mn}$
- Line series conductance: $g_{mn} = \frac{r_{mn}}{r_{mn}^2 + x_{mn}^2}$
- Line series susceptance: $b_{mn} = \frac{x_{mn}}{r_{mn}^2 + x_{mn}^2} > 0$
- Total charging susceptance: $b_{mn}^c > 0$

Line currents

$$\mathcal{I}_{mn} = y_{mn}(\mathcal{V}_m - \mathcal{V}_n) + j\frac{b^c_{mn}}{2}\mathcal{V}_m$$

$$\mathcal{I}_{mn} = \left(y_{mn} + j\frac{b^c_{mn}}{2}\right)\mathcal{V}_m - y_{mn}\mathcal{V}_n$$





Kirchoff's current law:

$$\mathcal{I}_m = \left(\sum_{n\neq m} y_{mn} + j\frac{b^c_{mn}}{2}\right)\mathcal{V}_m - \sum_{n\neq m} y_{mn}\mathcal{V}_n$$

Collect currents and voltages $\{\mathcal{I}_m, \mathcal{V}_m\}_{m=1}^N$ into $\mathbf{i}$, $\mathbf{v} \in \mathbb{C}^{N\times 1}$

Transformers and phase shifters are ignored in our analysis

Multivariate Ohm's law

Currents are linearly related to voltages: $\mathbf{i} = \mathbf{Y}\mathbf{v}$

Bus admittance matrix: fundamental in power systems operations

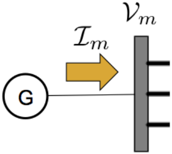
$$Y_{mn} = \begin{cases} \sum_{k \neq m} y_{mk} + j\frac{b^c_{mk}}{2} & , m = n \\ -y_{mn} & , \exists \text{ line } (m,n) \\ 0 & , \text{otherwise} \end{cases}$$

- symmetric ($Y_{mn} = Y_{nm}$); non-Hermitian ($Y_{mn} \neq Y^*_{nm}$)
- sparse: efficient computations and storage
- invertible if $b^c_{mn} \neq 0$ for at least one line; otherwise $\mathbf{Y}\mathbf{1} = \mathbf{0}$

Bus impedance matrix: $\mathbf{Z} := \mathbf{Y}^{-1} \quad (\mathbf{v} = \mathbf{Z}\mathbf{i})$

- non-sparse
- *not* the matrix of line impedances, i.e., $Z_{mn} \neq z_{mn} = \frac{1}{y_{mn}}$

Complex power



- Power $\mathcal{S}_m = \mathcal{S}_m^g - \mathcal{S}_m^d$ generated/consumed at bus m

$$\{\mathcal{S}_m = P_m + jQ_m = \mathcal{V}_m \mathcal{I}_m^*\}_{m=1}^N, \text{ and } \mathbf{i} = \mathbf{Y}\mathbf{v}$$

- Eliminate currents to get the multivariate power model

$$\mathbf{s} = \text{diag}(\mathbf{v})\mathbf{i}^* = \text{diag}(\mathbf{v})\mathbf{Y}^*\mathbf{v}^*$$

N complex equations in $2N$ complex unknowns

- Bus admittance matrix in rectangular coordinates $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$

- Similar expressions for power flow on line (m, n): $\mathcal{S}_{mn} = \mathcal{V}_m \mathcal{I}_{mn}^*$

Power flow equations

Voltages in **polar coordinates** ($\theta_{mn} = \theta_m - \theta_n$)

$$P_m = V_m \sum_{n=1}^{N} V_n \left(G_{mn} \cos\theta_{mn} + B_{mn} \sin\theta_{mn} \right)$$

$$Q_m = V_m \sum_{n=1}^{N} V_n \left(G_{mn} \sin\theta_{mn} - B_{mn} \cos\theta_{mn} \right)$$

dependence on phase differences only; reference bus $\theta_N = 0$

Voltages in **rectangular coordinates** (quadratic equations!)

$$P_m = V_{m,r} \sum_{n=1}^{N} (V_{n,r} G_{mn} - V_{n,i} B_{mn}) + V_{m,i} \sum_{n} (V_{n,i} G_{mn} + V_{n,r} B_{mn})$$

$$Q_m = V_{m,i} \sum_{n=1}^{N} (V_{n,r} G_{mn} - V_{n,i} B_{mn}) - V_{m,r} \sum_{n} (V_{n,i} G_{mn} + V_{n,r} B_{mn})$$

Power flow problem

There are $2N$ equations and $4N$ variables $\{(P_m, Q_m, V_m, \theta_m)\}_{m=1}^{N}$

Problem statement: Fixing the values of $2N$ variables, find the values of the rest $2N$ unknowns that satisfy the nonlinear power flow (PF) equations

Given values typically come from

- First N_d load buses (PQ buses) (P_m, Q_m)
- Next N_g generator buses (PV buses) (P_m, V_m)
- Reference bus $(V_N, \theta_N = 0)$

Number of buses $N = 1 + N_g + N_d$

Solving the power flow equations

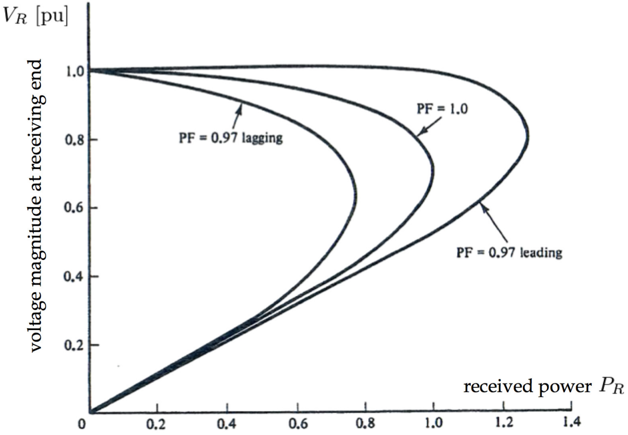
$$P_m = V_m \sum_n V_n \left(G_{mn}\cos\theta_{mn} + B_{mn}\sin\theta_{mn}\right), \; m = 1, \ldots, N_d + N_g = N - 1$$

$$Q_m = V_m \sum_n V_n \left(G_{mn}\sin\theta_{mn} - B_{mn}\cos\theta_{mn}\right), \; m = 1, \ldots, N_d$$

- Set of nonlinear equations in $\{(V_n, \theta_n)\}_{n=1}^N$ solved recursively
- Once voltages $\{(V_n, \theta_n)\}_{n=1}^N$ are found, any other quantity (injections, flows, currents, losses) can be calculated
- **Flat start** or **flat voltage profile**

  voltages usually initialized at $V_n = 1$ and $\theta_n = 0$ for all n
- PF solution is not unique!

'Nose' curve



Q: How is the famous nose curve derived?

Gauss-Seidel method

$$\mathcal{S}_m = \mathcal{V}_m \sum_{n=1}^{N} Y_{mn}^* \mathcal{V}_n^* \quad \Rightarrow \quad \mathcal{S}_m^* = \mathcal{V}_m^* \sum_{n=1}^{N} Y_{mn} \mathcal{V}_n \quad \Rightarrow$$

$$\left(\frac{\mathcal{S}_m}{\mathcal{V}_m}\right)^* = \sum_{n=1}^{m-1} Y_{mn}\mathcal{V}_n + Y_{mm}\mathcal{V}_m + \sum_{n=m+1}^{N} Y_{mn}\mathcal{V}_n \quad \Rightarrow$$

$$\mathcal{V}_m := \frac{1}{Y_{mm}}\left[\left(\frac{\mathcal{S}_m}{\mathcal{V}_m}\right)^* - \sum_{n=1}^{m-1} Y_{mn}\mathcal{V}_n - \sum_{n=m+1}^{N} Y_{mn}\mathcal{V}_n\right]$$

Gauss-Seidel iterations:

1. Initialize $\mathbf{v}_0$ at flat profile or at most recent grid state
2. Repeat until convergence $\|\mathbf{v}_{t+1} - \mathbf{v}_t\|_2 \leq \epsilon$

$$\mathcal{V}_m^{t+1} := \frac{1}{Y_{mm}}\left[\left(\frac{\mathcal{S}_m^{t}}{\mathcal{V}_m^{t}}\right)^* - \sum_{n=1}^{m-1} Y_{mn}\mathcal{V}_n^{t+1} - \sum_{n=m+1}^{N} Y_{mn}\mathcal{V}_n^{t}\right], \quad \forall\ m$$

   where S_m^t is either fixed or calculated from PF equations via $\mathbf{v}_t$

3. Normalize $\mathcal{V}_m^{t+1}$ to match given magnitude for PV buses

Impedance matrix method

Power flow problem involves two equations that can be combined:

$$\left.\begin{array}{lcl} \mathbf{s} = \text{diag}(\mathbf{v})\mathbf{i}^* & \Leftrightarrow & \mathbf{i} = [\text{diag}(\mathbf{v}^*)]^{-1}\mathbf{s}^* \\ \mathbf{i} = \mathbf{Y}\mathbf{v} & \Leftrightarrow & \mathbf{v} = \mathbf{Z}\mathbf{i} \end{array}\right\} \Rightarrow \; \mathbf{v} = \mathbf{Z}[\text{diag}(\mathbf{v}^*)]^{-1}\mathbf{s}^*$$

Jacobi-type iterations:

1. Initialize $\mathbf{v}_0$ at flat profile or at most recent grid state
2. Repeat until convergence $\|\mathbf{v}_{t+1} - \mathbf{v}_t\|_2 \leq \epsilon$

$$\mathbf{v}_{t+1} = \mathbf{Z}[\text{diag}(\mathbf{v}_t^*)]^{-1}\mathbf{s}_t^*$$

where entries of $\mathbf{s}_t$ are either known or calculated from PF equations via $\mathbf{v}_t$

Inversion of $\mathbf{Y}$; (close to) singularity of $\mathbf{Y}$ handled by eliminating the slack bus

Newton's method

- Newton-Raphson method aims at solving nonlinear equations: $\mathbf{f}(\mathbf{x}) = \mathbf{0}$
- At iteration $t+1$, function $\mathbf{f}(\mathbf{x})$ is linearized at $\mathbf{x}^t$

$$\mathbf{f}(\mathbf{x}) \approx \hat{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^t) + \mathbf{J}(\mathbf{x}^t)(\mathbf{x} - \mathbf{x}^t)$$

where $\mathbf{J}(\mathbf{x}^t)$ is the Jacobian matrix of $\mathbf{f}$

- Variable $\mathbf{x}^{t+1}$ is updated such that $\hat{\mathbf{f}}(\mathbf{x}^{t+1}) = \mathbf{0}$

$$\mathbf{x}^{t+1} := \mathbf{x}^t - [\mathbf{J}(\mathbf{x}^t)]^{-1}\mathbf{f}(\mathbf{x}^t)$$

- Newton's method in two steps (convergence to be studied later)

$$-\mathbf{J}(\mathbf{x}^t)\boldsymbol{\delta}^t = \mathbf{f}(\mathbf{x}^t) \quad \text{system of linear equations}$$

$$\mathbf{x}^{t+1} := \mathbf{x}^t + \boldsymbol{\delta}^t$$

Power flow via Newton's method

Equations involved in power flow problem:

$$\Delta P_m := \hat{P}_m - V_m \sum_k V_k \left(G_{mk} \cos\theta_{mk} + B_{mk} \sin\theta_{mk}\right) = 0, \;\; m = 1, \ldots, N_d + N_g$$

$$\Delta Q_m := \hat{Q}_m - V_m \sum_k V_k \left(G_{mk} \sin\theta_{mk} - B_{mk} \cos\theta_{mk}\right) = 0, \;\; m = 1, \ldots, N_d$$

or more compactly

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \Delta\mathbf{p}(\mathbf{x}) \\ \Delta\mathbf{q}(\mathbf{x}) \end{bmatrix} = \mathbf{0}$$

Variables involved in power flow problem

$$\mathbf{x} := \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \theta_1 \;\; \theta_2 \;\; \cdots \;\; \theta_{N-1} \;\; V_1 \;\; V_2 \;\; \cdots \;\; V_{N_d} \end{bmatrix}^\top$$

For Jacobian in NR, need to find: $-\frac{\partial \Delta P_m}{\partial \theta_n} = -\frac{\partial(\hat{P}_m - P_m)}{\partial \theta_n} = \frac{\partial P_m}{\partial \theta_n}$

Finding derivatives $\partial P_m/\partial \theta_n$

Repeating for convenience: $P_m = V_m \sum_k V_k \left(G_{mk}\cos\theta_{mk} + B_{mk}\sin\theta_{mk}\right)$

1. For $n \neq m$, we get

$$\frac{\partial P_m}{\partial \theta_n} = V_m V_n \left(G_{mn}\sin\theta_{mn} - B_{mn}\cos\theta_{mn}\right)$$

2. Notice similarity to summands in Q_m

$$Q_m = V_m \sum_{k\neq m} V_k \left(G_{mk}\sin\theta_{mk} - B_{mk}\cos\theta_{mk}\right) - B_{mm}V_m^2$$

3. For $n = m$, we get

$$\frac{\partial P_m}{\partial \theta_m} = -V_m \sum_{k\neq m} V_k \left(G_{mk}\sin\theta_{mk} - B_{mk}\cos\theta_{mk}\right) = -Q_m - B_{mm}V_m^2$$

Finding derivatives $\partial Q_m/\partial V_n$

Repeating for convenience: $Q_m = V_m \sum_k V_k \left(G_{mk}\sin\theta_{mk} - B_{mk}\cos\theta_{mk}\right)$

1. For $n \neq m$, we get

$$\frac{\partial Q_m}{\partial V_n} = V_m \left(G_{mn}\sin\theta_{mn} - B_{mn}\cos\theta_{mn}\right) \quad \Longrightarrow$$
$$V_n \frac{\partial Q_m}{\partial V_n} = V_m V_n \left(G_{mn}\sin\theta_{mn} - B_{mn}\cos\theta_{mn}\right) = \frac{\partial P_m}{\partial \theta_n}$$

2. For $n = m$, we get

$$V_m \frac{\partial Q_m}{\partial V_m} = Q_m - B_{mm}V_m^2$$

Multiplying $\partial Q_m/\partial V_n$ by V_n gives Jacobian matrix more symmetry

Finding derivatives $\partial Q_m/\partial\theta_n$

Repeating for convenience: $Q_m = V_m \sum_k V_k \left(G_{mk}\sin\theta_{mk} - B_{mk}\cos\theta_{mk}\right)$

1. For $n \neq m$, we have

$$\frac{\partial Q_m}{\partial \theta_n} = -V_m V_n (G_{mn}\cos\theta_{mn} + B_{mn}\sin\theta_{mn})$$

2. Notice similarity to summands in P_m

$$P_m = V_m \sum_{k\neq m} V_k \left(G_{mk}\cos\theta_{mk} + B_{mk}\sin\theta_{mk}\right) + G_{mm}V_m^2$$

3. For $n = m$, we get

$$\frac{\partial Q_m}{\partial \theta_m} = V_m \sum_{k\neq m} V_k \left(G_{mk}\cos\theta_{mk} + B_{mk}\sin\theta_{mk}\right) = P_m - G_{mm}V_m^2$$

Finding derivatives $\partial P_m / \partial V_n$

Repeating for convenience: $P_m = V_m \sum_k V_k \left(G_{mk} \cos\theta_{mk} + B_{mk} \sin\theta_{mk}\right)$

1. For $n \neq m$, we have

$$V_n \frac{\partial P_m}{\partial V_n} = V_m V_n (G_{mn} \cos\theta_{mn} + B_{mn} \sin\theta_{mn}) = -\frac{\partial Q_m}{\partial \theta_n}$$

2. For $n = m$, we get

$$\begin{aligned} V_m \frac{\partial P_m}{\partial V_m} &= V_m \sum_{k \neq m} V_k \left(G_{mk} \cos\theta_{mk} + B_{mk} \sin\theta_{mk}\right) + 2G_{mm} V_m^2 \\ &= P_m + G_{mm} V_m^2 \end{aligned}$$

Blocks of Jacobian matrix

$$\mathbf{H}^t_{(N-1)\times(N-1)} : H_{mn} = \frac{\partial P_m}{\partial \theta_n} = \begin{cases} V_m V_n (G_{mn} \sin\theta_{mn} - B_{mn}\cos\theta_{mn}), & n \neq m \\ -Q_m - B_{mm} V_m^2, & n = m \end{cases}$$

$$\mathbf{N}^t_{(N-1)\times N_d} : N_{mn} = V_n \frac{\partial P_m}{\partial V_n} = \begin{cases} -M_{mn}, & n \neq m \\ P_m + G_{mm} V_m^2, & n = m \end{cases}$$

$$\mathbf{M}^t_{N_d\times(N-1)} : M_{mn} = \frac{\partial Q_m}{\partial \theta_n} = \begin{cases} -V_m V_n (G_{mn} \cos\theta_{mn} + B_{mn}\sin\theta_{mn}), & n \neq m \\ P_m - G_{mm} V_m^2, & n = m \end{cases}$$

$$\mathbf{L}^t_{N_d\times N_d} : L_{mn} = V_n \frac{\partial Q_m}{\partial V_n} = \begin{cases} H_{mn}, & n \neq m \\ Q_m - B_{mm} V_m^2, & n = m \end{cases}$$

Newton's iterations

1. Initialize $\mathbf{v}_0$ at flat profile or at a recent grid state
2. For $t = 0, 1, \ldots,$ until convergence $\|\mathbf{v}_{t+1} - \mathbf{v}_t\|_2 \leq \epsilon$

   2.1 Evaluate Jacobian matrix at current state

   2.2 Find variable update by solving the linear system

   $$\begin{bmatrix} \mathbf{H}^t & \mathbf{N}^t \\ \mathbf{M}^t & \mathbf{L}^t \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\theta}^t \\ \Delta\mathbf{v}^t/\mathbf{v}^t \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{p}^t \\ \Delta\mathbf{q}^t \end{bmatrix}$$

   where division by $\mathbf{v}^t$ (known at iteration t) is for symmetry

   2.3 Update the state as

   $$\begin{bmatrix} \boldsymbol{\theta}^{t+1} \\ \mathbf{v}^{t+1} \end{bmatrix} := \begin{bmatrix} \boldsymbol{\theta}^t \\ \mathbf{v}^t \end{bmatrix} + \begin{bmatrix} \Delta\boldsymbol{\theta}^t \\ \Delta\mathbf{v}^t \end{bmatrix}$$

Fast decoupled power flow

Newton's iterations involve evaluating the Jacobian matrix and inverting it

Two approximations to save computations:

1. Keep the Jacobian constant by evaluating it at a specific point $\mathbf{x}$
2. Problem decouples by setting $\mathbf{M} = \mathbf{N} = \mathbf{0}$
3. After several approximations, matrices $\mathbf{H}$ and $\mathbf{L}$ simplify as

$$\mathbf{B}'\Delta\boldsymbol{\theta}^t = \Delta\mathbf{p}^t/\mathbf{v}^t$$

$$\mathbf{B}''\Delta\mathbf{v}^t = \Delta\mathbf{q}^t/\mathbf{v}^t$$

Matrices $\mathbf{B}'$ and $\mathbf{B}''$ are defined as

$$B'_{mn} = -x_{mn}^{-1}, \quad B'_{mm} = \sum_{n\neq m} x_{mn}^{-1} \qquad (b_{mn} \approx x_{mn}^{-1}, \text{no shunt, no voltage trans.})$$

$$B''_{mn} = -B_{mn}, \quad B''_{mm} = -B_{mm} \qquad (b_{mn} = \frac{x_{mn}}{r_{mn}^2 + x_{mn}^2}, \text{no phase shifters})$$

Specifications as quadratic functions

- Collect nodal voltages in rectangular coordinates in $\mathbf{v} \in \mathbb{C}^N$:

$$\mathbf{v} := [v_{1,r} + jv_{1,i} \quad \ldots \quad v_{N,r} + jv_{N,i}]^\top$$

- Power injections and squared voltage magn. are quadratic functions of $\mathbf{v}$:

$$P_m(\mathbf{v}) = \mathbf{v}^H \mathbf{M}_{P_m} \mathbf{v}$$

$$Q_m(\mathbf{v}) = \mathbf{v}^H \mathbf{M}_{Q_m} \mathbf{v}$$

$$V_m^2(\mathbf{v}) = \mathbf{v}^H \mathbf{M}_{V_m} \mathbf{v}$$

where matrices in blue are Hermitian symmetric $(\mathbf{M}_{P_m} = \mathbf{M}_{P_m}^H)$

- Every bus contributes two quadratic constraints/specifications on $\mathbf{v}$

Finding $\mathbf{M}$'s matrices

Voltage magnitude ($\mathbf{e}_m$ is the m-th canonical vector)

$$V_m^2(\mathbf{v}) = \mathcal{V}_m^* \mathcal{V}_m = \mathbf{v}^H \mathbf{e}_m \mathbf{e}_m^\top \mathbf{v} \quad \Rightarrow \quad \mathbf{M}_{V_m} = \mathbf{e}_m \mathbf{e}_m^\top$$

Complex power injection

$$S_m = \mathcal{V}_m \mathcal{I}_m^* = (\mathbf{v}^\top \mathbf{e}_m)(\mathbf{e}_m^\top \mathbf{i}^*) = \mathbf{v}^\top \mathbf{e}_m \mathbf{e}_m^\top \mathbf{Y}^* \mathbf{v}^* = \mathbf{v}^H \mathbf{Y}^* \mathbf{e}_m \mathbf{e}_m^\top \mathbf{v}$$

Active power

$$P_m = \frac{S_m + S_m^*}{2} = \mathbf{v}^H \mathbf{M}_{P_m} \mathbf{v} \quad \text{where} \quad \mathbf{M}_{P_m} = \frac{1}{2}\left(\mathbf{Y}^* \mathbf{e}_m \mathbf{e}_m^\top + \mathbf{e}_m \mathbf{e}_m^\top \mathbf{Y}\right)$$

Reactive power

$$Q_m = \frac{S_m - S_m^*}{2j} = \mathbf{v}^H \mathbf{M}_{Q_m} \mathbf{v} \quad \text{where} \quad \mathbf{M}_{Q_m} = \frac{1}{2j}\left(\mathbf{Y}^* \mathbf{e}_m \mathbf{e}_m^\top - \mathbf{e}_m \mathbf{e}_m^\top \mathbf{Y}\right)$$

Power flow as a feasibility problem

- System state as solution of feasibility problem

$$\begin{aligned} &\text{find} \quad \mathbf{v} \\ &\text{s.to} \quad \mathbf{v}^H \mathbf{M}_k \mathbf{v} = s_k, \quad k = 1:2N \end{aligned} \qquad \left[\text{note } \mathbf{v}^H \mathbf{M}_k \mathbf{v} = \text{Tr}(\mathbf{M}_k \mathbf{v}\mathbf{v}^H)\right]$$

- Introduce matrix variable $\mathbf{V} = \mathbf{v}\mathbf{v}^H$

$$\begin{aligned} &\text{find} \quad (\mathbf{v}, \mathbf{V}) \\ &\text{s.to} \quad \text{Tr}(\mathbf{M}_k \mathbf{V}) = s_k, \quad k = 1:2N \\ &\qquad\quad \mathbf{V} = \mathbf{v}\mathbf{v}^H \end{aligned}$$

- Eliminate variable $\mathbf{v}$; *non-convex* problem due to rank constraint

$$\begin{aligned} &\text{find} \quad \mathbf{V} \\ &\text{s.to} \quad \text{Tr}(\mathbf{M}_k \mathbf{V}) = s_k, \quad k = 1:2N \\ &\qquad\quad \mathbf{V} \succeq \mathbf{0}, \ \text{rank}(\mathbf{V}) = 1 \end{aligned}$$

Semidefinite program relaxation

- Drop rank constraint to get semidefinite program (SDP)

$$\begin{aligned} \text{find} \quad & \mathbf{V} \\ \text{s.to} \quad & \mathrm{Tr}(\mathbf{M}_k \mathbf{V}) = s_k, \quad k = 1, \ldots, 2N \\ & \mathbf{V} \succeq \mathbf{0} \end{aligned}$$

  which is a convex problem

- If the solution $\mathbf{V}_o$ is rank-one, the relaxation is said to be *exact*

- If exact, find $\mathbf{v}_o$ from $\mathbf{V}_o = \mathbf{v}_o \mathbf{v}_o^H$

- Relaxation is oftentimes exact under practical system conditions!

From feasibility to minimization

- Feasibility problem can be converted to the convex minimization problem

$$\min_{\mathbf{V}\succeq\mathbf{0}} \quad \mathrm{Tr}(\mathbf{M}\mathbf{V})$$
$$\text{s.to} \quad \mathrm{Tr}(\mathbf{M}_k\mathbf{V}) = s_k, \quad k = 1,\ldots,2N$$

- Design matrix $\mathbf{M}$ so that rank-one solutions are favored
  - selecting $\mathbf{M} = \mathbf{Y}^H\mathbf{Y}$ minimizes $\|\mathbf{i}\|_2^2$
  - selecting $\mathbf{M} = \mathbf{B}$ minimizes losses
  - both yield the "high-voltage solution" of the power flow equations

R. Madani, J. Lavaei, and R. Baldick, "Convexification of power flow problem over arbitrary networks," *in Proc. IEEE Conf. on Decision and Control*, Dec. 2015, Osaka, Japan.

DC power flow model

Power flow equations $$P_m = V_m \sum_n V_n \left(G_{mn} \cos\theta_{mn} + B_{mn} \sin\theta_{mn}\right)$$

Assumptions:

A1. Low r/x ratios in transmission lines (1/5-1/10 for 220-400kV)

$$r_{mn} \ll x_{mn} \rightarrow g_{mn} \ll b_{mn} \rightarrow \mathbf{G} \simeq \mathbf{0} \quad \text{and} \quad b_{mn} = \frac{x_{mn}}{r_{mn}^2 + x_{mn}^2}$$

A2. Small angle differences $\theta_m - \theta_n \simeq 0$; $\cos\theta_{mn} \simeq 1$ and $\sin\theta_{mn} \simeq \theta_{mn}$

A3. Voltage magnitudes close to unity (pu) $V_m \simeq 1$

DC power flow model:

[why called 'DC'?]

$$\boxed{P_m \simeq \sum_{n \neq m} b_{mn}(\theta_m - \theta_n)}$$

Coincides with 1st-order Taylor's series of P_m at $\mathbf{v}_{\text{flat}}$ under A1.

$\mathbf{B}$ matrix

Power injections (and flows) relate linearly to phase differences

$$P_m = \sum_{n:n\sim m} P_{mn} = \sum_{n:n\sim m} b_{mn}(\theta_m - \theta_n)$$

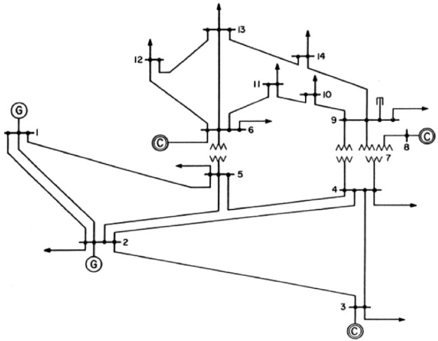
Multivariate power flow model: $\mathbf{p} = \mathbf{B}\boldsymbol{\theta}$

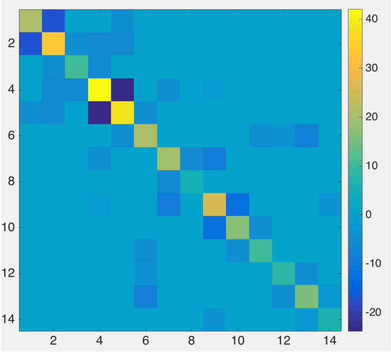
DC bus admittance matrix: (different from matrix $\mathbf{B}$ in $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$)

$$B_{mn} = \begin{cases} \sum_{n\neq m} b_{mn} & , m = n \\ -b_{mn} & , \exists \text{ line } (m,n) \\ 0 & , \text{otherwise} \end{cases}$$

- Real; symmetric; sparse; and positive semidefinite [Q: Why?]
- Lossless lines: $\mathbf{B}\mathbf{1}_N = \mathbf{0}_N \Rightarrow \mathbf{p}^T\mathbf{1}_N = 0$
- Oftentimes further simplify $b_{mn} = \frac{x_{mn}}{r_{mn}^2 + x_{mn}^2} \simeq \frac{1}{x_{mn}}$

Example for the IEEE 14-bus system






```
c = loadcase('case14'); % load case file
B = makeBdc(c); % B in sparse form; use B = full(B) if full form needed
imagesc(B);
axis square;
```