ECE 5314: Power System Operation & Control

Lecture 5: Economic Dispatch

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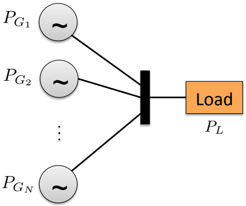
R2 A. Gomez-Exposito, A. J. Conejo, C. Canizares, *Electric Energy Systems: Analysis and Operation*, Chapter 5

Problem statement

- N power generation units
- serving a given load D
- P_i: power output of unit i [MW]
- $C_i(P_i)$: operation cost [$/h]

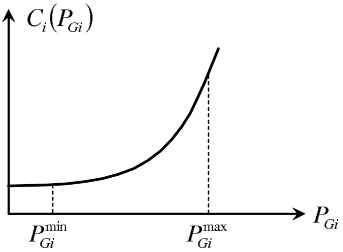


Find the most economic dispatch (ED) of units

$$\begin{aligned}\min_{\{P_i\}_{i=1}^N} \quad & \sum_{i=1}^N C_i(P_i) \\ \text{s.to} \quad & \sum_{i=1}^N P_i = D \\ & P_i^{\min} \leq P_i \leq P_i^{\max},\ i = 1, \ldots, N\end{aligned}$$

Convex problem if $C_i(P_i)$ are convex functions

Cost functions and energy markets



- Typical options for cost functions (linear, quadratic, piecewise linear)
- Who solves the economic dispatch?
- Fuel-cost curves or market bids
- Bilateral contracts and spot markets (2010)
- Day-ahead and real-time markets

Dispatch with linear costs

$$\begin{aligned} \min_{\{P_i\}_{i=1}^N} \quad & \sum_{i=1}^N a_i P_i \\ \text{s.to} \quad & \sum_{i=1}^N P_i = D \\ & P_i^{\min} \leq P_i \leq P_i^{\max}, \; i = 1, \ldots, N \end{aligned}$$

- solved as an LP or by simply sorting the cost coefficients
- uniqueness issues (case when $a_2 = a_3$ of the example)

Example:

$D = 350$MW

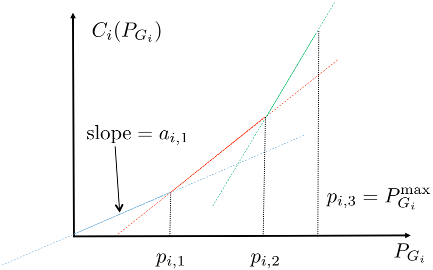
	$P_i^{\min}$ [MW]	$P_i^{\max}$ [MW]	a_i [$/MW]
G1	0	100	20
G2	0	400	25
G3	0	100	22
G4	0	200	18

Dispatch with convex piecewise-linear costs

The range $[0, P_i^{\max}]$ is divided into K_i blocks

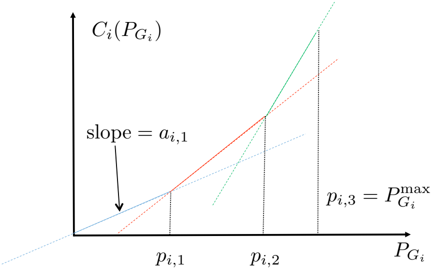
Cost $C_i(P_i)$ expressed as K_i pairs: $\{(p_{i,k}, a_{i,k}) = \text{(block size, price)}\}_{k=1}^{K_i}$

- Any power in $[0, p_{i,1}]$ has incremental cost $a_{i,1}$ $/MW
- Any power in $[p_{i,1}, p_{i,2}]$ has incremental cost $a_{i,2}$ $/MW ...



Cost $C_i(P_i)$ is convex if prices are increasing $(a_1 < \ldots < a_{i,K_i})$

Rewriting convex piecewise linear costs



Express each cost function as the pointwise maximum of linear functions:

$$C_i(P_i) = \max_{k \in \{1,\ldots,K_i\}} \{a_{i,k}P_i + b_{i,k}\}$$

Q.5.1 Show that $b_{i,k} = (a_{i,k-1} - a_{i,k})p_{i,k-1} + b_{i,k-1}$.

Using the epigraph form

dispatch with convex piecewise-linear costs

$$\min_{\{P_i\}} \sum_{i=1}^{N} \max_{k \in \{1,\ldots,K_i\}} \{a_{i,k} P_i + b_{i,k}\}$$

$$\text{s.to} \quad \sum_{i=1}^{N} P_i = D$$

$$0 \leq P_i \leq P_i^{\max}, \ \forall \ i$$

dispatch in epigraph form

$$\min_{\{P_i, t_i\}} \sum_{i=1}^{N} t_i$$

$$\text{s.to} \quad a_{i,k} P_i + b_{i,k} \leq t_i, \ \forall \ i, k$$

$$\sum_{i=1}^{N} P_i = D$$

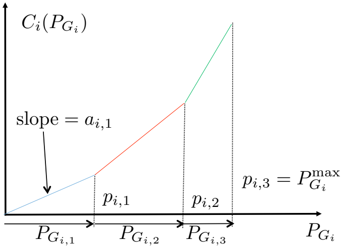
$$0 \leq P_i \leq P_i^{\max}, \ \forall \ i$$

An alternative formulation

- Introduce one variable per block

$$P_i = \sum_{k=1}^{K_i} P_{i,k}$$

- Let $p_{i,0} = 0$ and $p_{i,K_i} = P_i^{\max}$



$$\begin{aligned} \min_{\{P_{i,k}\}} \quad & \sum_{i=1}^{N} \sum_{k=1}^{K_i} a_{i,k} P_{i,k} \\ \text{s.to} \quad & \sum_{i=1}^{N} \sum_{k=1}^{K_i} P_{i,k} = D \\ & 0 \leq P_{i,k} \leq p_{i,k} - p_{i,k-1}, \ \forall i, k \end{aligned}$$

- At the optimum: $P_{i,k}^* = p_{i,k} - p_{i,k-1}$ if $P_{i,k+1}^* > 0$

Comparing the two formulations

Epigraph form

$$\min \quad \sum_{i=1}^{N} t_i$$

$$\text{s.to} \quad a_{i,k} P_i + b_{i,k} \leq t_i \;\forall\; i, k$$

$$\sum_{i=1}^{N} P_i = D$$

$$0 \leq P_i \leq P_i \;\forall\; i$$

variables: $2N$

equality constraints: 1

inequality constraints: $\sum_{i=1}^{N} K_i + 2N$

Form with one variable per block

$$\min \quad \sum_{i=1}^{N} \sum_{k=1}^{K_i} a_{i,k} P_{i,k}$$

$$\text{s.to} \quad \sum_{i=1}^{N} \sum_{k=1}^{K_i} P_{i,k} = D$$

$$0 \leq P_{i,k} \leq p_{i,k} - p_{i,k-1} \;\forall\; i, k$$

variables: $\sum_{i=1}^{N} K_i$

equality constraints: 1

inequality constraints: $2 \sum_{i=1}^{N} K_i$

solved by simply sorting prices

Dispatch with convex quadratic costs

$$\begin{aligned}
\min_{\{P_i\}_{i=1}^N} \quad & \sum_{i=1}^N (a_i P_i + c_i P_i^2) \\
\text{s.to} \quad & \sum_{i=1}^N P_i = D \\
& P_i^{\min} \leq P_i \leq P_i^{\max},\ i = 1, \ldots, N
\end{aligned}$$

- convex quadratic costs $c_i > 0$ [\$/(MW)^2]
- solved as a quadratic program (QP)
- no uniqueness issues

Lagrangian function

For *general* (strictly) convex and differentiable $C_i(P_i)$

Introduce **Lagrange multipliers**:

$$\min_{\{P_i\}_{i=1}^N} \quad \sum_{i=1}^N C_i(P_i)$$

$$\text{s.to} \quad \sum_{i=1}^N P_i = D \qquad \longleftarrow \quad \lambda$$

$$P_i^{\min} \leq P_i \leq P_i^{\max}, \; i = 1, \ldots, N \qquad \longleftarrow \quad \{(\underline{\mu}_i, \overline{\mu}_i)\}_{i=1}^N$$

Lagrangian function

$$L = \sum_{i=1}^N C_i(P_i) - \lambda \left(\sum_{i=1}^N P_i - D \right)$$
$$+ \sum_{i=1}^N \underline{\mu}_i (P_i^{\min} - P_i) + \sum_{i=1}^N \overline{\mu}_i (P_i - P_i^{\max})$$

Optimality conditions

For $i = 1, \ldots, N$,

1. $\sum_{i=1}^{N} P_i^* = D; \quad P_i^{\min} \leq P_i^* \leq P_i^{\max}$
2. $\underline{\mu}_i^* \geq 0; \quad \overline{\mu}_i^* \geq 0$
3. $C_i'(P_i^*) = \lambda^* - \overline{\mu}_i^* + \underline{\mu}_i^*$
4. $\underline{\mu}_i^*(P_i^{\min} - P_i^*) = 0; \quad \overline{\mu}_i^*(P_i^* - P_i^{\max}) = 0$

Conditions 3 and 4 can be equivalently written as

$$C_i'(P_i^*) = \begin{cases} \lambda^*, & \text{if } P_i^{\min} < P_i^* < P_i^{\max} \\ \lambda^* + \underline{\mu}_i^*, & \text{if } P_i^* = P_i^{\min} \\ \lambda^* - \overline{\mu}_i^*, & \text{if } P_i^* = P_i^{\max} \end{cases}$$

Optimality conditions simplified

Because $\underline{\mu}_i^*, \overline{\mu}_i^* \geq 0$, optimality conditions become

$$1. \quad \sum_{i=1}^{N} P_i^* = D$$

$$2.' \quad \begin{cases} C_i'(P_i^*) = \lambda^*, & \text{if } P_i^{\min} < P_i^* < P_i^{\max} \\ C_i'(P_i^*) \geq \lambda^*, & \text{if } P_i^* = P_i^{\min} \\ C_i'(P_i^*) \leq \lambda^*, & \text{if } P_i^* = P_i^{\max} \end{cases}$$

- Any $P_{G_1}^*, \ldots, P_{G_N}^*, \lambda^*$ satisfying the above conditions will be optimal
- $C_i'(P_i)$ is the *marginal* or *incremental* cost for unit i
- Optimal dispatch when all units not operating at their limits have the same incremental cost

Solving economic dispatch

- Strict convexity implies that $C_i'(P_i)$ is increasing in P_i
- Define the increasing function $f_i = (C_i')^{-1}$
- Given a λ, the $\{P_i\}_{i=1}^N$ satisfying condition 2' are expressed as

$$P_i(\lambda) = \begin{cases} f_i(\lambda), & \text{if } C_i'(P_i^{\min}) < \lambda < C_i'(P_i^{\max}) \\ P_i^{\min}, & \text{if } \lambda \leq C_i'(P_i^{\min}) \\ P_i^{\max}, & \text{if } \lambda \geq C_i'(P_i^{\max}) \end{cases}$$

- **Goal**: find λ^* such that $\{P_i\}_{i=1}^N$ satisfy condition 1 (power balance) too
- $P_i(\lambda)$ and $\sum_{i=1}^N P_i(\lambda) - D$ are increasing in λ
- Bisection on λ until $|\sum_{i=1}^N P_i(\lambda) - D| \leq \epsilon$ (tolerance)

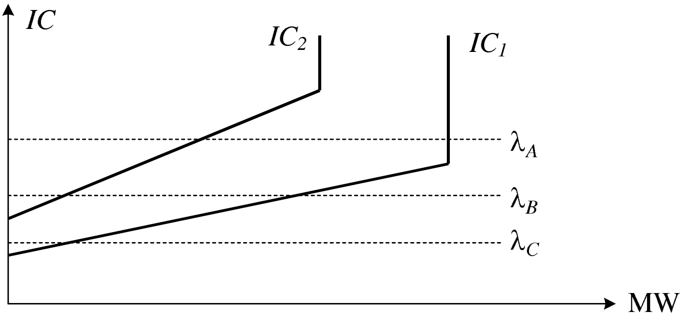
λ-iteration or bisection method

Given tolerance $\epsilon > 0$, start with $\underline{\lambda}$ and $\bar{\lambda}$ so that $\underline{\lambda} \leq \lambda^* \leq \bar{\lambda}$

for example: $\underline{\lambda} = \min_i C_i'(P_i^{\min})$ and $\bar{\lambda} = \max_i C_i'(P_i^{\max})$.

1. Set $\lambda = (\underline{\lambda} + \bar{\lambda})/2$
2. Find $P_i(\lambda)$ for all i as follows
   2.1 If $C_i'(P_i^{\min}) \geq \lambda$, set $P_i(\lambda) = P_i^{\min}$
   2.2 If $C_i'(P_i^{\max}) \leq \lambda$, set $P_i(\lambda) = P_i^{\max}$
   2.3 Otherwise, set $P_i(\lambda)$ as the solution to $C_i'(P_i) = \lambda$
3. If $\sum_{i=1}^N P_i(\lambda) - D > \epsilon$, set $\bar{\lambda} := \lambda$ and go to Step 1
4. If $\sum_{i=1}^N P_i(\lambda) - D < -\epsilon$, set $\underline{\lambda} := \lambda$ and go to Step 1
5. Else $\{P_i(\lambda)\}_{i=1}^N$ is the solution within the specified tolerance

Graphical illustration



λ-iteration method for two generators with quadratic costs

λ-iteration with convex piecewise linear costs

Convex piecewise linear $C_i(P_i)$ is non-differentiable

KKT conditions with constraints $P_i^{\min} \leq P_i \leq P_i^{\max}$ kept implicit

$$\text{Lagrangian function}: L(P_1, \ldots, P_N, \lambda) = \sum_{i=1}^{N} C_i(P_i) - \lambda \left(\sum_{i=1}^{N} P_i - D \right)$$

Optimality conditions:

1. $\displaystyle\sum_{i=1}^{N} P_i^* = D$

2. $\displaystyle P_i^* \in \arg \min_{P_i^{\min} \leq P_i \leq P_i^{\max}} \{C_i(P_i) - \lambda^* P_i\} \quad \forall\ i$

Step 2 of λ-iteration now becomes

$$\min_{P_i^{\min} \leq P_i \leq P_i^{\max}} \{C_i(P_i) - \lambda P_i\}$$

Economic interpretation

Optimality conditions with general costs

1. $\sum_{i=1}^{N} P_i^* = D$

2. $P_i^* \in \arg \min_{P_i^{\min} \leq P_i \leq P_i^{\max}} \{C_i(P_i) - \lambda^* P_i\} \quad \forall\ i$

Interpret multiplier λ [\$/MW] as the **price** at which unit i will be compensated for producing P_i MW

Given λ, unit i chooses P_i so its net cost $C_i(P_i) - \lambda P_i$ is minimized (net revenue maximized)

Optimal multiplier λ^* maximizes the total net revenue

Sensitivity interpretation

$$C(D) := \min_{\{P_i^{\min} \leq P_i \leq P_i^{\max}\}_{i=1}^N} \sum_{i=1}^N C_i(P_i)$$

$$\text{s.to} \quad \sum_{i=1}^N P_i = D \qquad \longleftarrow \lambda$$

Cost of generation as a function of demand D is a perturbation function!

If the problem is convex, then function $C(D)$ is convex

If $C(D)$ is differentiable and strong duality holds, then $C'(D) = \lambda^*(D)$

Marginal price λ^*: extra price one would pay for slight increase in demand

Participation factors

Consider an optimal dispatch P and a small load variation ΔD

Assume new dispatch does *not* alter which units are at $P_i^{\min}$ or $P_i^{\max}$

current ED

$$\sum_i P_i = D$$

$$\lambda = C_i'(P_i)$$

perturbed ED

$$\sum_i \Delta P_i = \Delta D$$

$$\Delta\lambda = C_i''(P_i)\Delta P_i$$

Participation factors: How much each unit contributes to serve the new load?

$$\frac{\Delta P_i}{\Delta D} = \frac{\Delta\lambda / C_i''(P_i)}{\sum_k \Delta\lambda / C_k''(P_k)} = \frac{1/C_i''(P_i)}{\sum_k 1/C_k''(P_k)} \in (0,1)$$

constant for quadratic costs

Used for fast generation in response to load variation $P_i^{\text{new}} = P_i + \frac{\Delta P_i}{\Delta D}\Delta D$

Elastic demand

Elastic demand is characterized by its utility function $U_j(D_j)$ [$/h]:

the benefit by consuming D_j MW for 1h

$$\begin{aligned}
\min_{\{P_i\}_{i=1}^N,\{D_j\}_{j=1}^M} \quad & \sum_{i=1}^N C_i(P_i) - \sum_{j=1}^M U_j(D_j) \\
\text{s.to} \quad & \sum_{i=1}^N P_i = \sum_{j=1}^M D_j \\
& P_i^{\min} \leq P_i \leq P_i^{\max}, \; i = 1, \ldots, N \\
& D_j^{\min} \leq D_j \leq D_j^{\max}, \; j = 1, \ldots, M
\end{aligned}$$

- demand bids $U_j(D_j)$ are submitted by utilities, load serving entities, aggregators, or industrial costumers
- convex problem for concave $U_j(D_j)$'s *(diminishing returns or buy in bulk)*

Transmission losses

Accounting for losses on transmission lines (3-5%):

$$\min_{\{P_i\}_{i=1}^N} \quad \sum_{i=1}^N C_i(P_i)$$

$$\text{s.to} \quad \sum_{i=1}^N P_i = P_{\text{loss}}(\mathbf{p}) \qquad \longleftarrow \quad \lambda$$

$$P_i^{\min} \leq P_i \leq P_i^{\max}, \; i = 1, \ldots, N$$

where $\mathbf{p} := [P_1 \; \ldots \; P_N]^\top$ captures generation and (fixed or elastic) demand

- Losses $P_{\text{loss}}(\mathbf{p})$ is a non-linear function of $\mathbf{p}$ (non-convex problem)
- In the past, modeled as quadratic functions: $P_{\text{loss}}(\mathbf{p}) = \mathbf{p}^\top \mathbf{B}\mathbf{p} + \mathbf{c}^\top \mathbf{p}$
- Now typically calculated by (successive) linearization of power flow equations or via the optimal power flow problem

Penalty factors

Lagrangian function (non-convex problem; KKT conditions are only necessary)

$$L = \sum_{i=1}^{N} C_i(P_i) - \lambda \left(\sum_{i=1}^{N} P_i - P_{\text{loss}}(\mathbf{p}) \right) + \sum_{i=1}^{N} \underline{\mu}_i (P_i^{\min} - P_i) + \sum_{i=1}^{N} \overline{\mu}_i (P_i - P_i^{\max})$$

Optimality conditions simplified:

1. $$\sum_{i=1}^{N} P_i^* = P_{\text{loss}}(\mathbf{p}^*)$$

2.′ $$\begin{cases} C_i'(P_i^*) = \lambda^* \left(1 - \left. \frac{\partial P_{\text{loss}}(\mathbf{p})}{\partial P_i} \right|_{\mathbf{p}=\mathbf{p}^*} \right), & \text{if } P_i^{\min} < P_i^* < P_i^{\max} \\ C_i'(P_i^*) \geq \lambda^* \left(1 - \left. \frac{\partial P_{\text{loss}}(\mathbf{p})}{\partial P_i} \right|_{\mathbf{p}=\mathbf{p}^*} \right), & \text{if } P_i^* = P_i^{\min} \\ C_i'(P_i^*) \leq \lambda^* \left(1 - \left. \frac{\partial P_{\text{loss}}(\mathbf{p})}{\partial P_i} \right|_{\mathbf{p}=\mathbf{p}^*} \right), & \text{if } P_i^* = P_i^{\max} \end{cases}$$

Marginal costs: $f_i C_i'(P_i^*)$ for **penalty factors** $f_i = \left(1 - \left. \frac{\partial P_{\text{loss}}(\mathbf{p})}{\partial P_i} \right|_{\mathbf{p}=\mathbf{p}^*} \right)^{-1}$

Solving economic dispatch with losses

Solve economic dispatch ignoring losses to get initial dispatch $\mathbf{p}^0$

At iteration $t = 0, 1, \ldots,$ do

1. Calculate $P_{\text{loss}}(\mathbf{p}^t)$ and $f_i^t := \left(1 - \left.\frac{\partial P_{\text{loss}}(\mathbf{p})}{\partial P_i}\right|_{\mathbf{p}^t}\right)^{-1}$ for $i = 1, \ldots, N$

2. Find $\mathbf{p}^{t+1}$ via λ-iteration with costs $\{f_i^t C_i\}_i$ and $\sum_{i=1}^N P_i = P_{\text{loss}}(\mathbf{p}^t)$

3. If $\|\mathbf{p}^{t+1} - \mathbf{p}^t\|_2 < \epsilon$ stop; otherwise go to Step 1

- heuristic manner to account for losses
- how to compute $\nabla P_{\text{loss}}(\mathbf{p})$?

Computing penalty factors

Active power injections are not independent (swing bus 1 makes up for losses)

$$\mathbf{p} = \begin{bmatrix} p_1(\tilde{\mathbf{p}}) \\ \tilde{\mathbf{p}} \end{bmatrix} \in \mathbb{R}^N$$

Active power losses: $P_{\text{loss}}(\mathbf{p}) = \mathbf{p}^\top \mathbf{1}_N = p_1(\tilde{\mathbf{p}}) + \tilde{\mathbf{p}}^\top \mathbf{1}_{N-1}$

Interest in finding $\nabla_{\mathbf{p}} P_{\text{loss}}(\mathbf{p})$ (assume $\frac{\partial P_{\text{loss}}(\mathbf{p})}{\partial p_1} = 0$)

Gradient vector for losses: $\nabla_{\tilde{\mathbf{p}}} P_{\text{loss}}(\tilde{\mathbf{p}}) = \mathbf{1} + \nabla_{\tilde{\mathbf{p}}} p_1(\tilde{\mathbf{p}})$

Resort to power flow equations to find $\nabla_{\tilde{\mathbf{p}}} p_1(\tilde{\mathbf{p}})$

Finding $\nabla_{\tilde{\mathbf{p}}} p_1(\tilde{\mathbf{p}})$ from power flow equations

power flow equations (dependence on voltage magnitudes has been ignored)

$$p_1 = f_1(\boldsymbol{\theta})$$

$$\tilde{\mathbf{p}} = \mathbf{f}_2(\boldsymbol{\theta})$$

where $\boldsymbol{\theta} := [\theta_2 \ \cdots \ \theta_N]^\top$ is the vector of voltage phases $(\theta_1 = 0)$

Jacobian matrix: $\frac{\partial \tilde{\mathbf{p}}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{f}_2(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{J}(\boldsymbol{\theta}) \in \mathbb{R}^{(N-1)\times(N-1)}$

Jacobian matrix of inverse function: under technical conditions
$\frac{\partial \boldsymbol{\theta}(\tilde{\mathbf{p}})}{\partial \tilde{\mathbf{p}}} = \frac{\partial \mathbf{f}_2^{-1}(\tilde{\mathbf{p}})}{\partial \tilde{\mathbf{p}}} = [\mathbf{J}(\boldsymbol{\theta}(\tilde{\mathbf{p}}))]^{-1} \in \mathbb{R}^{(N-1)\times(N-1)}$

Gradient vector:

$$\nabla_{\tilde{\mathbf{p}}} P_{\text{loss}}(\tilde{\mathbf{p}}) = \mathbf{1} + \left(\frac{\partial \boldsymbol{\theta}(\tilde{\mathbf{p}})}{\partial \tilde{\mathbf{p}}}\right)^\top \nabla_{\boldsymbol{\theta}} f_1(\boldsymbol{\theta}) = \mathbf{1} + [\mathbf{J}(\tilde{\mathbf{p}})]^{-\top} \nabla_{\boldsymbol{\theta}} f_1(\boldsymbol{\theta})$$