ECE 5314: Power System Operation & Control

Lecture 4: Lagrangian Duality

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R3 S. Boyd and L. Vandenberghe, *Convex Optimization*, Chapters 5.1-5.6.

Lagrangian function

primal problem: any problem in standard form (convexity not assumed yet)

$$
\begin{aligned}
p^* = \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\
\text{s.to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\
& h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p
\end{aligned}
$$

Lagrangian function: $L : \mathbb{R}^{n+m+p} \to \mathbb{R}$

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{i=1}^{p} \nu_i h_i(\mathbf{x})$$

- λ_i, ν_i: *Lagrange multipliers* or *dual variables*
- objective augmented with weighted sum of constraint functions

Dual function

Dual function: $g : \mathbb{R}^{m+p} \rightarrow \mathbb{R}$

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x}} f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{i=1}^{p} \nu_i h_i(\mathbf{x})$$

- can be $-\infty$ for some $(\boldsymbol{\lambda}, \boldsymbol{\nu})$
- g is concave even for nonconvex $f_i(\mathbf{x})$ or $h_i(\mathbf{x})$! [Why?]

Example: LP

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & \mathbf{G}\mathbf{x} \leq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

Lagrangian function: $L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = (\mathbf{c} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{A}^\top \boldsymbol{\nu})^\top \mathbf{x} - \boldsymbol{\lambda}^\top \mathbf{h} - \boldsymbol{\nu}^\top \mathbf{b}$

Dual function: $g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \begin{cases} -\boldsymbol{\lambda}^\top \mathbf{h} - \boldsymbol{\nu}^\top \mathbf{b} & , \ \mathbf{c} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{A}^\top \boldsymbol{\nu} = \mathbf{0} \\ -\infty & , \ \text{otherwise} \end{cases}$

Lower bound property (weak duality)

if $\boldsymbol{\lambda} \succeq \mathbf{0}$ and $\mathbf{x}$ is primal feasible, then

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leq f_0(\mathbf{x})$$

Proof: if $f_i(\mathbf{x}) \leq 0$, $h_i(\mathbf{x}) = 0$, and $\boldsymbol{\lambda} \succeq \mathbf{0}$,

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{y}} f_0(\mathbf{y}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{y}) + \sum_{i=1}^{p} \nu_i h_i(\mathbf{y}) \leq f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) \leq f_0(\mathbf{x})$$

Duality gap between primal feasible $\mathbf{x}$ and dual feasible $(\boldsymbol{\lambda} \succeq \mathbf{0}, \boldsymbol{\nu})$,

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) \leq p^*$$

dual feasible points yield lower bounds on optimal value!

Dual problem

find the best lower bound on p^*:

$$d^* = \max_{\boldsymbol{\lambda},\boldsymbol{\nu}} \; g(\boldsymbol{\lambda},\boldsymbol{\nu})$$
$$\text{s.to } \boldsymbol{\lambda} \succeq \mathbf{0}$$

- **dual problem** associated with primal problem
- always convex problem even for nonconvex primal
- **weak duality**: $d^* \leq p^*$
- $p^* - d^*$ is optimal duality gap

Strong duality

- for convex problems, we (usually) have strong duality

$$d^* = p^*$$

- then $(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$ serves as **certificate of optimality** for primal optimal $\mathbf{x}^*$

- many conditions or *constraint qualifications* guarantee strong duality for convex problems

- **Slater's condition**: if primal problem is convex and strictly feasible

$$f_i(\mathbf{x}) < 0, \quad i = 1, \ldots, m, \quad \mathbf{Ax} = \mathbf{b}$$

($f_i(\mathbf{x}) \leq 0$ is allowed for affine inequality constraints), then $d^* = p^*$

Dual of linear program

primal LP: n variables, m inequality and p equality constraints

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & \mathbf{G}\mathbf{x} \leq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

dual of LP is LP!

$$\begin{aligned} \max_{\boldsymbol{\lambda},\boldsymbol{\nu}} \quad & -\boldsymbol{\lambda}^\top \mathbf{h} - \boldsymbol{\nu}^\top \mathbf{b} \\ \text{s.to} \quad & \mathbf{c} + \mathbf{G}^\top \boldsymbol{\lambda} + \mathbf{A}^\top \boldsymbol{\nu} = \mathbf{0} \\ & \boldsymbol{\lambda} \succeq \mathbf{0} \end{aligned}$$

- $(m+p)$ variables, n equality and m nonnegativity constraints
- What happens when primal problem is unbounded below?
- strong duality holds always in LP, unless $p^* = +\infty$ and $d^* = -\infty$

Dual of quadratic program

primal QP: assume $\mathbf{P} \succ \mathbf{0}$ for simplicity

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^\top \mathbf{P} \mathbf{x} \\ \text{s.to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Lagrangian function: $\mathbf{x}^\top \mathbf{P} \mathbf{x} + \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b})$

- setting $\nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$, yields $\mathbf{x} = -\frac{1}{2}\mathbf{P}^{-1}\mathbf{A}^\top \boldsymbol{\lambda}$

Dual function: $g(\boldsymbol{\lambda}) = -\frac{1}{4}\boldsymbol{\lambda}^\top \mathbf{A}\mathbf{P}^{-1}\mathbf{A}^\top \boldsymbol{\lambda} - \mathbf{b}^\top \boldsymbol{\lambda}$

- concave quadratic; all $\boldsymbol{\lambda} \succeq \mathbf{0}$ are dual feasible

dual of QP is QP!: $\max \{g(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \succeq \mathbf{0}\}$

Complementary slackness

suppose $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*$ are primal dual feasible with zero duality gap

(hence, they are primal and dual optimal)

$$\begin{aligned} f_0(\mathbf{x}^*) = g(\boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) &= \min_{\mathbf{y}} f_0(\mathbf{y}) + \sum_{i=1}^m \lambda_i^* f_i(\mathbf{y}) + \sum_{i=1}^p \nu_i^* h_i(\mathbf{y}) \\ &\leq f_0(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i^* f_i(\mathbf{x}^*) \leq f_0(\mathbf{x}^*) \end{aligned}$$

Complementary slackness condition:

$$\sum_{i=1}^m \lambda_i^* f_i(\mathbf{x}^*) = 0 \quad \Longrightarrow \quad \lambda_i^* f_i(\mathbf{x}^*) = 0,\ i = 1, \ldots, m$$

- if i-th constraint inactive at optimum, then $\lambda_i^* = 0$
- if $\lambda_i^* > 0$ at optimum, then i-th constraint active at optimum

Lagrangian optimality

Suppose

- f_i and h_i are differentiable
- $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*$ are primal-dual optimal with zero duality gap

due to complementary slackness:

$$L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) = \min_{\mathbf{y}} L(\mathbf{y}, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$$

i.e., $\mathbf{x}^*$ minimizes $L(\mathbf{x}, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*)$; therefore $\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*) = \mathbf{0}$

$$\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\mathbf{x}^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(\mathbf{x}^*) = \mathbf{0}$$

Karush-Kuhn-Tucker optimality conditions

For differentiable functions, if $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*$ are primal-dual optimal with zero duality gap, then they satisfy the KKT conditions:

1. Primal feasibility: $f_i(\mathbf{x}^*) \leq 0$, $h_i(\mathbf{x}^*) = 0$
2. Dual feasibility: $\boldsymbol{\lambda}^* \succeq \mathbf{0}$
3. Lagrangian optimality:
   $\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(\mathbf{x}^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(\mathbf{x}^*) = \mathbf{0}$
4. Complementary slackness: $\lambda_i^* f_i(\mathbf{x}^*) = 0$

Conversely: for **convex** problems, if $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*$ satisfy the KKT conditions, then they are primal-dual optimal and strong duality holds

Examples

Minimization over the nonnegative orthant: $\min\{f_0(\mathbf{x}) : \mathbf{x} \succeq \mathbf{0}\}$

1. $\mathbf{x}^* \succeq \mathbf{0}$
2. $\boldsymbol{\lambda}^* \succeq \mathbf{0}$
3. $\nabla f_0(\mathbf{x}^*) - \boldsymbol{\lambda}^* = \mathbf{0}$
4. $x_i^* \lambda_i^* = 0$

$1'$. $\mathbf{x}^* \succeq \mathbf{0}$

$2'$. $\nabla f_0(\mathbf{x}^*) \succeq \mathbf{0}$

$3'$. $x_i^* [\nabla f_0(\mathbf{x}^*)]_i = 0$

Minimization with equality constraints: $\min\{f_0(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}\}$

$$\nabla f_0(\mathbf{x}^*) + \mathbf{A}^\top \boldsymbol{\nu}^* = \mathbf{0}$$

$$\mathbf{A}\mathbf{x}^* = \mathbf{b}$$

Q.4.1 Apply this for the quadratic cost $f_0(\mathbf{x}) = \mathbf{x}^\top \mathbf{x}$.

Implicit constraints

primal problem: (no convexity or differentiability assumptions)

$$\begin{aligned} p^* = \min \quad & f_0(\mathbf{x}) \\ \text{s.to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \\ & \mathbf{x} \in \mathcal{X}, \end{aligned}$$

implicit constraints $\mathbf{x} \in \mathcal{X}$ (we decide which constraints are kept implicit)

dual function: Lagrangian function is now minimized over $\mathcal{X}$

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x} \in \mathcal{X}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\mathbf{x} \in \mathcal{X}} \left\{ f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{i=1}^{p} \nu_i h_i(\mathbf{x}) \right\}$$

KKT optimality conditions (general form)

if $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*$ are primal-dual optimal with zero duality gap, then they satisfy:

1. Primal feasibility: $f_i(\mathbf{x}^*) \leq 0$, $h_i(\mathbf{x}^*) = 0$, $\mathbf{x}^* \in \mathcal{X}$
2. Dual feasibility: $\boldsymbol{\lambda}^* \succeq \mathbf{0}$
3. Lagrangian optimality: (includes $\mathcal{X}$, no assumptions on differentiability)

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathcal{X}} f_0(\mathbf{x}) + \sum_i \lambda_i^* f_i(\mathbf{x}) + \sum_i \nu_i^* h_i(\mathbf{x})$$

4. Complementary slackness: $\lambda_i^* f_i(x^*) = 0$

Conversely: for **convex** problems, if $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\nu}^*$ satisfy the above conditions, then they are (primal,dual) optimal and strong duality holds

Perturbation and sensitivity analysis

Perturbation function of an optimization problem

$$\begin{aligned} p(\mathbf{u}) = \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.to} \quad & f_i(\mathbf{x}) \leq u_i, \quad i = 1, \ldots, m \end{aligned}$$

Claim 1: If the problem is convex wrt $\mathbf{x}$, then $p(\mathbf{u})$ is a convex function

Claim 2: If $p(\mathbf{u})$ is differentiable at $\mathbf{u}_0$ and strong duality holds, then

$$\left.\frac{\partial p(\mathbf{u})}{\partial u_i}\right|_{\mathbf{u}=\mathbf{u}_0} = -\lambda_i^*$$

where λ_i^* is the optimal Lagrange multiplier for the i-th constraint