ECE 5314: Power System Operation & Control

Lecture 3: Convex Optimization Problems

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R3 S. Boyd and L. Vandenberghe, *Convex Optimization*, Chapters 4.1-4.4.

Optimization problem in standard form

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{aligned}$$

feasible set: the set of feasible points

optimal value: $f^\star = \inf_{\mathbf{x} \in \mathcal{X}} f_0(\mathbf{x})$ [Why $\inf$ and not simply $\min$?]

- $f^\star = -\infty$ (unbounded problem)
- $f^\star = +\infty$ (infeasible problem)

optimal point: a *feasible* point $\mathbf{x}$ attaining the optimum $f(\mathbf{x}) = f^\star$

optimal set: the set of optimal points

Feasibility problem

$$\begin{aligned} &\text{find} \quad \mathbf{x} \\ &\text{s.to} \quad f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ &\qquad\quad h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{aligned}$$

can be considered an optimization problem with $f_0(\mathbf{x}) = 0$:

$$\begin{aligned} &\min_{\mathbf{x}} \quad 0 \\ &\text{s.to} \quad f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ &\qquad\quad h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{aligned}$$

either find $\mathbf{x} \in \mathcal{X}$ $(f^\star = 0)$; or determine that $\mathcal{X} = \emptyset$ $(f^\star = \infty)$

Convex optimization problem

Convex optimization problem in standard form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & \mathbf{a}_i^\top \mathbf{x} = b_i, \quad i = 1, \ldots, p \end{aligned}$$

1. $f_0, f_1, \ldots, f_m$ are convex
2. equality constraints are affine (alternatively $\mathbf{A}\mathbf{x} = \mathbf{b}$)

Q.3.1 Show that the feasible set is convex.

Locally and globally optimal solutions

Consider problem $\min_{\mathbf{x}\in\mathcal{X}} f_0(\mathbf{x})$

Definitions:

- $\mathbf{x} \in \mathcal{X}$ is locally optimal if there exists an $R > 0$ such that

$$\text{for all } \mathbf{y} \in \mathcal{X} \text{ with } \|\mathbf{y} - \mathbf{x}\| \leq R \implies f_0(\mathbf{x}) \leq f_0(\mathbf{y})$$

- $\mathbf{x} \in \mathcal{X}$ is optimal (or simply optimal) if

$$\text{for all } \mathbf{y} \in \mathcal{X} \implies f_0(\mathbf{x}) \leq f_0(\mathbf{y})$$

Important properties:

1. For convex problems, any local solution is also global
2. If additionally $f_0(\mathbf{x})$ is strictly convex, there is at most one minimum
3. The optimal set $\mathcal{X}_{\text{opt}}$ is convex

An optimality criterion

Consider problem $\min_{\mathbf{x} \in C} f_0(\mathbf{x})$ with differentiable f_0 and convex C

Condition:

- If $\mathbf{x}$ is a local minimum, then

  $\nabla f_0(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \geq 0$ for all $\mathbf{y} \in C$

- The condition becomes sufficient

  when f_0 is convex

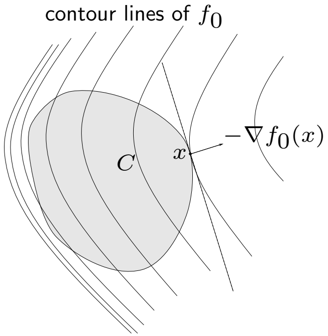


Figure: Source [R3]

Unconstrained problem $(C = \mathbb{R}^n)$: $\mathbf{x}$ is optimal iff $\nabla f_0(\mathbf{x}) = \mathbf{0}$

Epigraph trick

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}, t} \quad & t \\ \text{s.to} \quad & f_0(\mathbf{x}) \leq t \\ & f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \ldots, p \end{aligned}$$

The variables in second problem are $(\mathbf{x}, t)$

Convexity is preserved

Restriction and relaxation

original problem

$$f^* = \min_{\mathbf{x}} \quad f_0(\mathbf{x})$$
$$\text{s.to} \quad \mathbf{x} \in C$$

new problem

$$\tilde{f}^* = \min_{\mathbf{x}} \quad f_0(\mathbf{x})$$
$$\text{s.to} \quad \mathbf{x} \in \tilde{C}$$

The new problem is a

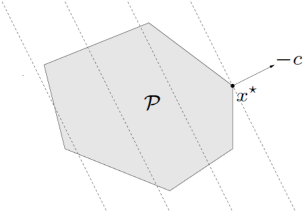
- **relaxation** of original if $C \subseteq \tilde{C}$, then $\tilde{f}^* \leq f^*$
- **restriction** of original if $\tilde{C} \subseteq C$, then $\tilde{f}^* \geq f^*$

Example: If $f_0(\mathbf{x})$ is convex and C is nonconvex, set $\tilde{C} = \text{conv}(C)$ to get a convex problem and a lower bound for the original nonconvex problem

Linear program (LP)

Linear cost; feasible set is a polyhedron

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & \mathbf{G}\mathbf{x} \preceq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$



standard form: widely used in LP literature and software (MATLAB, Sedumi)

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \succeq \mathbf{0} \end{aligned}$$

Converting LP to standard form

Inequality constraints: transform linear inequalities as

$$\mathbf{G}\mathbf{x} + \mathbf{s} = \mathbf{h} \quad \text{and} \quad \mathbf{s} \succeq \mathbf{0}$$

new vector $\mathbf{s}$ is called a *slack variable*

Unconstrained variables: decompose variable as

$$\mathbf{x} = \mathbf{x}^{+} - \mathbf{x}^{-} \quad \text{and} \quad \mathbf{x}^{+}, \mathbf{x}^{-} \succeq \mathbf{0}$$

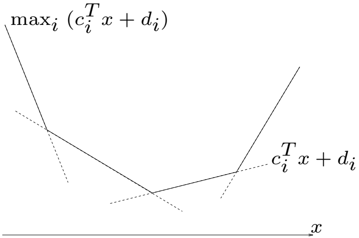
Problem in standard form

$$\begin{aligned} \min \quad & \mathbf{c}^{\top}\mathbf{x}^{+} - \mathbf{c}^{\top}\mathbf{x}^{-} \\ \text{s.to} \quad & \mathbf{G}\mathbf{x}^{+} - \mathbf{G}\mathbf{x}^{-} + \mathbf{s} = \mathbf{h} \\ & \mathbf{A}\mathbf{x}^{+} - \mathbf{A}\mathbf{x}^{-} = \mathbf{b} \\ & \mathbf{x}^{+} \succeq \mathbf{0},\ \mathbf{x}^{-} \succeq \mathbf{0},\ \mathbf{s} \succeq \mathbf{0} \end{aligned}$$

Piecewise-linear minimization

$$\min_{\mathbf{x}} \quad \max_{i=1,\ldots,K} (\mathbf{c}_i^\top \mathbf{x} + d_i)$$

$$\text{s.to} \quad \mathbf{A}\mathbf{x} \preceq \mathbf{b}$$



use epigraph trick to express problem as an LP

$$\min_{\mathbf{x},t} \quad t$$

$$\text{s.to} \quad \mathbf{c}_i^\top \mathbf{x} + d_i \leq t, \; i = 1, \ldots, K$$

$$\mathbf{A}\mathbf{x} \preceq \mathbf{b}$$

Minimizing a quadratic function

Convex problem iff $\mathbf{P} \succeq \mathbf{0}$

$$\min_{\mathbf{x}} f(\mathbf{x}) = \mathbf{x}^\top \mathbf{P}\mathbf{x} + 2\mathbf{q}^\top \mathbf{x} + r$$

nonconvex case $(\mathbf{P} \not\succeq \mathbf{0})$: unbounded below

proof: choose $\mathbf{x} = t\mathbf{v}$ and $t \to \infty$, where $\mathbf{P}\mathbf{v} = \lambda\mathbf{v}$ for $\lambda < 0$

convex case $(\mathbf{P} \succeq \mathbf{0})$: $\mathbf{x}$ is optimal if and only if

$$\nabla f(\mathbf{x}) = 2\mathbf{P}\mathbf{x} + 2\mathbf{q} = \mathbf{0}$$

- $\mathbf{q} \in \text{range}(\mathbf{P})$: $\mathbf{x}^* = -\mathbf{P}^\dagger \mathbf{q}$ is *a minimizer* and $f^* = r - \mathbf{q}^\top \mathbf{P}^\dagger \mathbf{q}$

  special case $\mathbf{P} \succ \mathbf{0}$: *the unique minimizer* $(\mathbf{P}^\dagger = \mathbf{P}^{-1})$

- $\mathbf{q} \notin \text{range}(\mathbf{P})$: unbounded below [Why?]

Least squares fit

Minimize (squared) Euclidean norm with $\mathbf{A} \in \mathbb{R}^{M \times N}$

$$\min_{\mathbf{x}} \; \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 = \mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x} - 2\mathbf{b}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{b}$$

Q.3.2 Show that the Hessian is psd, so the problem is convex

Geometrically: project $\mathbf{b}$ on $\mathrm{range}(\mathbf{A})$

Minimizer: set gradient equal to zero to get the *normal equations*

$$\mathbf{A}^\top \mathbf{A}\mathbf{x} = \mathbf{A}^\top \mathbf{b}$$

- system is always solvable since $\mathbf{A}^\top \mathbf{b} \in \mathrm{range}(\mathbf{A}^\top \mathbf{A})$
- if $\mathrm{rank}(\mathbf{A}) = N$, unique LS solution $\mathbf{x}_{\mathrm{ls}} = (\mathbf{A}^\top \mathbf{A})^{-1}\mathbf{A}^\top \mathbf{b}$

Quadratic program (QP)

$$\min_{\mathbf{x}} \quad \mathbf{x}^\top \mathbf{P}\mathbf{x} + 2\mathbf{q}^\top \mathbf{x} + r$$

$$\text{s.to} \quad \mathbf{G}\mathbf{x} \preceq \mathbf{h}, \;\; \mathbf{A}\mathbf{x} = \mathbf{b}$$

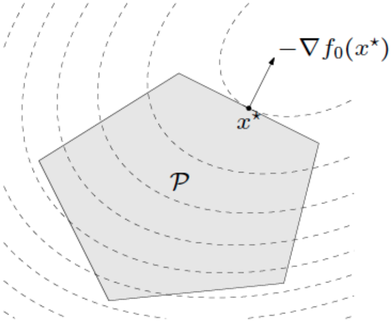


Figure: Source [R3]

- quadratic objective, linear constraints (inequalities and equalities)
- convex problem if $\mathbf{P} \succeq \mathbf{0}$
- NP-hard problem if $\mathbf{P} \not\succeq \mathbf{0}$

QCQP and SOCP

quadratically constrained quadratic program (QCQP):

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^\top \mathbf{P}_0 \mathbf{x} + 2\mathbf{q}_0^\top \mathbf{x} + r_0 \\ \text{s.to} \quad & \mathbf{x}^\top \mathbf{P}_i \mathbf{x} + 2\mathbf{q}_i^\top \mathbf{x} + r_i \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

convex if $\mathbf{P}_i \succeq \mathbf{0}$ for $i = 0, 1, \ldots, m$; NP-hard in general, otherwise

second-order cone programs (SOCP):

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{f}^\top \mathbf{x} \\ \text{s.to} \quad & \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^\top \mathbf{x} + d_i, \quad i = 1, \ldots, m \end{aligned}$$

Robust linear program

LP with uncertain parameters: $\mathbf{a}_i \in \mathcal{E}_i = \{\bar{\mathbf{a}}_i + \mathbf{P}_i\mathbf{u} : \ \|\mathbf{u}\|_2 \leq 1\}$

$$\min_{\mathbf{x}} \ \mathbf{c}^\top \mathbf{x}$$
$$\text{s.to} \ \ \mathbf{a}_i^\top \mathbf{x} \leq b_i \quad \text{for all } \mathbf{a}_i \in \mathcal{E}_i, \quad i = 1, \ldots, m$$

Robust LP has infinitely many constraints...

Key point: $\mathbf{a}_i^\top \mathbf{x} \leq b_i \ \forall \ \mathbf{a}_i \in \mathcal{E}_i \ \Leftrightarrow \ \max_{\|\mathbf{u}\|_2 \leq 1} \{\bar{\mathbf{a}}_i^\top \mathbf{x} + \mathbf{u}^\top \mathbf{P}_i^\top \mathbf{x}\} \leq b_i$

Robust LP becomes SOCP!

$$\min_{\mathbf{x}} \ \ \mathbf{c}^\top \mathbf{x}$$
$$\text{s.to} \ \ \bar{\mathbf{a}}_i^\top \mathbf{x} + \|\mathbf{P}_i^\top \mathbf{x}\|_2 \leq b_i, \quad i = 1, \ldots, m$$

Semidefinite program (SDP)

Semidefinite program in **standard form** (symmetric $\mathbf{A}_i$ with $i = 0, \ldots, p$)

$$\begin{aligned} \min_{\mathbf{X}} \quad & \mathrm{Tr}(\mathbf{A}_0 \mathbf{X}) \\ \text{s.to} \quad & \mathrm{Tr}(\mathbf{A}_i \mathbf{X}) = b_i, \quad i = 1, \ldots, p \\ & \mathbf{X} \succeq 0 \end{aligned}$$

LMI constraint is convex

SDP in **inequality form**

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & x_1 \mathbf{A}_1 + \ldots + x_n \mathbf{A}_n \preceq \mathbf{A}_0 \end{aligned}$$

Although LMIs correspond to a set of polynomial inequalities, they can be handled efficiently by modern solvers

Maximum eigenvalue minimization

$$\min_{\mathbf{x}} \ \lambda_{\max}(\mathbf{A}(x))$$

where $\mathbf{A}(\mathbf{x}) = \mathbf{A}_0 + x_1\mathbf{A}_1 + \ldots + x_m\mathbf{A}_m, \quad \mathbf{A}_i \in \mathbb{S}^n$

problem can be cast as an SDP:

$$\begin{aligned} \min_{\mathbf{x},t} \quad & t \\ \text{s.to} \quad & \mathbf{A}(\mathbf{x}) \preceq t\mathbf{I} \end{aligned}$$

Conic problem hierarchy

Model generality: LP $<$ QP $<$ QCQP $<$ SOCP $<$ SDP

Solution efficiency: LP $>$ QP $>$ QCQP $>$ SOCP $>$ SDP

Q.3.3 Show that that LP $\subset$ QP $\subset$ QCQP $\subset$ SOCP $\subset$ SDP

where symbol $\subset$ means *"special case of"*.

Example: an SOCP constraint can be expressed as SDP constraint as

$$\|\mathbf{A}\mathbf{x}+\mathbf{b}\|_2 \leq \mathbf{c}^\top\mathbf{x}+d \quad \Longleftrightarrow \quad \begin{bmatrix} (\mathbf{c}^\top\mathbf{x}+d)\mathbf{I} & \mathbf{A}\mathbf{x}+\mathbf{b} \\ (\mathbf{A}\mathbf{x}+\mathbf{b})^\top & \mathbf{c}^\top\mathbf{x}+d \end{bmatrix} \succeq 0$$

Hint: the above follows from property of Schur's complement