ECE 5314: Power System Operation & Control

Lecture 2: Convex Sets and Convex Functions

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R3 S. Boyd and L. Vandenberghe, *Convex Optimization*, Chapters 2.1-2.3, 3.1-3.3.

What is an optimization problem?

Minimization of a function subject to constraints on its variables

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.to} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \quad \text{(\textbf{inequality constraints})} \\ & h_j(\mathbf{x}) = 0, \quad j = 1, \ldots, p \quad \text{(\textbf{equality constraints})} \end{aligned}$$

- vector of **unknowns** or **variables** $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^\top$
- **objective or cost function** $f_0(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$
- **constraint functions** $g_i(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h_j(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$
- **feasible set**: the set of points satisfying all constraints

$$\mathcal{X} := \{\mathbf{x} : g_i(\mathbf{x}) \leq 0, \ i = 1, \ldots, m; \ h_j(\mathbf{x}) = 0, \ j = 1, \ldots, p\}$$

A simple example

$$\min_{x} \quad (x_1 - 2)^2 + (x_2 - 1)^2$$

$$\text{s.to} \quad x_1^2 - x_2 \leq 0$$

$$x_1 + x_2 \leq 2$$

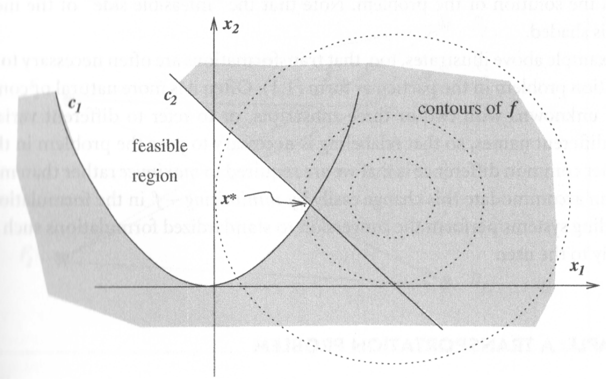
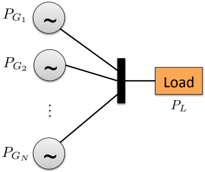


Figure: Nocedal-Wright, Numerical Optimization

Economic dispatch problem



- N generation units serving load P_L
- power output of unit i is P_{G_i} [MW]
- generation cost for unit is $C_i(P_{G_i})$ [$/h]

Problem: find the most economical power schedule

$$\begin{aligned} \min_{\{P_{G_i}\}_{i=1}^N} \quad & \sum_{i=1}^N C_i(P_{G_i}) \\ \text{s.to} \quad & \sum_{i=1}^N P_{G_i} = P_L \\ & P_{G_i} \geq 0,\ i = 1, \ldots, N \end{aligned}$$

Difficult versus easy problems

Convex vs. nonconvex: dividing line between easy and difficult problems

Convex problem: convex objective $f_0(\mathbf{x})$ and convex feasible set $\mathcal{X}$

$$\min_{\mathbf{x}\in\mathcal{X}} f_0(\mathbf{x})$$

Features of convex problems:

1. Every local minimum is a global minimum
2. Computationally tractable
   - computation time grows gracefully with problem size
   - non-heuristic stopping criteria and provable lower bounds
3. Occur often in engineering; yet sometimes hard to recognize

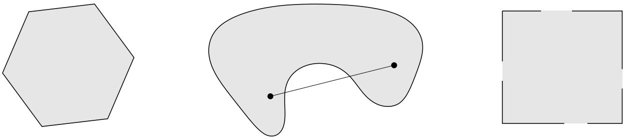
Convex sets

$\mathcal{X} \subseteq \mathbb{R}^n$ is **convex** if

$$\mathbf{x}, \mathbf{y} \in \mathcal{X} \quad \Longrightarrow \theta \mathbf{x} + (1-\theta)\mathbf{y} \in \mathcal{X} \quad \text{for all } \theta \in [0,1]$$

geometrically: $\mathbf{x}, \mathbf{y} \in \mathcal{X} \Rightarrow$ line segment from $\mathbf{x}$ to $\mathbf{y}$ belongs to $\mathcal{X}$

Examples: which are convex?



Q.2.1 Show that $\mathcal{X} = \{\mathbf{x} : \mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{b} \;\text{ for some }\; \mathbf{v} \in \mathbb{R}^m\}$ is convex.

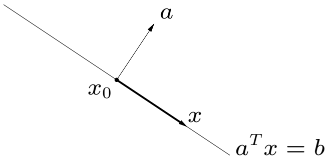
Q.2.2 Show that $\mathcal{X} = \{\mathbf{x} : \mathbf{B}\mathbf{x} = \mathbf{d}\}$ is convex.

Hyperplanes and halfspaces

hyperplane $\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} = b\}$

alternative representation $\{\mathbf{x} \mid \mathbf{a}^\top (\mathbf{x} - \mathbf{x}_0) = 0\}$

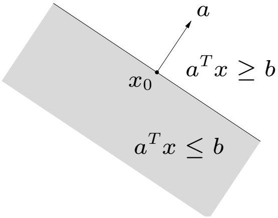
$\mathbf{a}$ is *normal* vector; $\mathbf{x}_0$ lies on hyperplane



halfspace $\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \leq b\}$

alternative representation $\{\mathbf{x} \mid \mathbf{a}^\top (\mathbf{x} - \mathbf{x}_0) \leq 0\}$

$\mathbf{a}$ is *outward* normal vector; $\mathbf{x}_0$ lies on boundary



Q.2.3 Show that both sets are convex.

Set operations that preserve convexity

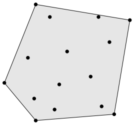
Intersection: the intersection of convex sets is also a convex set!

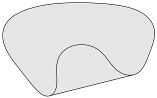
Q.2.4 How about unions or differences of convex sets?

Convex hull: $\mathrm{conv}(\mathcal{X})$ is the set of all convex combinations of the points in $\mathcal{X}$

- Convex combination of $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is
  $\mathbf{x}_\theta = \theta_1 \mathbf{x}_1 + \ldots + \theta_k \mathbf{x}_k$ with $\theta_i \geq 0$ and $\sum_{i=1}^k \theta_i = 1$

- Examples:





Q.2.5 If $\mathcal{X} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathbf{e}_i$ are the *canonical vectors* in $\mathbb{R}^3$, find $\mathrm{conv}(\mathcal{X})$?
Repeat for $\mathcal{X} = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$?

Polyhedra

Polyhedron is the solution set of finitely many linear inequalities and equalities

$$\mathcal{P} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \preceq \mathbf{b},\ \mathbf{C}\mathbf{x} = \mathbf{d}\}$$

Symbol $\preceq$ for component-wise inequality. Equalities as two inequalities.

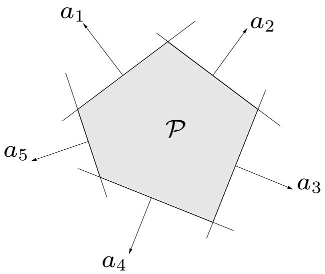


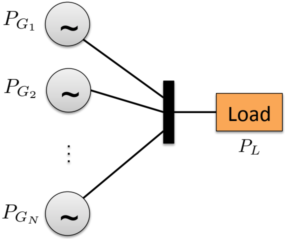
Figure: Source [R3]

A bounded polyhedron is called a **polytope**

Economic dispatch

- N generators serve load P_L
- generation costs $C_i(P_{G_i})$ \$/h

$$\min_{\{P_{G_i}\}_{i=1}^N} \quad \sum_{i=1}^N C_i(P_{G_i})$$
$$\text{s.to} \quad \sum_{i=1}^N P_{G_i} = P_L$$
$$0 \leq P_{G_i} \ \forall\ i$$



- Is the feasible set convex? Polyhedron?
- What if units have production limits, i.e., $P_{G_i} \leq P_{G_i}^{\max}$?

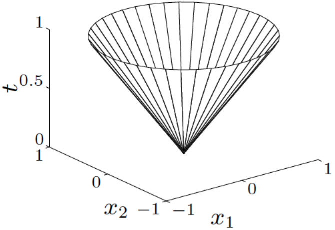
Norm balls and cones

Norm ball: $\mathcal{B} = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_c\| \leq 1\} \subset \mathbb{R}^N$ is convex

example: ℓ_p-norms in $\mathbb{R}^2$

Norm cone: $\mathcal{C} = \{(\mathbf{x}, t) : \|\mathbf{x}\| \leq t\} \subset \mathbb{R}^{N+1}$ is a convex cone

example: *second-order cone* or Lorentz cone $S = \{(\mathbf{x}, t) : \|\mathbf{x}\|_2 \leq t\}$



Q.2.6 The **second-order cone (SOC)** constr. $\|\mathbf{A}\mathbf{x} + \mathbf{b}\|_2 \leq \mathbf{c}^\top \mathbf{x} + d$ is convex

Ellipsoids

$$\mathcal{E}_1 = \left\{\mathbf{x} : (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{A}^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\right\} \text{ where } \mathbf{A} \in \mathbb{S}^n_{++} \text{ and } \mathbf{x}_c \in \mathbb{R}^n \text{ (center)}$$

- semiaxis length: $\sqrt{\lambda_i}$; λ_i eigenvalues of $\mathbf{A}$
- semiaxis directions: eigenvectors of $\mathbf{A}$

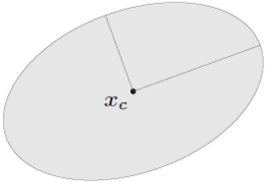


Figure: Source [R3]

Q.2.7 Show that an ellipsoid is a convex set.

Q.2.8 Find matrix $\mathbf{B}$ so that $\mathcal{E}_2 = \{\mathbf{B}\mathbf{u} + \mathbf{x}_c : \|\mathbf{u}\|_2 \leq 1\}$ is an alternative representation for ellipsoid $\mathcal{E}_1$.

Linear matrix inequalities

Symmetric matrices: $\mathbb{S}^n = \{\mathbf{X} \in \mathbb{R}^{n\times n} : \mathbf{X} = \mathbf{X}^\top\}$ (set of linear equalities)

Symmetric PSD cone: $\mathbb{S}^n_+ = \{\mathbf{X} \in \mathbb{S}^n : \mathbf{X} \succeq \mathbf{0}\}$ is a convex cone

$$\mathbf{X} \in \mathbb{S}^n_+ \quad \Longleftrightarrow \quad \mathbf{z}^\top \mathbf{X}\mathbf{z} \geq 0 \text{ for all } \mathbf{z} \in \mathbb{R}^n$$

(intersection of infinite number of halfspaces)

Example:

$$\mathbb{S}^2_+ := \left\{ (x,y,z) : \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0 \right\}$$

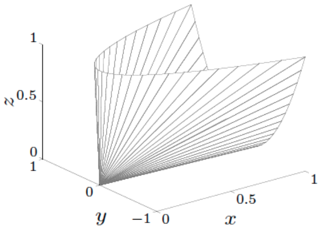


Figure: Source [R3]

Convex functions

- Function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if its domain is convex set and for all $\mathbf{x}, \mathbf{y}$:

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \text{ for all } \theta \in [0,1]$$

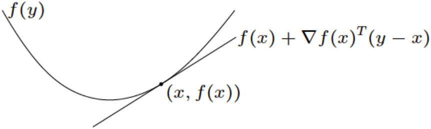
Figure: Source [R3]

- f is *concave* if $-f$ is convex
- f is *strictly convex* if strict inequality for $\theta \in (0,1)$

First- and second-order conditions for convexity

1st-order condition: differentiable f is convex iff

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \text{ for all } \mathbf{x}, \mathbf{y} \in \operatorname{dom} f$$



- first-order (Taylor's series) approximation of f is a global underestimator

2nd-order conditions: twice differentiable f with convex $\operatorname{dom} f$:

- f is convex iff $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for all $\mathbf{x} \in \operatorname{dom} f$
- if $\nabla^2 f(\mathbf{x}) \succ \mathbf{0}$ for all $\mathbf{x} \in \operatorname{dom} f$, then f is strictly convex

Operations that preserve convexity

nonnegative multiple: f convex, $\alpha \geq 0 \implies \alpha f$ convex

finite sum: f_1, f_2 convex $\implies f_1 + f_2$ convex

pointwise maximum: f_1, f_2 convex $\implies \max\{f_1(\mathbf{x}), f_2(\mathbf{x})\}$ convex

partial minimization if $f(\mathbf{x}, \mathbf{y})$ is convex in $(\mathbf{x}, \mathbf{y})$ and C is a convex set, then

$$g(\mathbf{x}) = \min_{\mathbf{y} \in C} f(\mathbf{x}, \mathbf{y}) \text{ is convex}$$

affine transformation of domain: f is convex $\implies f(\mathbf{A}\mathbf{x} + \mathbf{b})$ convex

Function examples

Examples in $\mathbb{R}$:

- x^α is convex on $\mathbb{R}_{++}$ for $\alpha \geq 1$, $\alpha \leq 0$; concave for $\alpha \in [0, 1]$
- $e^{\alpha x}$ is convex; $\log x$ is concave
- $|x|$, $\max\{0, x\}$, $\max\{0, -x\}$ are convex

Examples in $\mathbb{R}^n$:

- linear and affine functions are both convex and concave!
- vector norms are convex
- piecewise linear functions $f(\mathbf{x}) = \max_i\{\mathbf{a}_i^\top \mathbf{x} + b_i\}$ are convex

Q.2.9 Show three of the above claims.