ECE 5314: Power System Operation & Control

Lecture 0: Mathematical Background

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R3 Boyd and Vandenberghe, *Convex Optimization*, Appendix A.

Vectors

- Notation for vectors: $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \in \mathbb{R}^N$
- A *linear function* of $\mathbf{x}$ can be expressed as the *inner product*:

$$f_1(\mathbf{x}) = \sum_{i=1}^{N} b_i x_i = \mathbf{b}^\top \mathbf{x}$$

  where $^\top$ denotes transposition, i.e., $\mathbf{b}^\top = [b_1 \ \cdots \ b_N]$.
- The *gradient* of a multivariate function is the vector of partial derivatives:

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1} \ \cdots \ \frac{\partial f}{\partial x_N}\right]^\top$$

Q.0.1 Show that $\nabla f_1(\mathbf{x}) = \mathbf{b}$.

Q.0.2 Write $f(\mathbf{x}) = 2x_1 + 3x_2 - x_4$ as an inner product and find its gradient

Q.0.3 Express $\sum_{i=1}^N x_i$ and x_2 as inner products of $\mathbf{x}$

Vector norms

A vector norm is a function $\|.\| : \mathbb{R}^N \to \mathbb{R}$ satisfying the following three properties for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, and $a \in \mathbb{R}$:

1. *Positive definiteness:* $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
2. *Scaling:* $\|a\mathbf{x}\| = |a| \cdot \|\mathbf{x}\|$
3. *Triangle inequality:* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

ℓ_p norms

For $\mathbf{x} \in \mathbb{R}^N$ and $p \geq 1$,

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

1. $p = 2$: Euclidean norm
2. $p = 1$: sum-abs-values $\|\mathbf{x}\|_1 = \sum_i |x_i|$
3. $p = \infty$: max-abs-value $\|\mathbf{x}\|_\infty = \lim_{p\to\infty} \|\mathbf{x}\|_p = \max_i |x_i|$

Q.0.4 Find $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ for $\mathbf{x} = [3\ 0\ \ -4]^\top$.

Q.0.5 Show that $\mathbf{x}^\top \mathbf{x} = \|\mathbf{x}\|_2^2$.

Norm inequalities

Cauchy-Schwartz inequality

$$\mathbf{x}^\top \mathbf{y} \leq \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2$$

hold with equality iff $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Hölder's inequality

$$\mathbf{x}^\top \mathbf{y} \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad p \geq 1.$$

Comparing norms $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$

Q.0.6 How does Hölder's inequality apply for $p = 1$?

Q.0.7 Show all three norms are equal for $\mathbf{x} = [c\ 0\ \cdots\ 0]^\top$ for any $c \in \mathbb{R}$.

Matrices

Notation for matrices: $$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{21} & A_{22} & \cdots & A_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NM} \end{bmatrix} \in \mathbb{R}^{N \times M}.$$

Q.0.8 What is the matrix transpose $\mathbf{A}^\top$?

Matrix-vector product: If $\mathbf{b} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} \in \mathbb{R}^{N \times M}$, then

$$b_i = \sum_{j=1}^{M} A_{ij} x_j$$

If $\mathbf{A}_{:,j}$ denotes the j-th column of $\mathbf{A}$, verify that

$$\begin{aligned} \mathbf{b} = \mathbf{A}\mathbf{x} &= \mathbf{A}_{:,1} x_1 + \ldots + \mathbf{A}_{:,M} x_M \\ &= \sum_{j=1}^{M} \mathbf{A}_{:,j} x_j. \end{aligned}$$

Quadratic functions

Every homogeneous *quadratic function* of $\mathbf{x}$ can be expressed as follows

$$\begin{aligned} f_2(\mathbf{x}) &= \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}x_ix_j \\ &= \sum_{i=1}^{N} x_i \left(\sum_{j=1}^{N} A_{ij}x_j\right) \\ &= \sum_{i=1}^{N} x_i[\mathbf{A}\mathbf{x}]_i = \mathbf{x}^\top\mathbf{A}\mathbf{x} \quad \text{for some} \quad \mathbf{A}\in\mathbb{R}^{N\times N} \end{aligned}$$

Q.0.9 Express $x_1^2 - 2x_2^2$ as $\mathbf{x}^\top\mathbf{A}\mathbf{x}$.

Q.0.10 Express $x_1^2 - 2x_1x_2$ as $\mathbf{x}^\top\mathbf{A}\mathbf{x}$.

Q.0.11 Express $x_1^2 - 2x_1x_2 + 2x_2$ as $\mathbf{x}^\top\mathbf{A}\mathbf{x} + \mathbf{b}^\top\mathbf{x}$.

Q.0.12 If $\mathbf{A}$ is symmetric, show that $\nabla f_2(\mathbf{x}) = 2\mathbf{A}\mathbf{x}$.

Square matrices

- *Symmetric matrix*: $\mathbf{A} = \mathbf{A}^\top$ ($A_{ij} = A_{ji}$ for all (i, j))

  Q13: Show that $\mathbf{x}^\top \mathbf{A}\mathbf{x} = \mathbf{x}^\top \mathbf{A}_s \mathbf{x}$ where $\mathbf{A}_s = \frac{\mathbf{A}+\mathbf{A}^\top}{2}$

  ($\mathbf{A}_s$ is symmetric even if $\mathbf{A}$ is not)

- *Trace*: $\mathrm{Tr}(\mathbf{A}) = \sum_{i=1}^N A_{ii}$ (sum of diagonal elements)

- *Inner product*: $\mathrm{Tr}(\mathbf{A}\mathbf{B}^\top) = \sum_{i,j} A_{ij} B_{ij}$

- *Orthonormal matrices*: $\mathbf{A}\mathbf{A}^\top = \mathbf{A}^\top \mathbf{A} = \mathbf{I}$

Hessian and Jacobian matrices

Hessian matrix: the matrix of second-order partial derivatives of $f : \mathbb{R}^N \rightarrow \mathbb{R}$

$$\nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_N^2} \end{bmatrix}$$

Q.0.14 For symmetric $\mathbf{A}$, show that $\nabla^2 f_2(\mathbf{x}) = 2\mathbf{A}$.

Jacobian matrix: its rows are the gradients of $\mathbf{f} : \mathbb{R}^N \rightarrow \mathbb{R}^M$

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \nabla f_1(\mathbf{x})^\top \\ \vdots \\ \nabla f_M(\mathbf{x})^\top \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \cdots & \frac{\partial f_M}{\partial x_N} \end{bmatrix}$$

Q.0.15 What is the Jacobian matrix of $\mathbf{f}(\mathbf{x}) = 2\mathbf{A}\mathbf{x}$? Note $\mathbf{f} : \mathbb{R}^N \rightarrow \mathbb{R}^N$

Eigenvalue decomposition

- *Eigenvalue/eigenvector pair* $(\lambda, \mathbf{v})$ of $\mathbf{A}$:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad \text{for} \quad \mathbf{v} \neq \mathbf{0}$$

- For every diagonalizable $\mathbf{A}$ (linearly independent eigenvectors)

$$\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$$

  eigenvectors as columns of $\mathbf{V}$; eigenvalues as entries of diagonal $\boldsymbol{\Lambda}$

- For a symmetric matrix:

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$$

  where $\mathbf{U}$ is orthonormal and $\boldsymbol{\Lambda}$ is diagonal and real

Q.0.15 Use MATLAB's `eig` to compute the eigenvalue decomposition for the symmetric matrices $\mathbf{A}_s$ obtained from Q.0.9-Q.0.11.

Singular value decomposition

- Defined even for non-square matrices:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

  where $\mathbf{U}$ and $\mathbf{V}$ are orthonormal and $\mathbf{\Sigma}$ diagonal matrix

- *singular values*: $\sigma_i = \sqrt{\lambda_i(\mathbf{A}\mathbf{A}^\top)}$
- *left (right) singular vectors* are the eigenvectors of $\mathbf{A}\mathbf{A}^\top$ $(\mathbf{A}^\top\mathbf{A})$
- *Rank of a matrix*: number of non-zero singular values

Q.0.16 Show that $\sigma_i(\mathbf{A}) = |\lambda_i(\mathbf{A})|$ for symmetric $\mathbf{A}$.

Positive definite matrices

- If all eigenvalues are positive, symmetric $\mathbf{A}$ is *positive definite* $\mathbf{A} \succ \mathbf{0}$
- If all eigenvalues are non-negative, symmetric $\mathbf{A}$ is *positive semi-definite* $\mathbf{A} \succeq \mathbf{0}$

- *Square root*: $\mathbf{A}^{1/2} = \mathbf{U}\sqrt{\mathbf{\Lambda}}\mathbf{U}^\top$
- Matrix $\mathbf{A}$ is positive (semi-)definite iff $\mathbf{x}^\top\mathbf{A}\mathbf{x} > 0$ (≥ 0) for all $\mathbf{x}$.

Q.0.17 Are matrices $\mathbf{A}_s$ in Q.0.9-Q.0.11 positive definite or positive semi-definite?

Schur complement

For invertible $\mathbf{A}$ and *symmetric* matrix $\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix}$

Define **Schur complement** as $\mathbf{S} = \mathbf{C} - \mathbf{B}^\top \mathbf{A}^{-1} \mathbf{B}$ (Appendix A.5.5 of R3)

Q.0.18 Show that $\mathbf{X} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{0} \end{bmatrix} \iff \mathbf{S}\mathbf{y}_1 = \mathbf{b}_1$

Properties

1. $\det(\mathbf{X}) = \det(\mathbf{A}) \det(\mathbf{S})$
2. $\mathbf{X} \succ \mathbf{0}$ iff $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{S} \succ \mathbf{0}$
3. Assume $\mathbf{A} \succ \mathbf{0}$. Then $\mathbf{X} \succeq \mathbf{0}$ iff $\mathbf{S} \succeq \mathbf{0}$

Vector spaces of a matrix

Range space: $\text{range}(\mathbf{A}) = \{\mathbf{x} : \ \mathbf{x} = \mathbf{A}\mathbf{v} \text{ for } \mathbf{v} \in \mathbb{R}^N\} \subseteq \mathbb{R}^M$

- vectors that are linear combinations of the columns of $\mathbf{A}$
- first $\text{rank}(\mathbf{A})$ columns of $\mathbf{U}$ form a basis for $\text{range}(\mathbf{A})$

Null space: $\text{null}(\mathbf{A}) = \{\mathbf{x} : \ \mathbf{A}\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^N$

- vectors perpendicular to all rows of $\mathbf{A}$
- a basis for $\text{null}(\mathbf{A})$ are the last $N - \text{rank}(\mathbf{A})$ columns of $\mathbf{V}$

Fundamental theorem of linear algebra:

$$\text{range}(\mathbf{A}) = (\text{null}(\mathbf{A}^\top))^\perp$$

i.e., the vectors in $\text{range}(\mathbf{A})$ are orthogonal to the vectors in $\text{null}(\mathbf{A}^\top)$

Taylor's Series Expansion and Mean Value Theorem

- Univariate function (yields linear and quadratic approximations)

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\ldots = \sum_{i=1}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

- Multivariate function:

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0))^\top(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^\top\nabla^2 f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)$$

- *Mean value theorem*: There exist y and z between x and x_0 such that

$$f(x) = f(x_0) + f'(y)(x-x_0)$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(z)}{2}(x-x_0)^2$$

MVT generalizes to multivariate functions.

Open and closed sets

- *Ball* of radius $\epsilon > 0$ around $\mathbf{x}$ is $\{\mathbf{y} : \|\mathbf{x} - \mathbf{y}\| \leq \epsilon\}$.

- A point $\mathbf{x}$ is an *interior point* of a set $\mathcal{S}$ if $\mathbf{x} \in \mathcal{S}$ and there exists a ball around $\mathbf{x}$ that is contained entirely in $\mathcal{S}$

- *Open set*: if every point in $\mathcal{S}$ is an interior point

  Examples: $(0, 1)$, interior of a circle

- *Closed set*: if its compliment set is open

  Examples: $[0, 1]$, circle

- $\mathbb{R}^N$ and $\emptyset$ are both closed and open sets!