ECE 5984: Power Distribution System Analysis

Lecture 9: Three-Phase Transformer Models

Reference: Textbook, Chapter 8

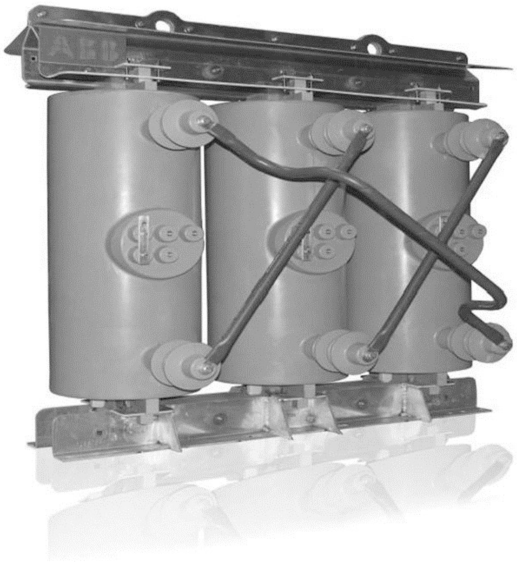
Instructor: V. Kekatos

VirginiaTech
Invent the Future

Distribution system transformers

- Found at the substation and in-line

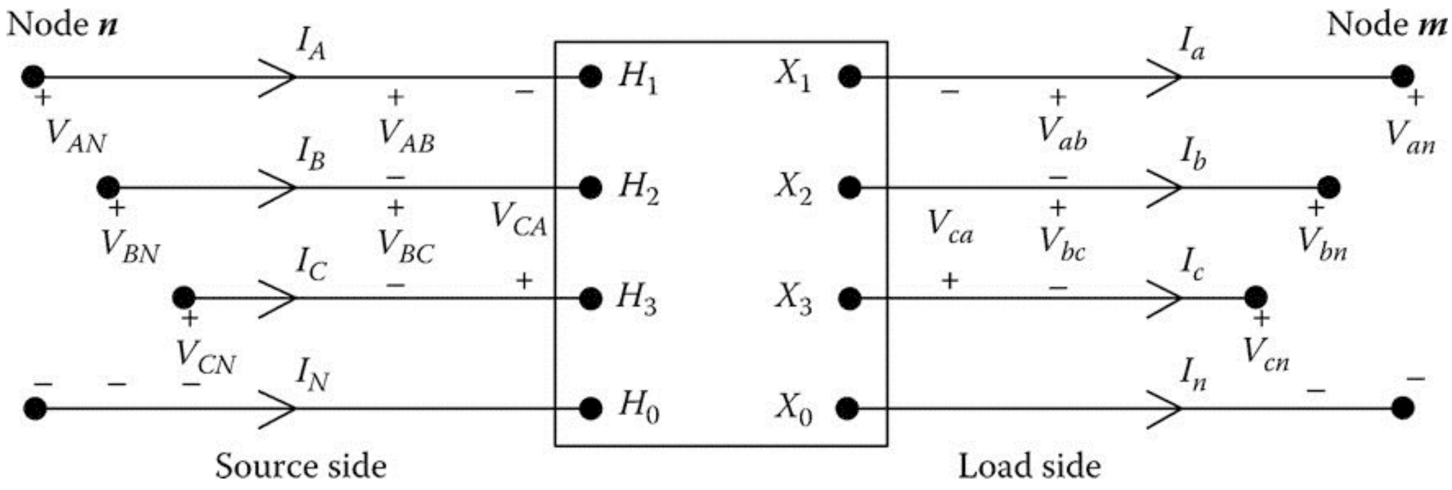
substation transformer



pole-mounted transformer

pad-mounted single-phase transformer

Generalized matrices



Book's notation

- ABCD model

$$\mathbf{v}_n = \mathbf{A}\mathbf{v}_m + \mathbf{B}\mathbf{i}_m$$
$$\mathbf{i}_n = \mathbf{C}\mathbf{v}_m + \mathbf{D}\mathbf{i}_m$$

(backward update)

$$\mathbf{v}_n = [a_t]\mathbf{v}_m + [b_t]\mathbf{i}_m$$
$$\mathbf{i}_n = [c_t]\mathbf{v}_m + [d_t]\mathbf{i}_m$$

- Forward (EF) model

$$\mathbf{v}_m = \mathbf{E}\mathbf{v}_n - \mathbf{F}\mathbf{i}_m$$

(forward update)

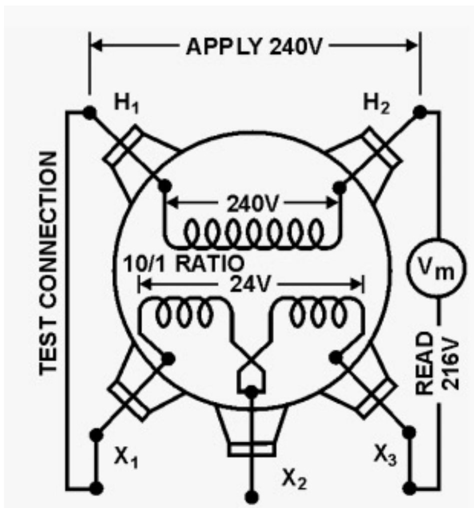
$$\mathbf{v}_m = [A_t]\mathbf{v}_n - [B_t]\mathbf{i}_m$$

Conventions

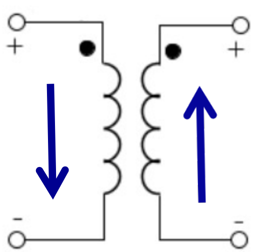
- ANSI/IEEE Std. C57.12.00 for Delta-Wye transformer connections

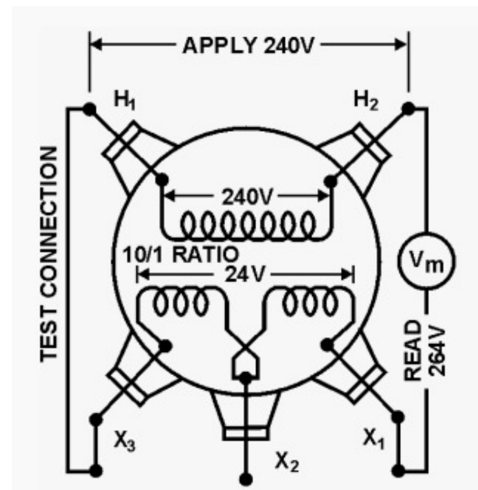
Voltages and currents on the high-voltage side lead by 30 degrees

- *Distribution transformers of <200 kVA and HV<8.66 kV have additive polarity*

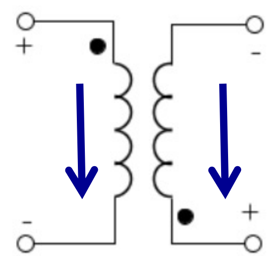


subtractive





additive



[source: electrical-engineering-portal.com]

Three-phase connections

- Multi-phase transformers usually implemented by connecting 1ϕ transformers

- Transformer connections

1) Grounded Wye – grounded Wye
2) Delta – grounded Wye (step-up)
3) Delta – grounded Wye (step-down)
4) Ungrounded Wye – Delta (step-down)
5) Ungrounded Wye – Delta (step-up)
6) Grounded Wye – Delta (step-down)
7) Delta-Delta
8) Open Wye – open Delta
9) Open Delta – open Delta

Voltage conversions (review)

- *LN voltages* $\mathbf{v} := \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix}$ *LL voltages* $\tilde{\mathbf{v}} := \begin{bmatrix} V_{ab} \\ V_{bc} \\ V_{ca} \end{bmatrix}$

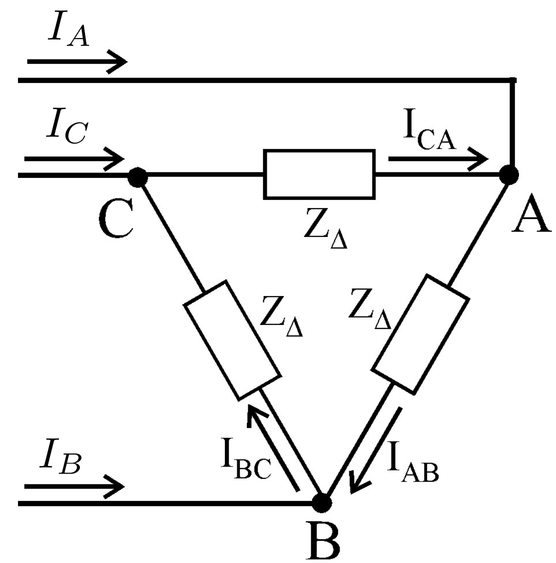
- LN-to-LL conversion $\tilde{\mathbf{v}} = \mathbf{D}_f \mathbf{v}, \quad \mathbf{D}_f := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ *singular matrix!*

- LL voltages are zero-sum: $\mathbf{1}^\top \tilde{\mathbf{v}} = V_{ab} + V_{bc} + V_{ca} = 0$ *even for unbalanced conditions*

- Given LL voltages, recover *equivalent* LN voltages $\mathbf{v} = \mathbf{W}\tilde{\mathbf{v}}, \quad \mathbf{W} := \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

- Vector of LN voltages represents
  - line-to-ground for grounded Wye
  - line-to-neutral for ungrounded Wye
  - 'equivalent' line-to-neutral for Delta connections

Current conversions (review)



- *Line currents* $\mathbf{i} := \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}$ *phase currents* $\tilde{\mathbf{i}} := \begin{bmatrix} I_{ab} \\ I_{bc} \\ I_{ca} \end{bmatrix}$

- Phase to line conversion

$$\mathbf{i} = \begin{bmatrix} I_{ab} - I_{ca} \\ I_{bc} - I_{ab} \\ I_{ca} - I_{bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tilde{\mathbf{i}} = \mathbf{D}_f^\top \tilde{\mathbf{i}}$$

- If line currents exit triangle (delta source), vector $\mathbf{i}$ gets negative sign or $\tilde{\mathbf{i}} := \begin{bmatrix} I_{ba} \\ I_{cb} \\ I_{ac} \end{bmatrix}$

- Singularity (shift-invariance) can be waived by fixing the sum of delta currents

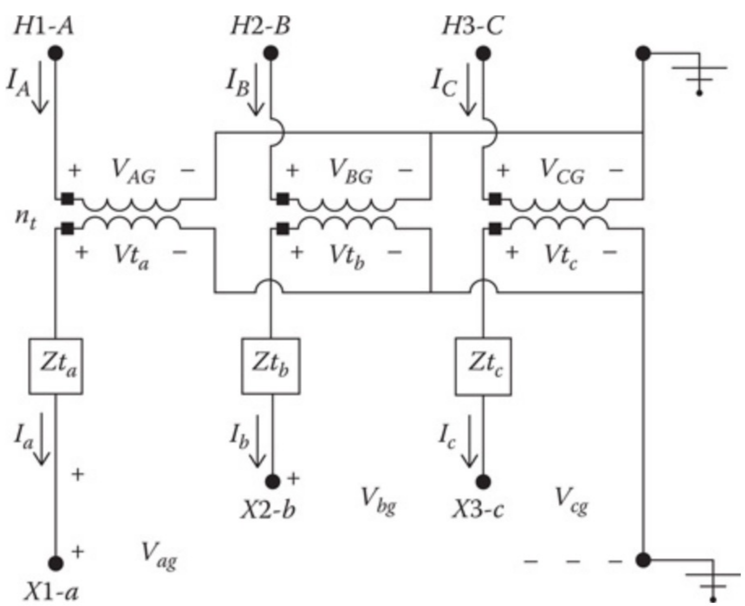
- Given line currents, recover *equivalent* delta currents

$$\tilde{\mathbf{i}} = \mathbf{W}^\top \mathbf{i}$$

- Textbook follows a different derivation and finds matrix $\boldsymbol{L}$ with a zero column

1) $Y_G - Y_G$ connection

- Turns ratios are typically identical across all phases $n_t = \frac{N_{\text{primary}}}{N_{\text{secondary}}}$



- Transformations $\frac{V_{AG}}{V_{ta}} = n_t$ and $\frac{I_A}{I_a} = \frac{1}{n_t}$

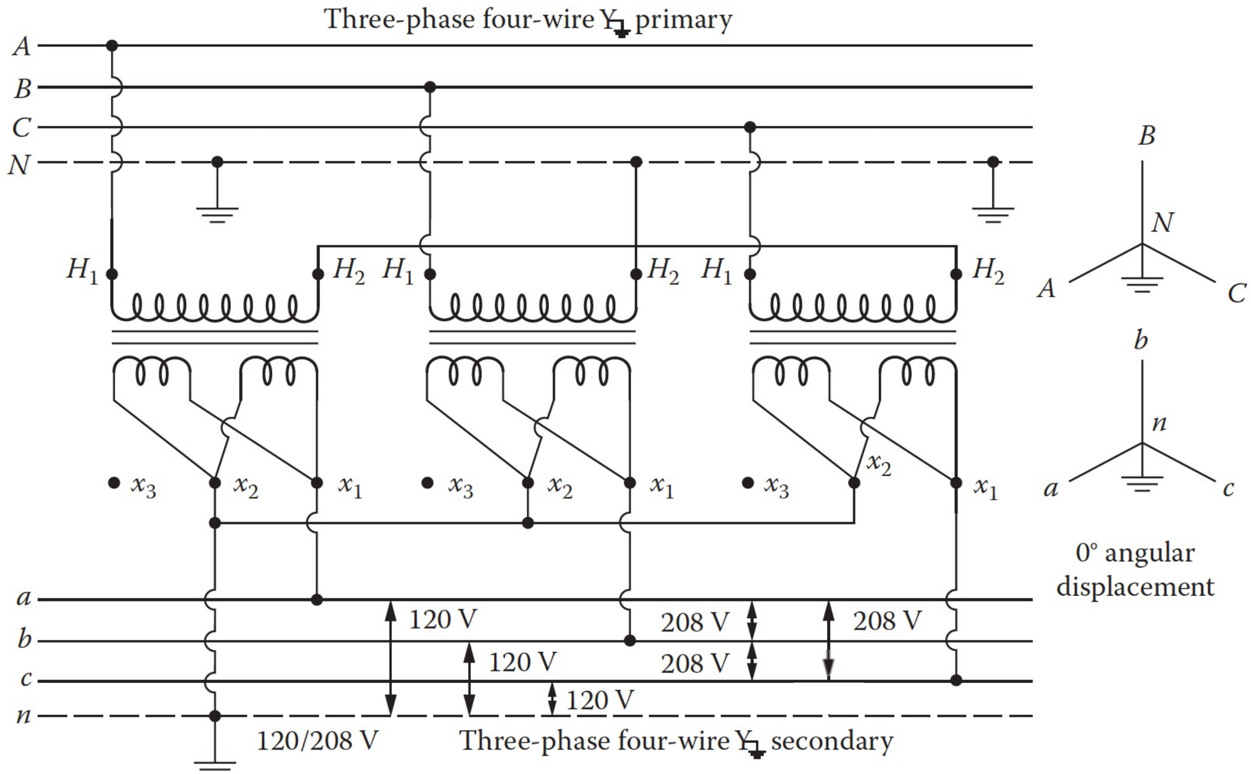
- Forward model
$$\mathbf{v}_m = \mathbf{v}_t - \mathrm{dg}(\mathbf{z})\mathbf{i}_m = \frac{1}{n_t}\mathbf{v}_n - \mathrm{dg}(\mathbf{z})\mathbf{i}_m$$

diagonal matrix with vector **z** *on its main diagonal*

- Impedances may not be equal; transformers of different power ratings

- Backward model $\mathbf{i}_n = \frac{1}{n_t}\mathbf{i}_m$

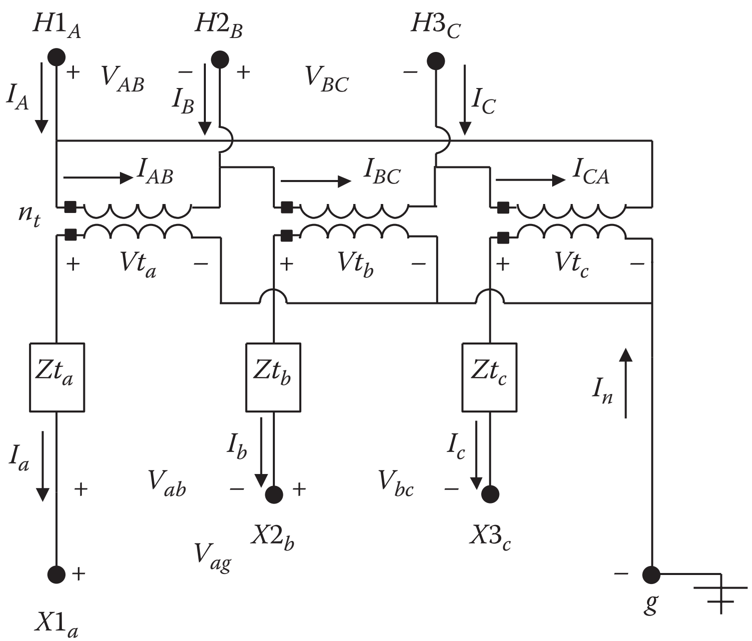
Discussion on $Y_G - Y_G$ connection

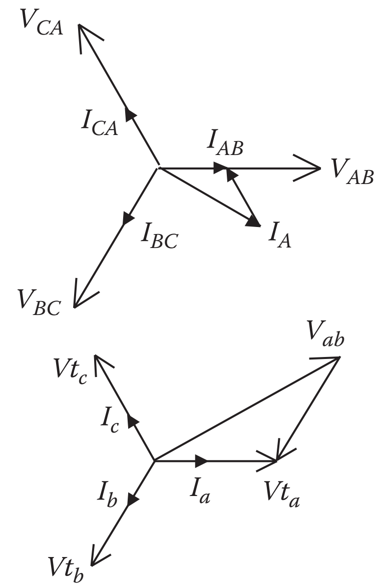


- Very common in three-phase four-wire systems
- Useful in voltage upgrades: a 2.4kV delta feeder can be converted to 4W-Y (4.16kV) by reconnecting the same transformers from Delta-Delta to Wye-Wye!
- Issues with third harmonics unless solidly grounded (no grounding impedance)

2) Step-up $\Delta - Y_{\mathrm{G}}$ connection

$$n_t = \frac{V_{\mathrm{LL,primary}}}{V_{\mathrm{LN,secondary}}}$$

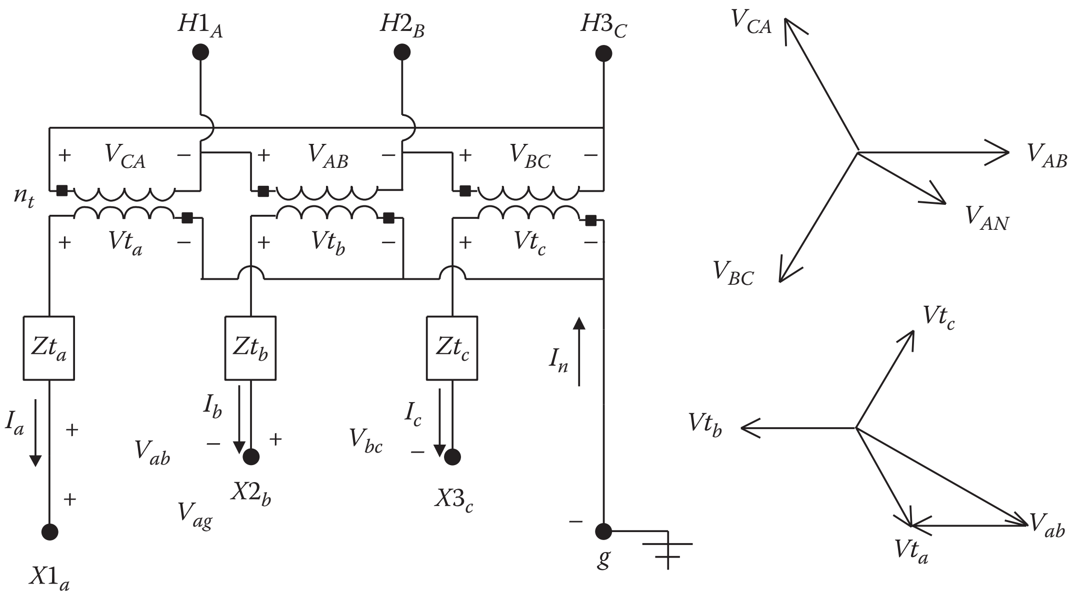




phasor diagrams assume balanced conditions

- How can we determine it is a step-up without knowing the turns ratio?
- Voltage and current transformations $\tilde{\mathbf{v}}_n = n_t \mathbf{v}_t$ and $\tilde{\mathbf{i}}_n = \begin{bmatrix} I_{AB} \\ I_{BC} \\ I_{CA} \end{bmatrix} = \frac{1}{n_t}\mathbf{i}_m$
- Sanity check (for *ideal* transformer) $\tilde{\mathbf{i}}_n^H \tilde{\mathbf{v}}_n = \left(\frac{1}{n_t}\mathbf{i}_m^H\right)(n_t \mathbf{v}_t) = \mathbf{i}_m^H \mathbf{v}_m$
- Backward model $\mathbf{i}_n = \mathbf{D}_f^\top \tilde{\mathbf{i}}_n = \frac{1}{n_t}\mathbf{D}_f^\top \mathbf{i}_m$
- Forward model $\mathbf{v}_m = \mathbf{v}_t - \mathrm{dg}(\mathbf{z})\mathbf{i}_m = \frac{1}{n_t}\mathbf{D}_f \mathbf{v}_n - \mathrm{dg}(\mathbf{z})\mathbf{i}_m$

3) Step-down $\Delta - Y_{\mathrm{G}}$ connection



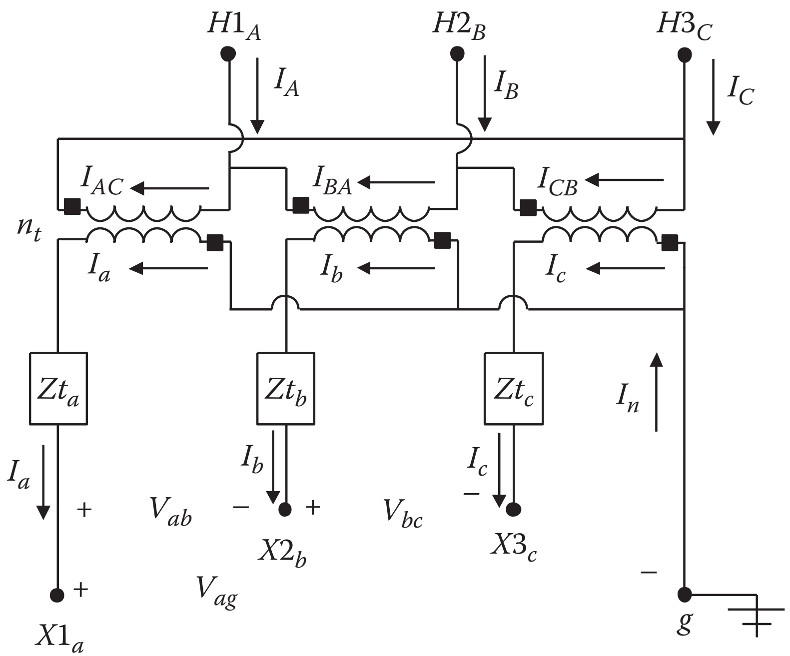
why step-down?

- Voltage transform $\tilde{\mathbf{v}}_n = \begin{bmatrix} V_{AB} \\ V_{BC} \\ V_{CA} \end{bmatrix} = -n_t \mathbf{A}_v \mathbf{v}_t$ where $\mathbf{A}_v := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ *permutation matrix*
- *Key properties* p1) $\mathbf{A}_v^{-1} = \mathbf{A}_v^\top$

  p2) $-\mathbf{A}_v^\top \mathbf{D}_f = \mathbf{D}_f^\top$
- Forward model

$$\begin{aligned} \mathbf{v}_m &= \mathbf{v}_t - \mathrm{dg}(\mathbf{z})\mathbf{i}_m \\ &= -\frac{1}{n_t}\mathbf{A}_v^\top \tilde{\mathbf{v}}_n - \mathrm{dg}(\mathbf{z})\mathbf{i}_m \\ &= -\frac{1}{n_t}\mathbf{A}_v^\top \mathbf{D}_f \mathbf{v}_n - \mathrm{dg}(\mathbf{z})\mathbf{i}_m = \frac{1}{n_t}\mathbf{D}_f^\top \mathbf{v}_n - \mathrm{dg}(\mathbf{z})\mathbf{i}_m \end{aligned}$$

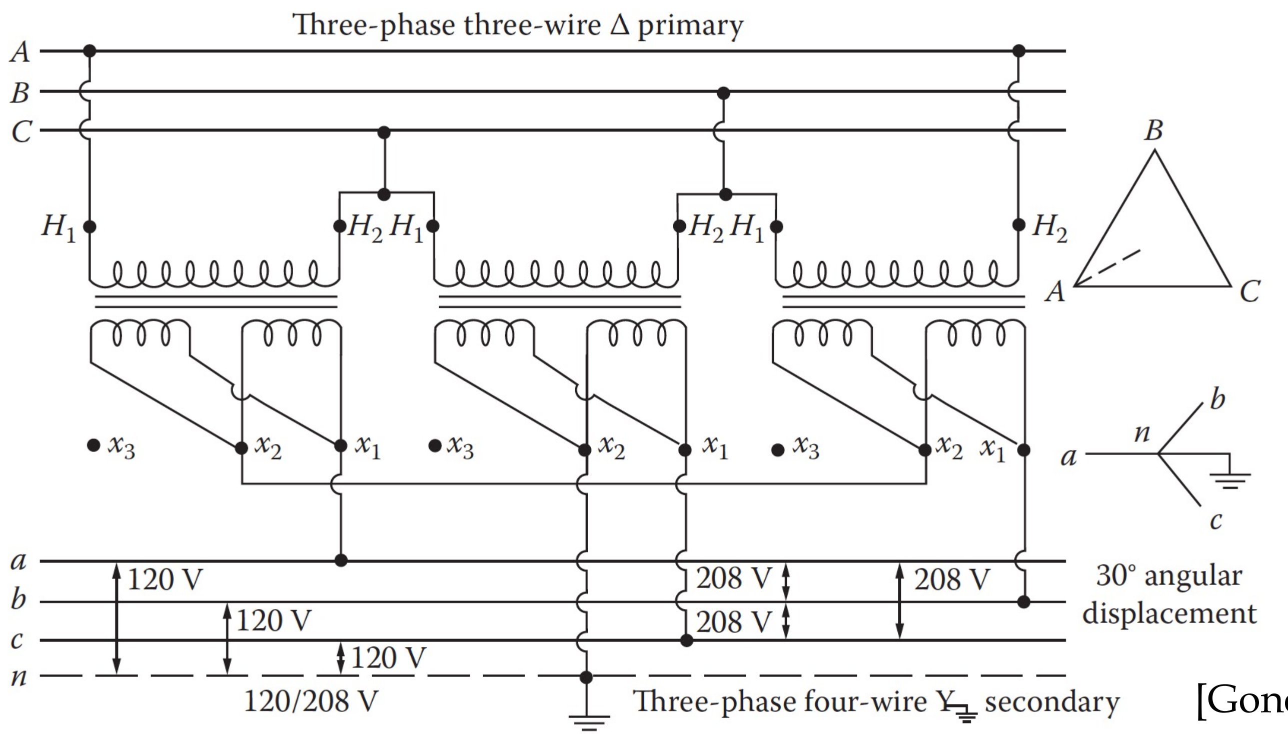
Step-down $\Delta - Y_{\mathrm{G}}$ connection



- Current transformation $$\tilde{\mathbf{i}}_n = \begin{bmatrix} I_{BA} \\ I_{CB} \\ I_{AC} \end{bmatrix} = \frac{1}{n_t}\mathbf{A}_v \mathbf{i}_m$$

- Backward model $$\begin{aligned} \mathbf{i}_n &= -\mathbf{D}_f^\top \tilde{\mathbf{i}}_n \\ &= -\frac{1}{n_t}\mathbf{D}_f^\top \mathbf{A}_v \mathbf{i}_m \\ &= \left(-\mathbf{A}_v^\top \mathbf{D}_f\right)^\top \mathbf{i}_m = \frac{1}{n_t}\mathbf{D}_f \mathbf{i}_m \end{aligned}$$

Discussion on $\Delta - Y$ connection



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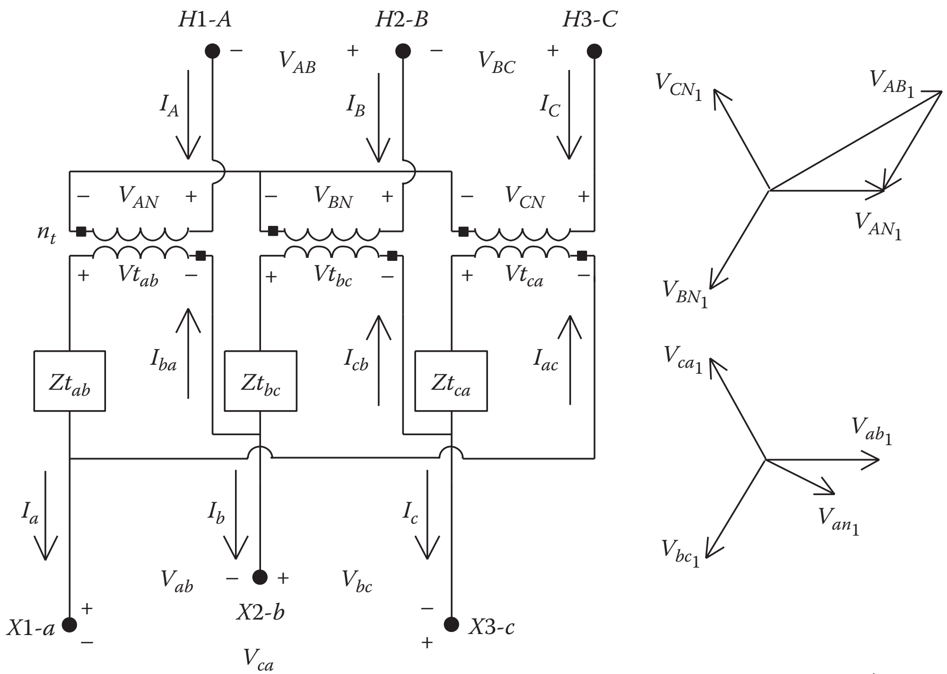
- Typical distribution substation connection
- Easier balancing of (large) 1ϕ loads across all three transformers
- Cannot operate with two transformers (no open delta – open Y)

4) Step-down $Y - \Delta$ connection

ungrounded Y



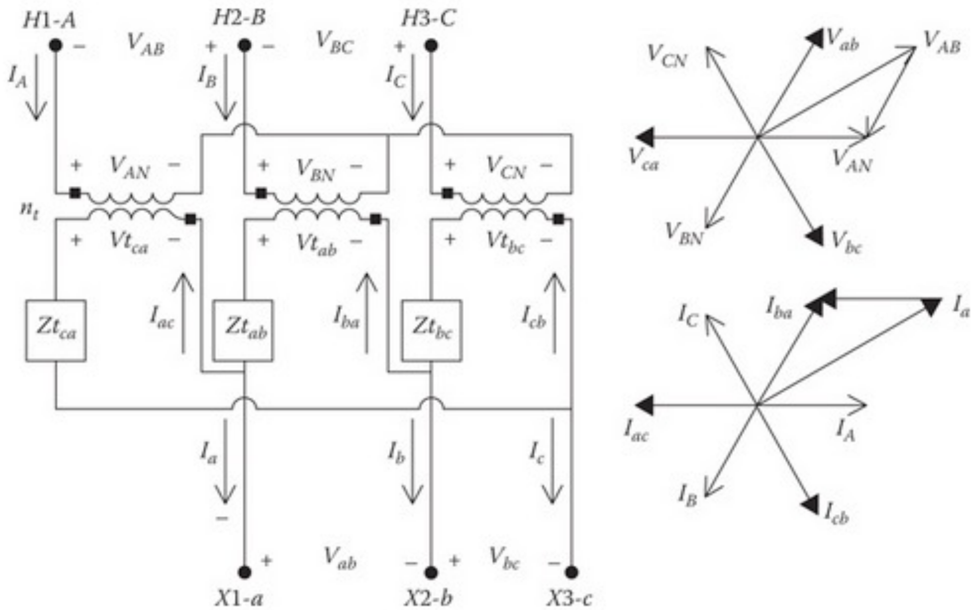
- Voltage and current transformations $\mathbf{v}_n = n_t \tilde{\mathbf{v}}_t$ and $\mathbf{i}_n = \frac{1}{n_t}\tilde{\mathbf{i}}_m = \frac{1}{n_t}\begin{bmatrix} I_{ba} \\ I_{cb} \\ I_{ac} \end{bmatrix}$
- *Key point: phase currents in delta are zero-sum* $\mathbf{i}_m = \mathbf{D}_f^\top \tilde{\mathbf{i}}_m \quad \Rightarrow \quad \tilde{\mathbf{i}}_m = \mathbf{W}^\top \mathbf{i}_m$
- Backward model $\mathbf{i}_n = \frac{1}{n_t}\mathbf{L}\mathbf{i}_m$

  matrix $\mathbf{L}$ can be either $\mathbf{W}^\top$ or the matrix in (8.63) of book

- Forward model $$\mathbf{v}_m = \mathbf{W}\left(\tilde{\mathbf{v}}_t - \mathrm{dg}(\mathbf{z})\tilde{\mathbf{i}}_m\right)$$
$$= \frac{1}{n_t}\mathbf{W}\mathbf{v}_n - \mathbf{W}\mathrm{dg}(\mathbf{z})\mathbf{L}\mathbf{i}_m$$

5) Step-up $Y - \Delta$ connection

ungrounded Y

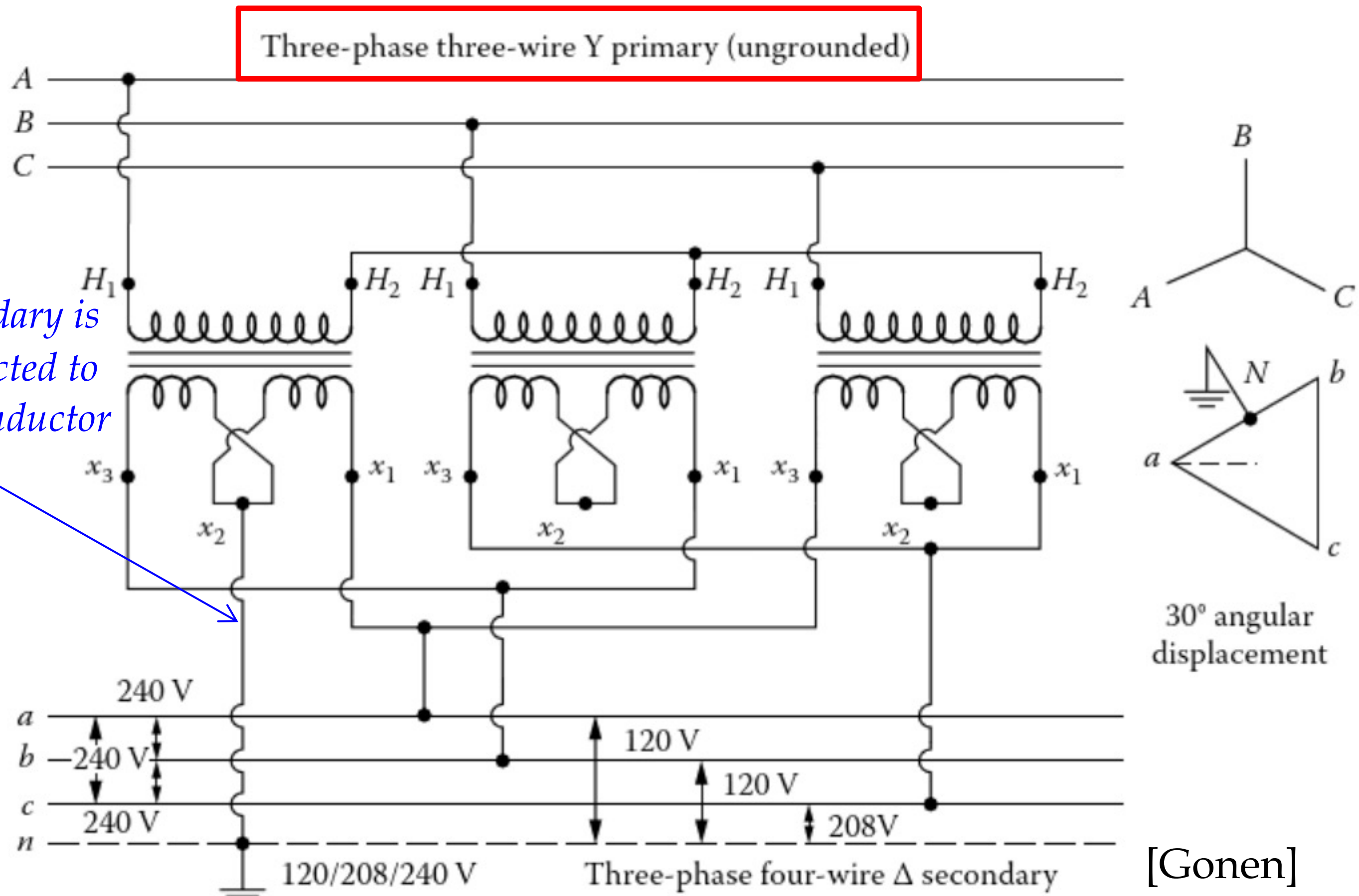


key connection difference with step-down?

- *Similar* analysis observing $\mathbf{v}_n = -n_t \mathbf{A}_v^\top \tilde{\mathbf{v}}_t$ and $\mathbf{i}_n = -\frac{1}{n_t}\mathbf{A}_v^\top \tilde{\mathbf{i}}_m$ where $\tilde{\mathbf{i}}_m = \begin{bmatrix} I_{ba} \\ I_{cb} \\ I_{ac} \end{bmatrix}$
- Backward model $\mathbf{i}_n = -\frac{1}{n_t}\mathbf{A}_v^\top \mathbf{L}\mathbf{i}_m$
- Forward model $\mathbf{v}_m = -\frac{1}{n_t}\mathbf{W}\mathbf{A}_v\mathbf{v}_n - \mathbf{W}\mathrm{dg}(\mathbf{z})\mathbf{L}\mathbf{i}_m$

Discussion on $Y - \Delta$ connection

mid-tap of one secondary is grounded and connected to secondary neutral conductor



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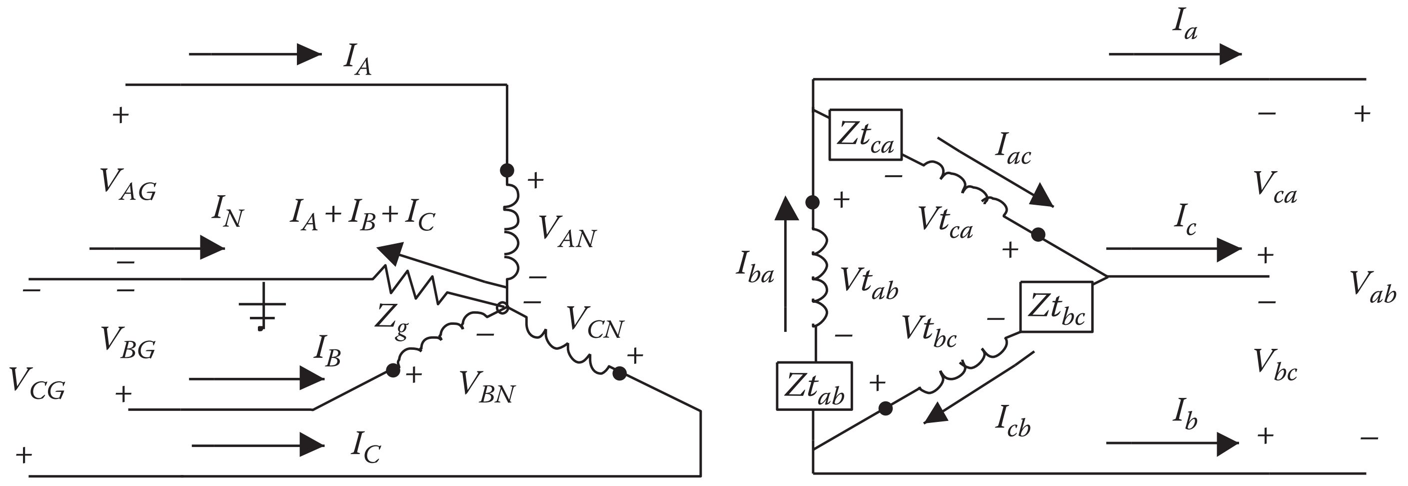
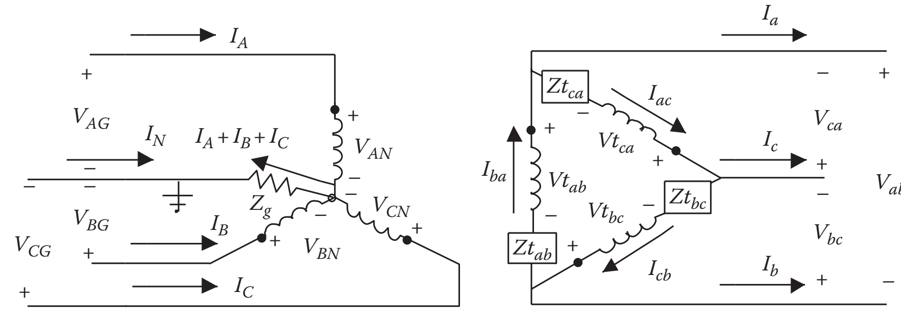
6) Step-down $Y_G - \Delta$ connection



- Challenging part is to relate delta currents to secondary line currents
- Exploit the fact that LL voltages across the delta loop are zero-sum

Step-down $Y_G - \Delta$ connection



- Current transformation $\mathbf{i}_n = \frac{1}{n_t}\tilde{\mathbf{i}}_m \qquad (1)$

- Voltage transformation $\mathbf{v}_n = n_t \tilde{\mathbf{v}}_t \qquad (2)$

- Non-ideal secondary $\tilde{\mathbf{v}}_m = \tilde{\mathbf{v}}_t - \mathrm{dg}(\mathbf{z})\tilde{\mathbf{i}}_m \qquad (3)$

- LG voltages

$$\begin{aligned}\bar{\mathbf{v}}_n &= \mathbf{v}_n + z_g \mathbf{1}\mathbf{1}^\top \mathbf{i}_n \overset{(1),(2)}{\Longrightarrow} \\ &= n_t \tilde{\mathbf{v}}_t + \frac{z_g}{n_t}\mathbf{1}\mathbf{1}^\top \tilde{\mathbf{i}}_m \overset{(3)}{\Longrightarrow} \\ &= n_t \tilde{\mathbf{v}}_m + \left(n_t \mathrm{dg}(\mathbf{z}) + \frac{z_g}{n_t}\mathbf{1}\mathbf{1}^\top\right)\tilde{\mathbf{i}}_m \qquad (4)\end{aligned}$$

- Solve for secondary LL voltages $\tilde{\mathbf{v}}_m = \frac{1}{n_t}\bar{\mathbf{v}}_n - \left(\mathrm{dg}(\mathbf{z}) + \frac{z_g}{n_t^2}\mathbf{1}\mathbf{1}^\top\right)\tilde{\mathbf{i}}_m \qquad (5)$

- Solve for secondary equivalent LN voltages

$$\mathbf{v}_m = \mathbf{W}\tilde{\mathbf{v}}_m = \frac{1}{n_t}\mathbf{W}\bar{\mathbf{v}}_n - \mathbf{W}\left(\mathrm{dg}(\mathbf{z}) + \frac{z_g}{n_t^2}\mathbf{1}\mathbf{1}^\top\right)\tilde{\mathbf{i}}_m \qquad (6)$$

Step-down $Y_G - \Delta$ connection

- *Goal*: express delta currents in terms of line currents
- There are two independent linear equations from $\mathbf{i}_m = \mathbf{D}_f^\top \tilde{\mathbf{i}}_m$
- Get one more equation from zero-sum LL voltages across delta

$$\mathbf{1}^\top \tilde{\mathbf{v}}_m = 0 \quad \overset{(5)}{\Longrightarrow} \quad \mathbf{1}^\top \bar{\mathbf{v}}_n = \left(n_t \mathbf{z} + \frac{3 z_g}{n_t} \mathbf{1} \right)^\top \tilde{\mathbf{i}}_m$$

- Put the three equations together

$$\begin{bmatrix} I_a \\ I_b \\ \mathbf{1}^\top \bar{\mathbf{v}}_n \end{bmatrix} = \mathbf{K}^{-1} \tilde{\mathbf{i}}_m \text{ where } \mathbf{K}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ n_t z_{ab} + \frac{3 z_g}{n_t} & n_t z_{bc} + \frac{3 z_g}{n_t} & n_t z_{ca} + \frac{3 z_g}{n_t} \end{bmatrix}$$

- Solve for delta currents if $\mathbf{K} = [\mathbf{k}_1 \ \mathbf{k}_2 \ \mathbf{k}_3]$

$$\tilde{\mathbf{i}}_m = [\mathbf{k}_1 \ \mathbf{k}_2 \ \mathbf{0}] \, \mathbf{i}_m + \mathbf{k}_3 \mathbf{1}^\top \bar{\mathbf{v}}_n \qquad (7)$$

delta currents depend on line currents *and* primary voltages!

Step-down $Y_G - \Delta$ connection

- Substitute (7) into (6) to get the forward update with

$$\mathbf{E} = \mathbf{W}\left(\frac{1}{n_t}\mathbf{I} - \left(\mathrm{dg}(\mathbf{z}) + \frac{z_g}{n_t}\mathbf{1}\mathbf{1}^\top\right)\mathbf{k}_3\mathbf{1}^\top\right)$$

$$\mathbf{F} = -\mathbf{W}\left(\mathrm{dg}(\mathbf{z}) + \frac{z_g}{n_t}\mathbf{1}\mathbf{1}^\top\right)[\mathbf{k}_1\ \mathbf{k}_2\ \mathbf{0}]$$

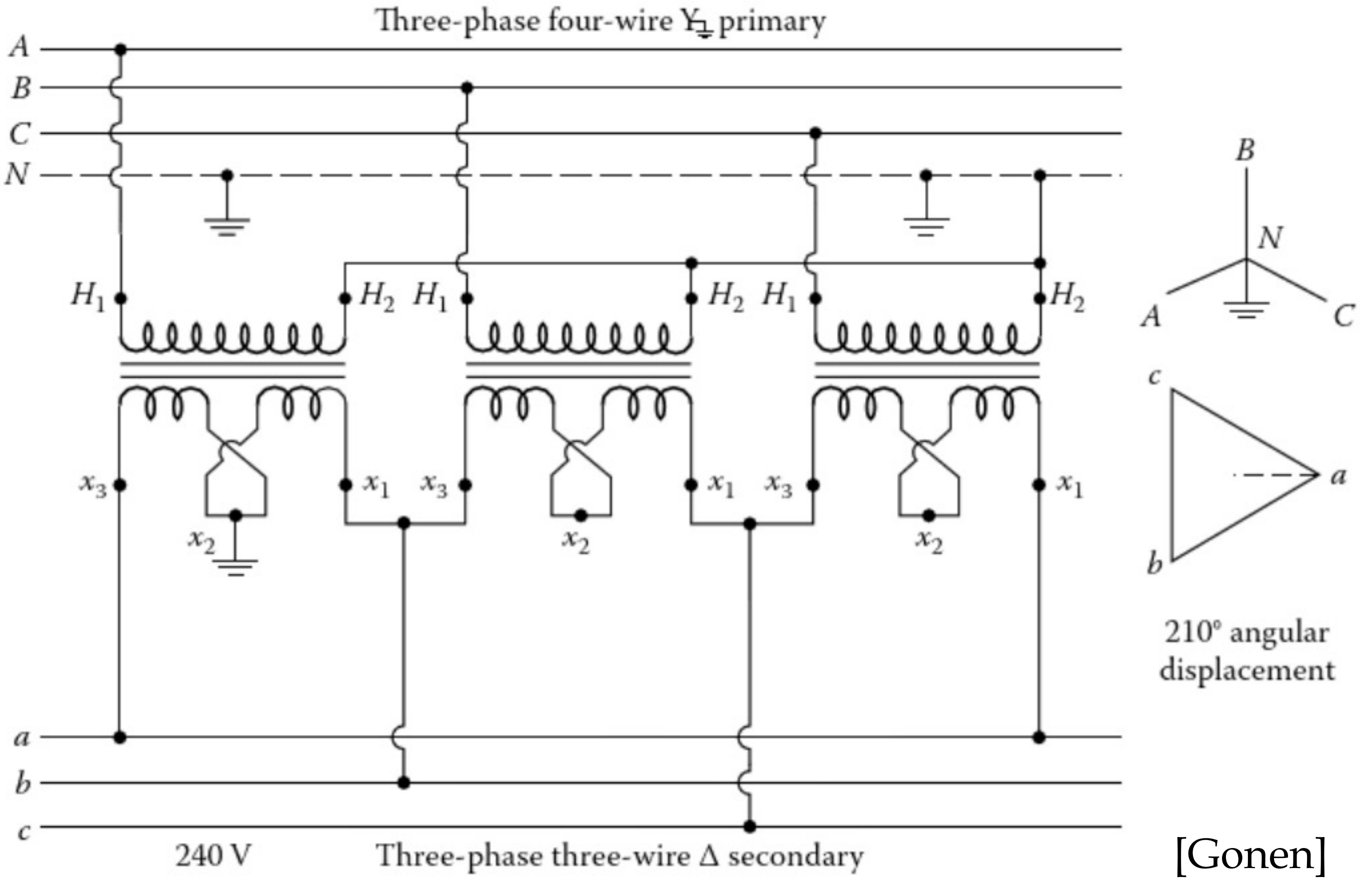
- Substitute (7) into (1) to get the backward update with

$$\mathbf{i}_n = \mathbf{C}'\bar{\mathbf{v}}_n + \mathbf{F}\mathbf{i}_m \quad \text{where} \quad \mathbf{C}' = \frac{1}{n_t}\mathbf{k}_3\mathbf{1}^\top \quad \text{and} \quad \mathbf{D} = \frac{1}{n_t}[\mathbf{k}_1\ \mathbf{k}_2\ \mathbf{0}]$$

- Matrix **C** usually multiplies secondary voltages $\mathbf{v}_{\mathrm{m}}$. However, this equation can still be used during backward sweep as primary voltages are known and fixed during this update.

- This is the only connection with a **C** matrix!

Discussion on $Y_G - \Delta$ connection



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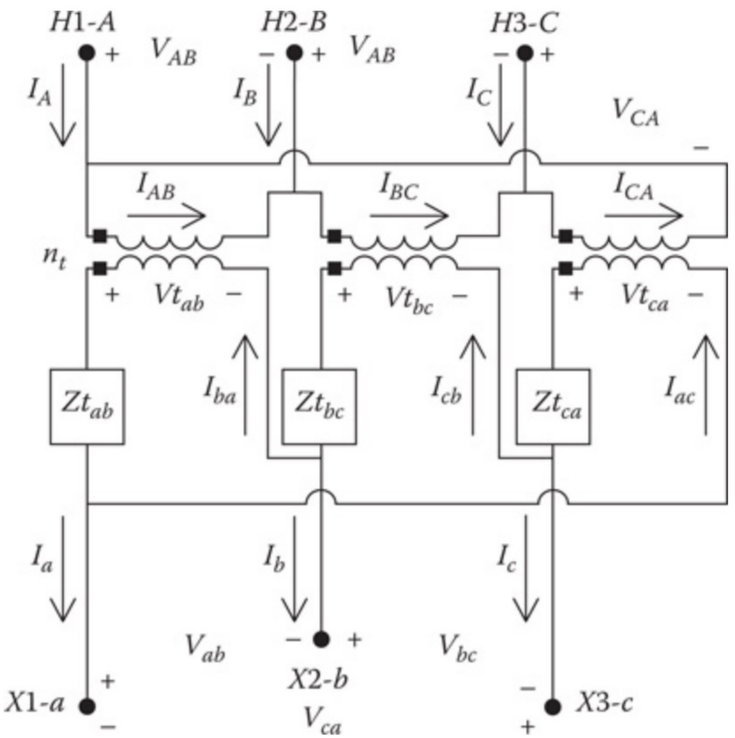
7) $\Delta - \Delta$ connection



- Use zero-sum voltage drop across delta to relate phase to line currents

- Backward model

$$\tilde{\mathbf{i}}_n = \frac{1}{n_t}\tilde{\mathbf{i}}_m \;\Rightarrow\; \mathbf{D}_f^\top \tilde{\mathbf{i}}_n = \frac{1}{n_t}\mathbf{D}_f^\top \tilde{\mathbf{i}} \;\Rightarrow\; \mathbf{i}_n = \frac{1}{n_t}\mathbf{i}_m$$

- Voltage transformations

$$\tilde{\mathbf{v}}_m = \tilde{\mathbf{v}}_t - \mathrm{dg}(\mathbf{z})\tilde{\mathbf{i}}_m = \frac{1}{n_t}\tilde{\mathbf{v}}_n - \mathrm{dg}(\mathbf{z})\tilde{\mathbf{i}}_m \;\Rightarrow$$

$$\mathbf{1}^\top \tilde{\mathbf{v}}_m = \mathbf{1}^\top \tilde{\mathbf{v}}_t - \mathbf{1}^\top \mathrm{dg}(\mathbf{z})\tilde{\mathbf{i}}_m \;\Rightarrow\; \mathbf{z}^\top \tilde{\mathbf{i}}_m = 0$$

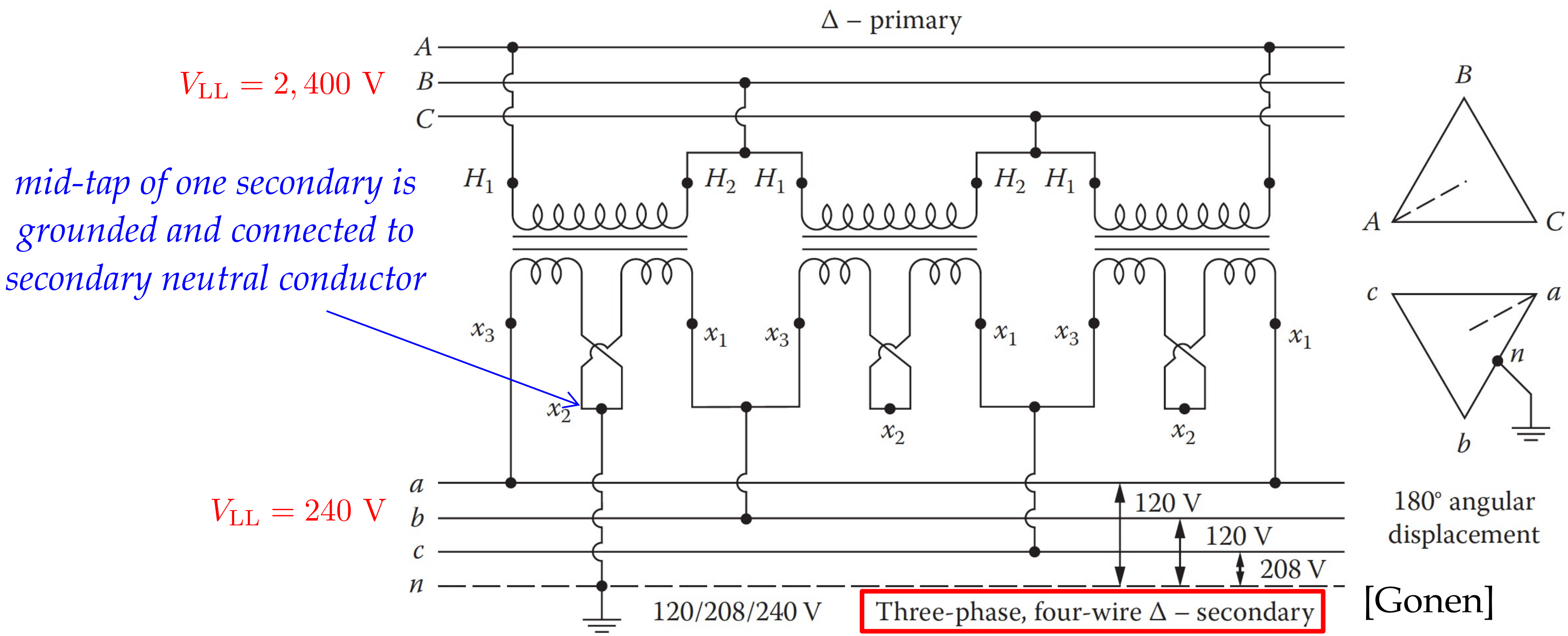
- Stack previous equation with two from $\mathbf{i}_m = \mathbf{D}_f^\top \tilde{\mathbf{i}}_m = 0$

$$\begin{bmatrix} I_a \\ I_b \\ 0 \end{bmatrix} = \mathbf{K}^{-1}\tilde{\mathbf{i}}_m \text{ where } \mathbf{K}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ z_{ab} & z_{bc} & z_{ca} \end{bmatrix} \text{ and } \mathbf{K} = [\mathbf{k}_1 \;\mathbf{k}_2\; \mathbf{k}_3]$$

- Forward model

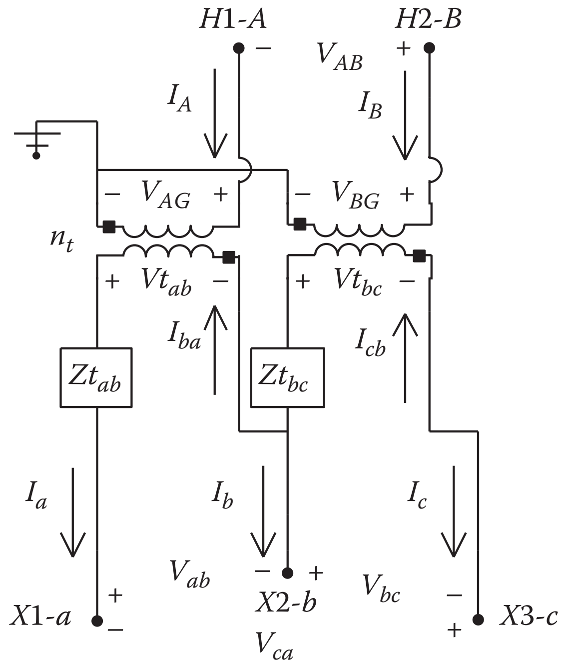
$$\mathbf{v}_m = \mathbf{W}\tilde{\mathbf{v}}_m = \mathbf{E}\mathbf{v}_n - \mathbf{F}\mathbf{i}_m \quad \text{where} \quad \mathbf{E} = \frac{1}{n_t}\mathbf{W}\mathbf{D}_f \text{ and } \mathbf{F} = \mathbf{W}\mathrm{dg}(\mathbf{z})[\mathbf{k}_1 \;\mathbf{k}_2\; \mathbf{0}]$$

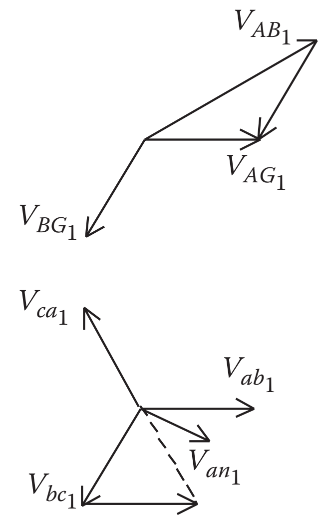
Discussion on $\Delta - \Delta$ connection



- Used in three-wire delta systems
- 180° displacement due to additive polarity; 0° displacement for subtractive polarity
- *Load connections*
  - large 3ϕ to delta (240V)
  - large 1ϕ to one of deltas (240V) or cn (208V)
  - small 1ϕ to an or bn (120V)

8) Open Y – open Δ connection



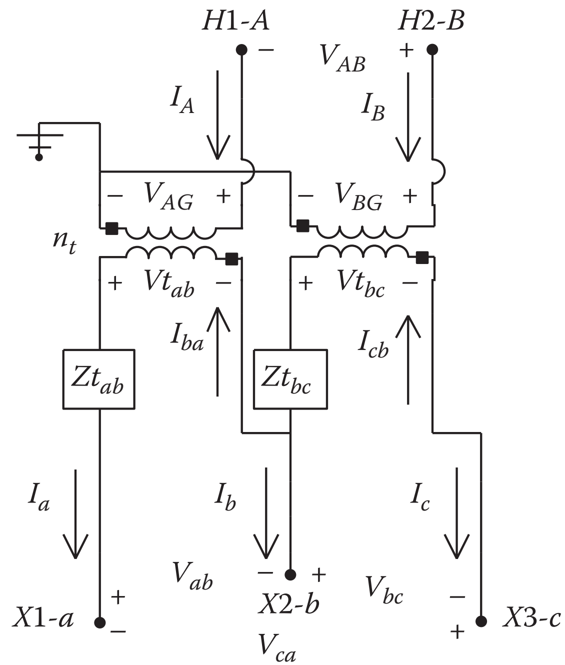


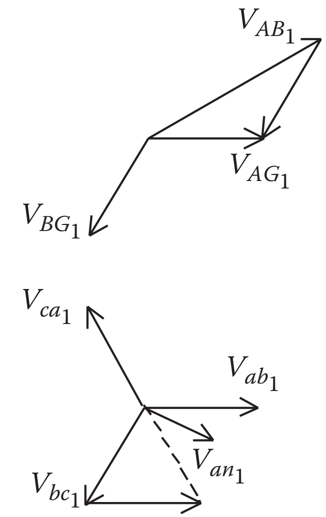
- *Small* 3ϕ load (motor) plus 1ϕ load (lighting)
- 2ϕ *(2-line) primary* and two transformers
- If 1ϕ load is connected on *ab*, the 'lighting' transformer is on *AG*
- Primary phase of lighting transformer (usually larger kVA) determines *leading/lagging connection*

Open Y – open Δ connection (cont'd)

- Backward model

$$\mathbf{i}_n = \frac{1}{n_t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{i}_m$$





- Secondary voltages

$$V_{ab} = V_{t,ab} - I_a z_{ab} = \frac{1}{n_t}V_{AN} - I_a z_{ab}$$

$$V_{bc} = V_{t,bc} + I_c z_{bc} = \frac{1}{n_t}V_{BN} + I_c z_{bc}$$

$$V_{ca} = -V_{ab} - V_{bc}$$

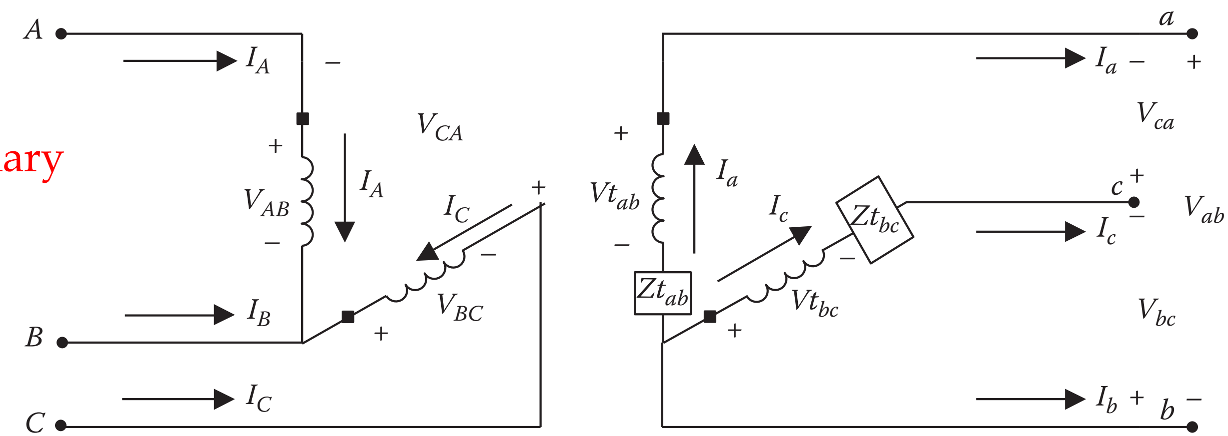
- Collecting in matrix form $\quad \tilde{\mathbf{v}}_m = \frac{1}{n_t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}\mathbf{v}_n + \begin{bmatrix} -z_{ab} & 0 & 0 \\ 0 & 0 & z_{bc} \\ z_{ab} & 0 & -z_{bc} \end{bmatrix}\mathbf{i}_m$

- Forward model

$$\mathbf{v}_m = \mathbf{W}\tilde{\mathbf{v}}_m = \frac{1}{3n_t}\begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix}\mathbf{v}_n - \frac{1}{3}\begin{bmatrix} 2z_{ab} & 0 & -z_{bc} \\ -z_{ab} & 0 & -z_{bc} \\ -z_{ab} & 0 & 2z_{bc} \end{bmatrix}\mathbf{i}_m$$

9) Open Δ – open Δ connection

3-line primary



- Backward model $\mathbf{i}_n = \frac{1}{n_t}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{i}_m$

- Secondary LL voltages

$$V_{ab} = \frac{1}{n_t} V_{AB} - I_a z_{ab}$$

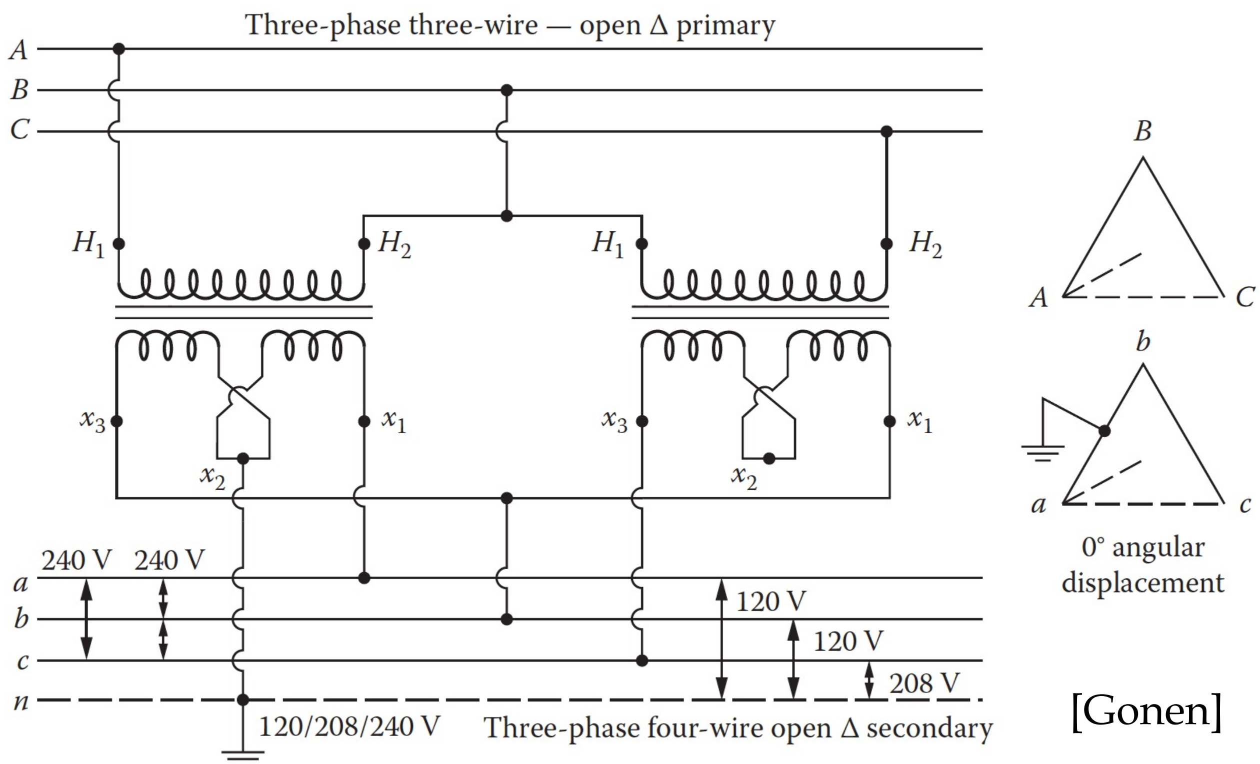
$$V_{bc} = \frac{1}{n_t} V_{BC} - I_c z_{bc}$$

$$V_{ca} = -V_{ab} - -V_{bc}$$

- Forward model

$$\mathbf{v}_m = \mathbf{W}\tilde{\mathbf{v}}_m = \frac{1}{n_t}\mathbf{W}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}\mathbf{D}_f \mathbf{v}_n - \mathbf{W}\begin{bmatrix} Z_{ab} & 0 & 0 \\ 0 & 0 & Z_{bc} \\ -Z_{ab} & 0 & -Z_{bc} \end{bmatrix}\mathbf{i}_m$$

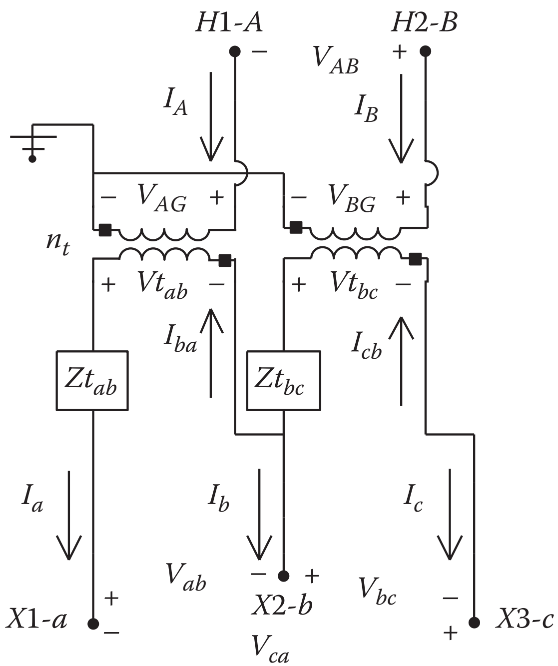
Discussion on open Δ or V connection



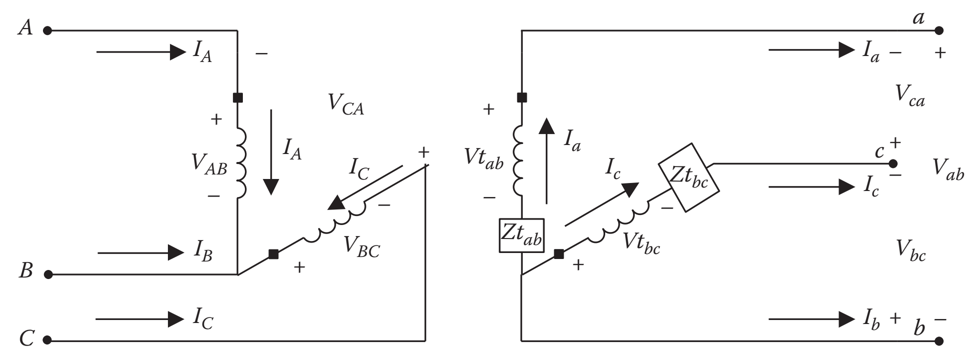
[Gonen]

- Used in emergency or when load is expected to grow

Comparing two 'open' connections



- Both use two transformers and can serve 3ϕ loads
- One feeds from a 2ϕ primary; the other from a 3ϕ one



- Compare to the corresponding 3-tranformer bank:
  - LL voltages remain unchanged
  - line currents become phase (delta) currents
  - to comply with power rating, load has to be scaled down by $\frac{1}{\sqrt{3}} = 57.7\%$
- Hence, a bank with two transformers serves only 57.7% rather than 2/3=66.6% of the full-bank capacity