ECE 5984: Power Distribution System Analysis

Lecture 13: Distribution Grid Optimization under Uncertainty

Reference: see publications list at the end

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Invent the Future

Outline

1) Local, centralized, and decentralized inverter control

2) LDF-OPF deterministic formulation

3) Robust optimization

4) Chance-constrained optimization

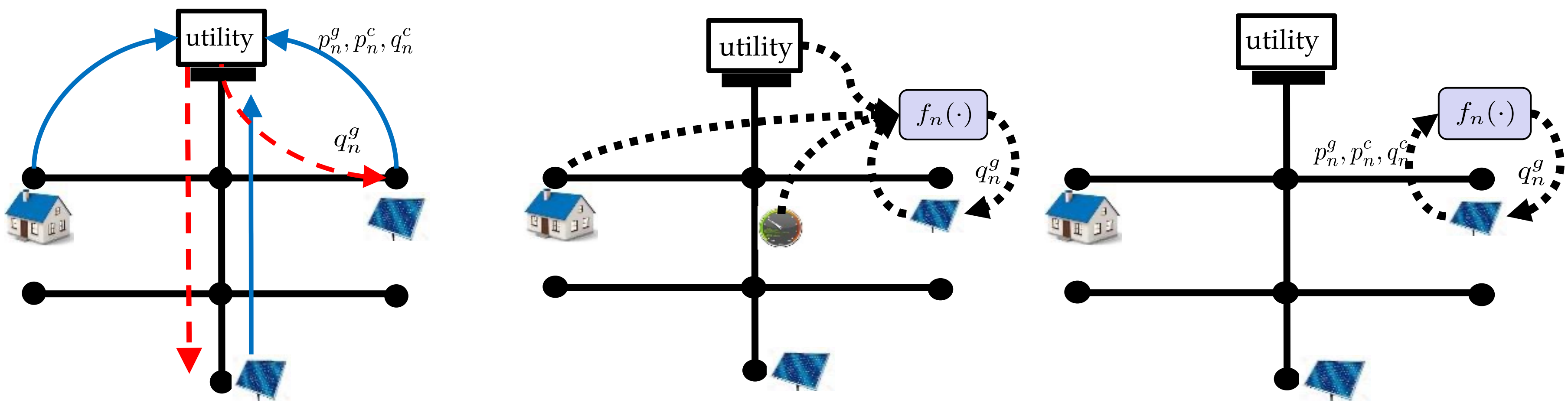
5) Scenario-based optimization

6) Multi-parametric programming

7) Polytopic approximation of quadratic constraints

Reactive power control schemes

- Categorized based on required communication



centralized ***decentralized*** ***local***

✓ high cyber requirements
✓ optimal
✓ decisions may be obsolete due to delays

✓ moderate computations
✓ high communication

✓ no cyber requirements
✓ suboptimal

Deterministic OPF under LDF

- Lecture 11 posed inverter control as SOCP/SDP under AC grid model

- To ease computations, consider OPF under LDF model

$$
\begin{aligned}
\min_{\mathbf{p},\mathbf{q}}\ & f(\mathbf{p},\mathbf{q}) \\
\text{s.t. } & \underline{\mathbf{v}} \leq \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q} + v_0\mathbf{1} \leq \bar{\mathbf{v}} && \text{voltage limits} \\
& \underline{\mathbf{q}} \leq \mathbf{q} \leq \bar{\mathbf{q}} && \text{resource constraints} \\
& \underline{\mathbf{p}} \leq \mathbf{p} \leq \bar{\mathbf{p}} && \text{(DER inverters)}
\end{aligned}
$$

- Depending on cost, we may get *linear program (LP)* or *quadratic program (QP)*

$$f_v(\mathbf{p},\mathbf{q}) = \|\mathbf{v} - v_0\mathbf{1}\|_2^2 \simeq \|\mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q}\|_2^2 \quad \text{or} \quad f_\ell(\mathbf{p},\mathbf{q}) \simeq \mathbf{p}^\top\mathbf{R}\mathbf{p} + \mathbf{q}^\top\mathbf{X}\mathbf{q}$$

voltage deviations   *ohmic losses on lines*

- Other objectives/constraints can be envisioned (EVs, batteries, TCLs)

- Previous setup assumes grid conditions are *fixed (deterministic) and known*

- *What if* we are trying to
  - control inverter **q** while solar **p** changes?
  - control inverters (**p**,**q**) while load (**p**,**q**) is uncertain?
  - design an inverter rule **q=Dp** for random solar **p**?

Optimization under uncertainty

- Consider a *parameterized* convex optimization problem [6]

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & g(\mathbf{x}, \boldsymbol{\theta}) \leq 0 \end{aligned}$$

- Having a *single constraint* is wlog (point-wise max of all)

- *Robust optimization:* satisfy constraint for all values within an uncertainty range

$$g(\mathbf{x}, \boldsymbol{\theta}) \leq 0 \text{ for all } \boldsymbol{\theta} \in \boldsymbol{\Theta}$$

need uncertainty range

- *Chance-constrained optimization:* satisfy constraint with some (high) probability

$\Pr\{\boldsymbol{\theta} \in \boldsymbol{\Theta} : g(\mathbf{x}, \boldsymbol{\theta}) \leq 0\} \geq \alpha$ for say $\alpha = 0.95 \text{ or } 0.99$

need pdf for uncertain parameters and being able to integrate over

- *Scenario-based optimization:* we only have some scenarios for uncertain variables

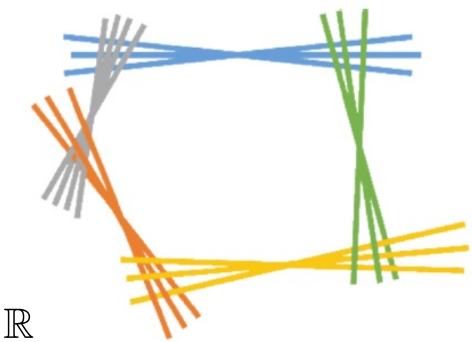
$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & g(\mathbf{x}, \boldsymbol{\theta}_s) \leq 0 \quad s = 1, \ldots, S \end{aligned}$$

need scenarios (forecasts or historical data)

enforcing constraint for all scenarios may be too conservative or impossible (infeasible)

Robust linear programming

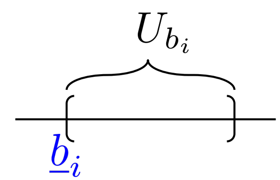
- Linear program involving uncertainties in constraints



$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.to} \quad \mathbf{a}_i^\top \mathbf{x} \leq b_i \quad \text{for all} \quad \mathbf{a}_i \in U_{\mathbf{a}_i}, b_i \in U_{b_i} \;\; i = 1, \ldots, M$$

- Given uncertainty sets for problem parameters $U_{\mathbf{a}_i} \subseteq \mathbb{R}^N, U_{b_i} \subseteq \mathbb{R}$

- Solve LP for *worst-case* of $(\mathbf{a}_i, b_i)'s$
    - Replace b_i with the *smallest* value over $U_{b_i} : \underline{b}_i$
    - Replace $\mathbf{a}_i^\top \mathbf{x}$ with the *largest* value over $U_{\mathbf{a}_i} : \max_{\mathbf{a}_i \in U_{\mathbf{a}_i}} \mathbf{a}_i^\top \mathbf{x}$



- Worst-case formulation [2]

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.to} \quad \max_{\mathbf{a}_i \in U_{\mathbf{a}_i}} \mathbf{a}_i^\top \mathbf{x} \leq \underline{b}_i \qquad i = 1, \ldots, M$$

Uncertainty sets

- *Polytopic* uncertainty set $U_{\mathbf{a}_i} = \{\mathbf{a}_i | \mathbf{D}_i \mathbf{a}_i \leq \mathbf{d}_i\}$

  – use *duality* to replace the internal maximization [2]

$$\begin{array}{lll} \min\limits_{\mathbf{x}, \{\boldsymbol{\lambda}_i\}_{i=1}^M} & \mathbf{c}^\top \mathbf{x} & \\ \text{s.to} & \boldsymbol{\lambda}_i^\top \mathbf{d}_i \leq \underline{b}_i, & i = 1, \ldots, M \\ & \mathbf{D}_i^\top \boldsymbol{\lambda}_i = \mathbf{x}, & i = 1, \ldots, M \\ & \boldsymbol{\lambda}_i \geq \mathbf{0}, & i = 1, \ldots, M \end{array}$$

  – worst-case formulation remains a *linear program (LP)*

- *Ellipsoidal* uncertainty set $U_{\mathbf{a}_i} = \{\bar{\mathbf{a}}_i + \mathbf{P}_i \mathbf{u} | \; \|\mathbf{u}\|_2 \leq 1\}$

  – internal maximization has explicit solution [2] $\max\limits_{\mathbf{a}_i \in U_{\mathbf{a}_i}} \mathbf{a}_i^\top \mathbf{x} = \bar{\mathbf{a}}_i^\top + \|\mathbf{P}_i^\top \mathbf{x}\|_2$

$$\begin{array}{lll} \min\limits_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} & \\ \text{s.to} & \bar{\mathbf{a}}_i^\top \mathbf{x} + \|\mathbf{P}_i^\top \mathbf{x}\|_2 \leq \underline{b}_i & i = 1, \ldots, M \end{array}$$

  – worst-case formulation becomes a *second-order cone program (SOCP)*

Voltage regulation using chance constraints

- Suppose need to design linear inverter control rules $\mathbf{q} = \mathbf{D}\mathbf{p}$ for some (diagonal) $\mathbf{D}$

- Solar injections can be modeled as random variables, e.g., $\mathbf{p} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

then reactive power injections are also Gaussian $\mathbf{q} \sim \mathcal{N}(\mathbf{D}\boldsymbol{\mu}, \mathbf{D}\boldsymbol{\Sigma}\mathbf{D})$

- *Stochastic optimization* deals with costs and objectives that use random variables

deterministic OPF (fixed $\mathbf{p}$*)*

$$\min_{\mathbf{D}:\mathbf{q}=\mathbf{D}\mathbf{p}} f(\mathbf{q}) = \mathbf{q}^\top \mathbf{R}\mathbf{q} + \mathbf{p}^\top \mathbf{R}\mathbf{p}$$

$$\text{s.to } \mathbf{v} = \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q} + v_0\mathbf{1}$$

$$\underline{\mathbf{v}} \leq \mathbf{v} \leq \bar{\mathbf{v}}$$

stochastic OPF (random $\mathbf{p}$*)*

$$\min_{\mathbf{D}:\mathbf{q}=\mathbf{D}\mathbf{p}} \mathbb{E}[f(\mathbf{q})]$$

$$\text{s.to } \mathbf{v} = \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q} + v_0\mathbf{1}$$

chance constraints

$$\begin{cases} \Pr\{v_n \leq \bar{v}_n\} \geq \alpha_i, & n = 1, \ldots, N \\ \Pr\{v_n \geq -\bar{v}_n\} \geq \alpha_i, & n = 1, \ldots, N \end{cases}$$

- Average ohmic losses expressed as quadratic function of $\mathbf{D}$

$$\mathbb{E}[\mathbf{p}^\top \mathbf{R}\mathbf{p}] = \text{Trace}\left(\mathbf{R}\mathbb{E}[\mathbf{p}\mathbf{p}^\top]\right) = \text{Trace}(\mathbf{R}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^\top \mathbf{R}\boldsymbol{\mu} = \text{constant}$$

$$\mathbb{E}[\mathbf{q}^\top \mathbf{R}\mathbf{q}] = \text{Trace}\left(\mathbf{R}\mathbb{E}[\mathbf{q}\mathbf{q}^\top]\right) = \text{Trace}(\mathbf{R}\mathbf{D}\boldsymbol{\Sigma}\mathbf{D}) + \boldsymbol{\mu}^\top \mathbf{D}\mathbf{R}\mathbf{D}\boldsymbol{\mu}$$

Chance constraints

- Chance constraints under Gaussian pdf yield second-order conic constraints [4]

- Constraint $v_n \leq \overline{v}_n$ under statistical model $\mathbf{p} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{q} \sim \mathcal{N}(\mathbf{D}\boldsymbol{\mu}, \mathbf{D}\boldsymbol{\Sigma}\mathbf{D})$

$$\mathbf{v} = \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q} + v_0\mathbf{1} \quad \Longrightarrow \quad v_n = \mathbf{r}_n^\top \mathbf{p} + \mathbf{x}_n^\top \mathbf{q} + v_0 \sim \mathcal{N}(\mu_n, \sigma_n^2)$$

voltage at bus n

where $\mu_n = (\mathbf{r}_n + \mathbf{D}\mathbf{x}_n)^\top \boldsymbol{\mu} + v_0$

and $\sigma_n^2 = (\mathbf{r}_n + \mathbf{D}\mathbf{x}_n)^\top \boldsymbol{\Sigma} (\mathbf{r}_n + \mathbf{D}\mathbf{x}_n)$

- Reformulate chance (probability) constraint as

CDF of standard Gaussian

$$\Pr(v_n \leq \bar{v}_n) = \Pr\left(\underbrace{\frac{v_n - \mu_n}{\sigma_n}}_{\sim \mathcal{N}(0,1)} \leq \frac{\bar{v}_n - \mu_n}{\sigma_n}\right) = \Phi\left(\frac{\bar{v}_n - \mu_n}{\sigma_n}\right) \geq \alpha_i \iff \frac{\bar{v}_n - \mu_n}{\sigma_n} \geq \Phi^{-1}(\alpha_i) = \text{const.}$$

$$\Phi^{-1}(\alpha_i)\sigma_n + \mu_n \leq \bar{v}_n \iff \Phi^{-1}(\alpha_i)\|\boldsymbol{\Sigma}^{1/2}(\mathbf{r}_n + \mathbf{D}\mathbf{x}_n)\|_2 + (\mathbf{r}_n + \mathbf{D}\mathbf{x}_n)^\top \boldsymbol{\mu} + v_0 \leq \bar{v}_n$$

second-order cone (SOC) constraint

- Problem has been reformulated to an SOCP

- If $\mathbf{p}$ follows a *log-concave pdf* (more general family of pdf's than Gaussian), the feasible set is still *convex* but not intersection of SOCs [5]

Scenario-based optimization

- Suppose you solve scenario-based optimization over $\mathbf{x} \in \mathbb{R}^N$

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.to} \quad & g(\mathbf{x}, \boldsymbol{\theta}_s) \leq 0 \qquad s = 1, \ldots, S \end{aligned}$$

- How many scenarios are needed to ensure $\Pr\{g(\mathbf{x}, \boldsymbol{\theta}) \leq 0\} \geq \alpha$?

- If you sample S scenarios with [6]

$$S \geq \left\lceil \frac{2}{1-\alpha} \ln \frac{1}{1-\beta} + 2N + \frac{2N}{1-\alpha} \ln \frac{2}{1-\alpha} \right\rceil \quad \text{for some } \beta \in (0,1)$$

then with probability β

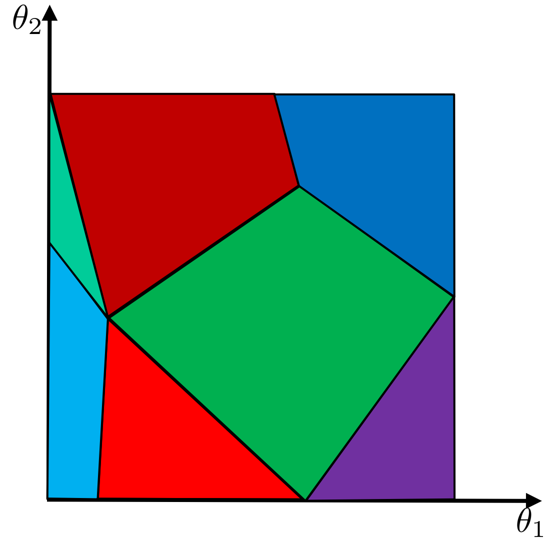
  - either problem is infeasible
  - or problem is feasible (and found solution satisfies original chance constraint)

- Holds for constraint function g that is convex in $\mathbf{x}$ and any dependence on $\boldsymbol{\theta}$

Multi-parametric programming

- *Multiparametric quadratic (MPQP)* program over $\boldsymbol{\theta} \in \Theta$

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}\mathbf{x}^\top \mathbf{H}\mathbf{x} + (\mathbf{C}\boldsymbol{\theta} + \mathbf{d})^\top \mathbf{x} \\ \text{s.to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{E}\boldsymbol{\theta} + \mathbf{b} \qquad : \boldsymbol{\lambda} \end{aligned} \tag{QP}$$



- Need to solve it for all (or many) $\boldsymbol{\theta} \in \Theta$

- Space Θ can be partitioned into *critical regions* $\Theta_k' s$ [7]

- Each region Θ_k :

  i) is described by a *polytope* $\Theta_k := \{\boldsymbol{\theta} : \ \mathbf{N}_k\boldsymbol{\theta} + \mathbf{t}_k \le \mathbf{0}\}$

  ii) same constraints become *active*

  iii) optimal primal/dual solutions $\mathbf{x}^* = \mathbf{L}_k\boldsymbol{\theta} + \mathbf{r}_k$ and $\boldsymbol{\lambda}^* = \mathbf{M}_k\boldsymbol{\theta} + \mathbf{s}_k$

- Once Θ_k is identified, easily solve the QP's related to *all* $\theta \in \Theta_k$!

Polytopic approximation of quadratic constraints

- Most of previous schemes assumed constraints are linear in $(\mathbf{x}, \boldsymbol{\theta})$

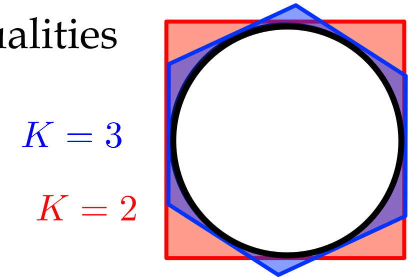
- If optimizing p+jq (solar curtailment), need to enforce kVA inverter ratings

$$|s| = |p + jq| = \sqrt{p^2 + q^2} \leq \overline{s}$$

- Bound magnitude of s as $\cos\left(\frac{\phi}{2}\right)|s| \leq f(s) \leq |s|$

$$f(s) = \max_{k=1,\ldots,K} |\cos(k\phi)p + \sin(k\phi)q|, \quad \phi = \frac{\pi}{K}$$

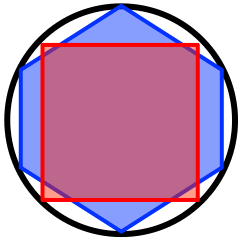
- Outer approximation (relaxation) as intersection of linear inequalities



$$-\overline{s} \leq \cos(k\phi)p + \sin(k\phi)q \leq \overline{s}, \quad \phi = \frac{\pi}{K}, \; k = 1, \ldots, K$$

with relative accuracy of $1 - \cos\left(\frac{\pi}{2K}\right)$

- Inner approximation (restriction) as $\frac{f(s)}{\cos(\phi/2)} \leq \overline{s}$



- kVA ratings of lines/transformers can be handled similarly

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