ECE 5984: Power Distribution System Analysis

Lecture 11: DistFlow and LinDistFlow

Reference: see publications list at the end

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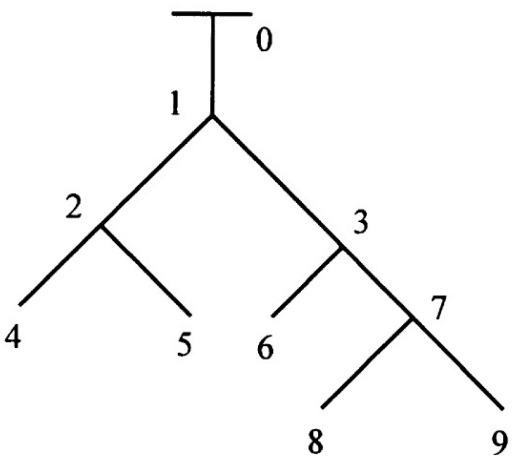
Outline

1. Branch flow model (BFM)
2. DistFlow model
3. DistFlow model for power flow
4. DistFlow model for optimal power flow
5. LinDistFlow model for approximate analysis

Branch-bus incidence matrix

- *Single-phase* and *radial* feeder represented by tree graph

  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with $|\mathcal{N}| = N + 1$ and $|\mathcal{E}| = L = N$



$|\mathcal{N}| = 10$ and $|\mathcal{E}| = L = N = 9$

breadth-first
(vs. depth-first)
numbering

- Branch-bus incidence matrix

buses

lines

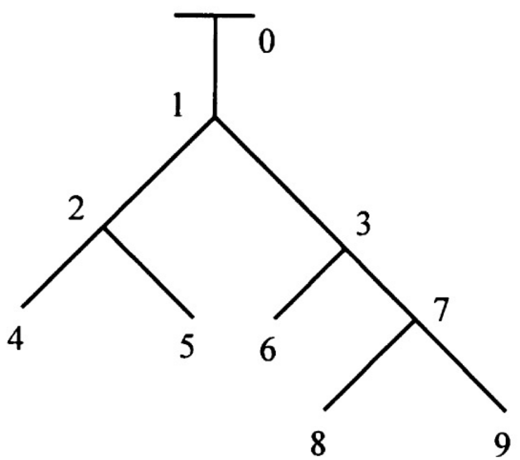
reduced branch-bus incidence matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & -1 & & & & & & & & \\ & 1 & -1 & & & & & & & \\ & 1 & & -1 & & & & & & \\ & & 1 & & -1 & & & & & \\ & & 1 & & & -1 & & & & \\ & & & 1 & & & -1 & & & \\ & & & 1 & & & & -1 & & \\ & & & & & & & 1 & -1 & \\ & & & & & & & 1 & & -1 \end{bmatrix}$$

$$\tilde{\mathbf{A}} = [\mathbf{a}_0 \;\; \mathbf{A}]$$

- The line feeding bus n is indexed as line n-th

Branch-bus incidence matrix (cont'd)



reduced branch-bus incidence matrix

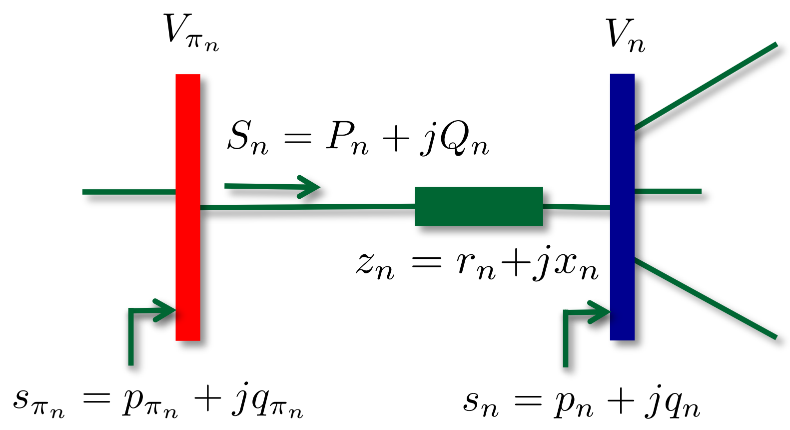
$$\mathbf{A} = \begin{bmatrix} -1 & & & & & & & & \\ 1 & -1 & & & & & & & \\ 1 & & -1 & & & & & & \\ & 1 & & -1 & & & & & \\ & 1 & & & -1 & & & & \\ & & 1 & & & -1 & & & \\ & & 1 & & & & -1 & & \\ & & & & & & 1 & -1 & \\ & & & & & & 1 & & -1 \end{bmatrix}$$

- Branch-bus incidence matrix $\tilde{\mathbf{A}} = [\mathbf{a}_0 \;\; \mathbf{A}]$

$$\begin{aligned} \tilde{\mathbf{A}}\mathbf{1}_{N+1} &= \mathbf{0} \Rightarrow \\ \mathbf{a}_0 + \mathbf{A}\mathbf{1}_N &= \mathbf{0} \\ \mathbf{1}_N &= -\mathbf{A}^{-1}\mathbf{a}_0 \end{aligned}$$

$$\mathbf{F} = \mathbf{A}^{-1} = \begin{vmatrix} -1 & & & & & & & & \\ -1 & -1 & & & & & & & \\ -1 & & -1 & & & & & & \\ -1 & -1 & & -1 & & & & & \\ -1 & -1 & & & -1 & & & & \\ -1 & & -1 & & & -1 & & & \\ -1 & & -1 & & & & -1 & & \\ -1 & & -1 & & & & -1 & -1 & \\ -1 & & -1 & & & & -1 & & -1 \end{vmatrix}$$

Branch flow model (BFM)



Line n feeding bus n from its parent bus π_n

- Branch flow equations on $\mathbf{x}(\mathbf{s}) = (\mathbf{S}, \mathbf{I}, \mathbf{V}, s_0)$

$$
\begin{aligned}
V_{\pi_n} - V_n &= z_n I_n \\
S_n &= V_{\pi_n} I_n^* \\
S_n - z_n |I_n|^2 + s_n &= \sum_{k:\ n \to k} S_k
\end{aligned}
$$

- Boundary conditions?

- Given **s**, solve *2L+N+1* equations in *2L+N+1* complex unknowns [3]

- Equivalent with typical bus injection model (BIM); a.k.a. power flow equations

Branch flow model *squared*

- Introduce squared voltage and current magnitudes

$$v_n = |V_n|^2 \quad \text{and} \quad \ell_n = |I_n|^2$$

- Rearrange power injection equations

$$S_n - z_n|I_n|^2 + s_n = \sum_{k:\ n\to k} S_k \quad \Rightarrow \quad \begin{aligned} \sum_{k:\ n\to k} P_k &= p_n + P_n - r_n\ell_n \\ \sum_{k:\ n\to k} Q_k &= q_n + Q_n - x_n\ell_n \end{aligned}$$

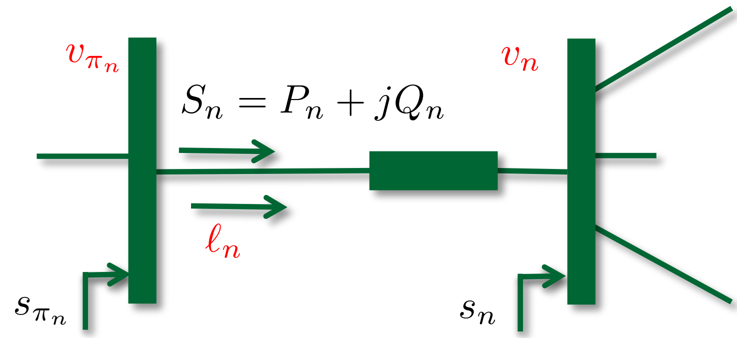
- Ohm's law squared (multiply both sides by complex conjugate)

$$V_{\pi_n} - V_n = z_n I_n \quad \Rightarrow \quad \begin{aligned} V_n &= V_{\pi_n} - z_n I_n \Rightarrow \\ V_n V_n^* &= (V_{\pi_n} - z_n I_n)(V_{\pi_n} - z_n I_n)^* \Rightarrow \\ v_n &= v_{\pi_n} - 2\mathrm{Re}\left[z_n^* V_{\pi_n} I_n^*\right] + |z_n|^2 \ell_n \Rightarrow \\ v_n &= v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2)\ell_n \end{aligned}$$

- Definition of complex power flow squared

$$S_n = V_{\pi_n} I_n^* \quad \Rightarrow \quad \ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}$$

Relaxed branch flow model



- Relaxed BFM on $\mathbf{y}(\mathbf{s}) := (\mathbf{S}, \boldsymbol{\ell}, \mathbf{v}, p_0, q_0)$

$$\sum_{k:\ n\to k} P_k = p_n + P_n - r_n\ell_n$$

$$\sum_{k:\ n\to k} Q_k = q_n + Q_n - x_n\ell_n$$

$$v_n = v_{\pi_n} - 2r_nP_n - 2x_nQ_n + (r_n^2 + x_n^2)\ell_n$$

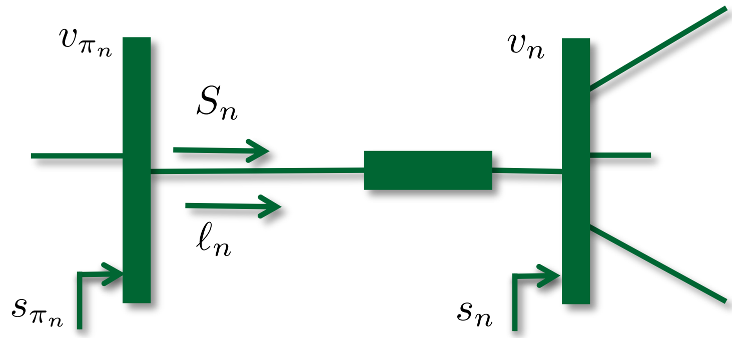
$$\ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}$$

$$v_n = |V_n|^2 \ \text{ and } \ \ell_n = |I_n|^2$$

current and voltage phases have been dropped!

- Boundary conditions?
- Current mags. can be eliminated; equations remain nonlinear
- Given **s**, solve *2(L+N+1)* equations in *3L+N+2* real unknowns [1]-[2]
- In radial grids, we get *4N+2* equations in *4N+2* real unknowns
- Unique solution for practical networks with $v_0 \simeq 1$ and small $\{(r_n, x_n)\}$

Recovering phases



- After the relaxed branch flow equations have been solved [3]

- Recover voltage phases

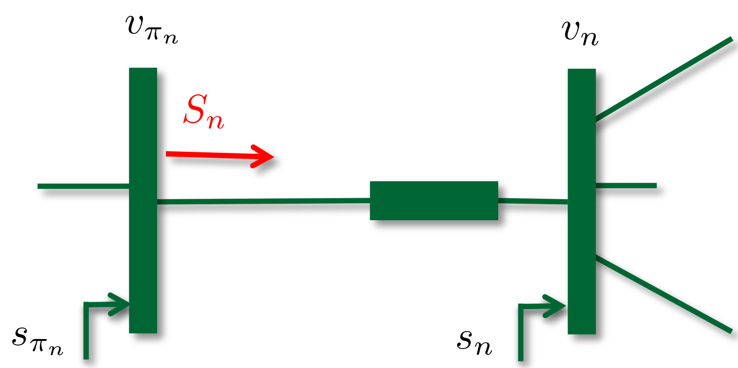
$$
\begin{aligned}
V_{\pi_n} - V_n &= z_n I_n \Rightarrow \\
V_{\pi_n}^* - V_n^* &= z_n^* I_n^* \Rightarrow \\
V_{\pi_n} V_n^* &= v_{\pi_n} - z_n^* S_n \Rightarrow \\
\theta_{\pi_n} - \theta_n &= \angle(v_{\pi_n} - z_n^* S_n)
\end{aligned}
$$

linear system can be inverted only when L=N

- Recover current phasors $I_n = \left(\frac{S_n}{V_{\pi_n}}\right)^*$

Linearized distribution flow (LinDistFlow)

- Approximate model to overcome the complexity of quadratic equations [1]-[2]
- Derived from forward DistFlow model upon dropping terms related to losses



DistFlow (forward form)

$$\sum_{k:\ n\to k} P_k = p_n + P_n - r_n\ell_n$$

$$\sum_{k:\ n\to k} Q_k = q_n + Q_n - x_n\ell_n$$

$$v_n = v_{\pi_n} - 2r_nP_n - 2x_nQ_n + (r_n^2 + x_n^2)\ell_n$$

$$\ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}$$

LinDistFlow

$$\sum_{k:\ n\to k} P_k \simeq p_n + P_n$$

$$\sum_{k:\ n\to k} Q_k \simeq q_n + Q_n$$

$$v_n \simeq v_{\pi_n} - 2r_nP_n - 2x_nQ_n$$

Voltage drop and line power flows are approximately ***linearly*** *related to power injections*

Comparison to Lecture 3

- Drop in squared voltage magnitudes from LDF $v_{\pi_n} - v_n \simeq 2r_n P_n + 2x_n Q_n$

- Drop in voltage magnitudes from chapter 3 $|V_{\pi_n}| - |V_n| \simeq \mathrm{Re}\{z_n I_n\}$

- How are these two approximations related?

- Consider first-order Taylor series expansion around $|V_0| = 1$ (in per unit wlog)

$$\begin{aligned} v_n &= |V_n|^2 \simeq |V_0|^2 + 2|V_0|\left(|V_n| - |V_0|\right) \\ &= 1 + 2(|V_n| - 1) = 2|V_n| - 1 \end{aligned} \quad \Longrightarrow \quad v_{\pi_n} - v_n \simeq 2\left(|V_{\pi_n}| - |V_n|\right)$$

$$r_n P_n + x_n Q_n = \mathrm{Re}\{z_n S_n^*\} = \mathrm{Re}\{z_n I_n V_{\pi_n}^*\} \simeq \mathrm{Re}\{z_n I_n\}$$

- Equivalent useful approximation $|V_{\pi_n}| - |V_n| \simeq r_n P_n + x_n Q_n$

LDF in compact form

- Express LDF in matrix-vector notation

$$\sum_{k:\ n\to k} \hat{P}_k \simeq p_n + \hat{P}_n$$
$$\sum_{k:\ n\to k} \hat{Q}_k \simeq q_n + \hat{Q}_n$$
$$\hat{v}_n \simeq \hat{v}_{\pi_n} - 2r_n\hat{P}_n - 2x_n\hat{Q}_n$$

$$\mathbf{D}_r = \mathrm{dg}(\mathbf{r}), \mathbf{D}_x = \mathrm{dg}(\mathbf{x})$$

$$\mathbf{p} = \mathbf{A}^\top \hat{\mathbf{P}}$$
$$\mathbf{q} = \mathbf{A}^\top \hat{\mathbf{Q}}$$
$$\mathbf{A}\hat{\mathbf{v}} + v_0\mathbf{a}_0 = 2\mathbf{D}_r\hat{\mathbf{P}} + 2\mathbf{D}_x\hat{\mathbf{Q}}$$

$$\begin{aligned}\hat{\mathbf{v}} &= v_0\mathbf{1} + 2\mathbf{F}\mathbf{D}_r\mathbf{F}^\top\mathbf{p} + 2\mathbf{F}\mathbf{D}_x\mathbf{F}^\top\mathbf{q}\\ &= v_0\mathbf{1} + 2\mathbf{R}\mathbf{p} + 2\mathbf{X}\mathbf{q}\end{aligned}$$

- Matrices $(\mathbf{R}, \mathbf{X})$ are *symmetric positive definite* and have *positive entries*

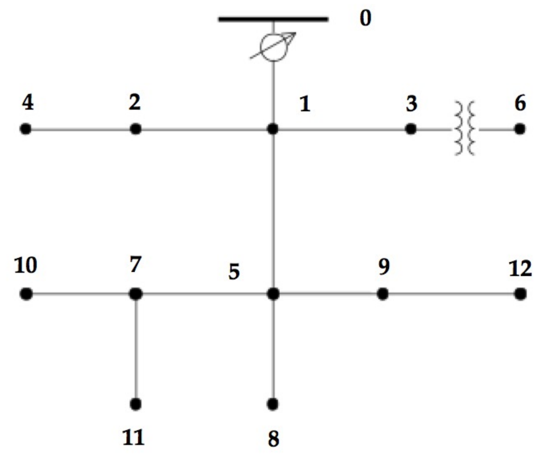
- Both matrices are almost equally important

DISTRIBUTION LINE RESISTANCE-TO-REACTANCE RATIOS

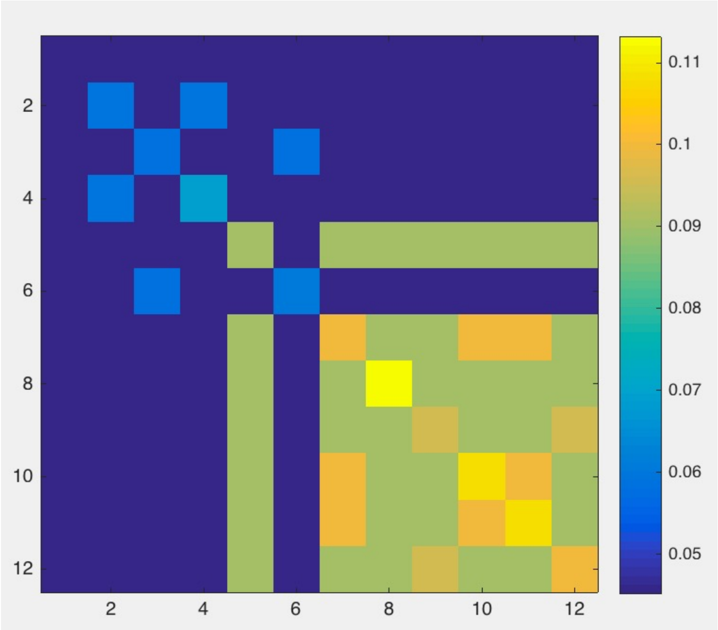
Feeder	$\alpha_{\min}$	$\alpha_{\max}$	mean	std	median
IEEE 34-bus	1.00	1.88	1.41	0.29	1.37
IEEE 37-bus	1.48	2.70	2.72	0.45	1.93
IEEE 123-bus	0.42	2.02	0.74	0.38	0.97

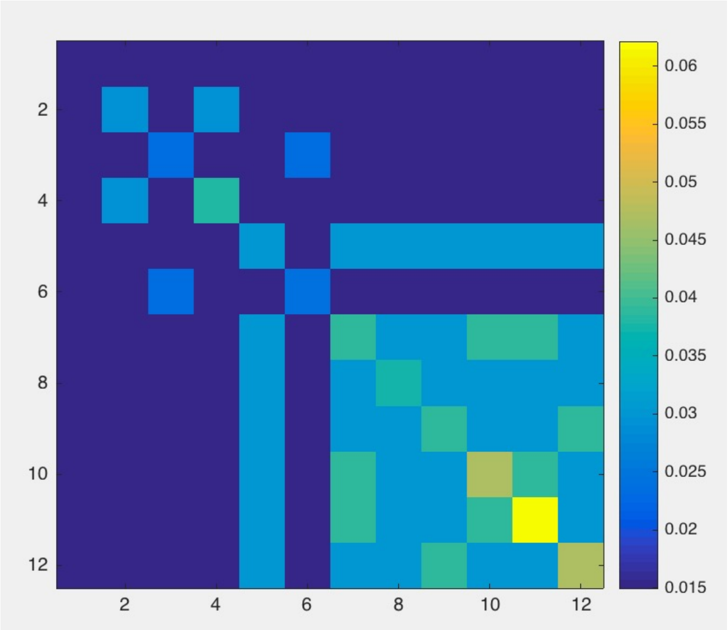
IEEE 13-bus feeder

- Assume transposed lines; average diagonal and off-diagonal entries; take positive-sequence impedance



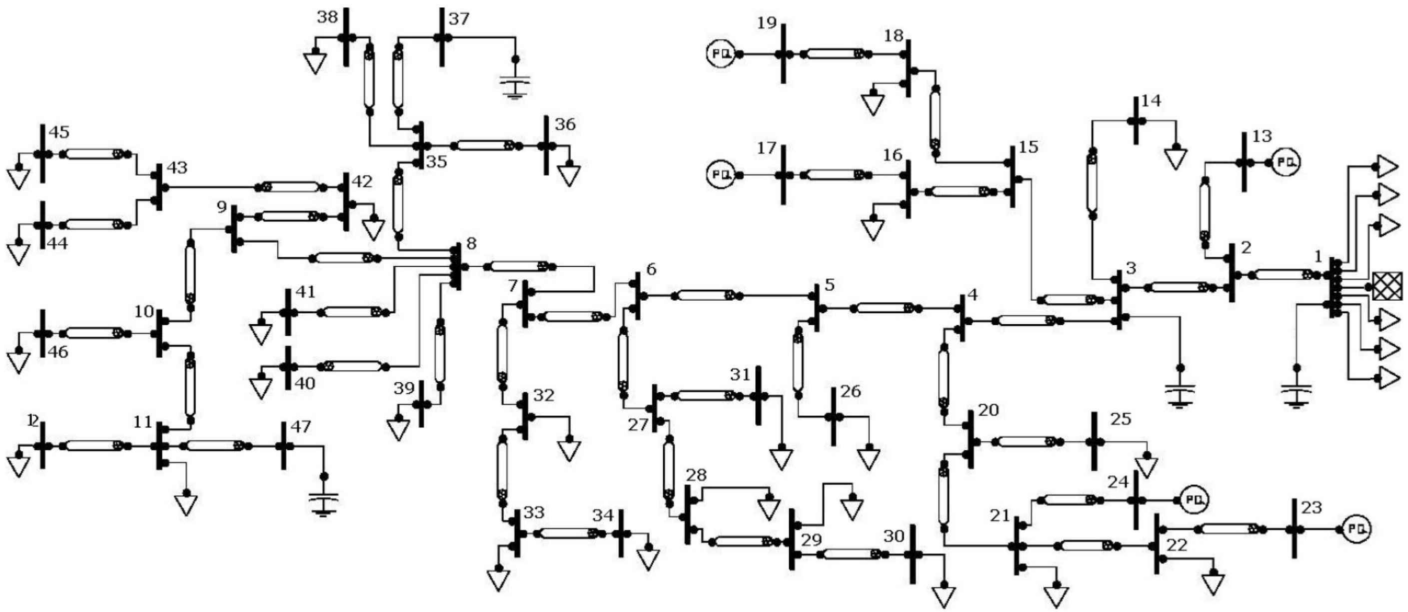
$$\hat{\mathbf{v}} = v_0\mathbf{1} + 2\mathbf{R}\mathbf{p} + 2\mathbf{X}\mathbf{q}$$



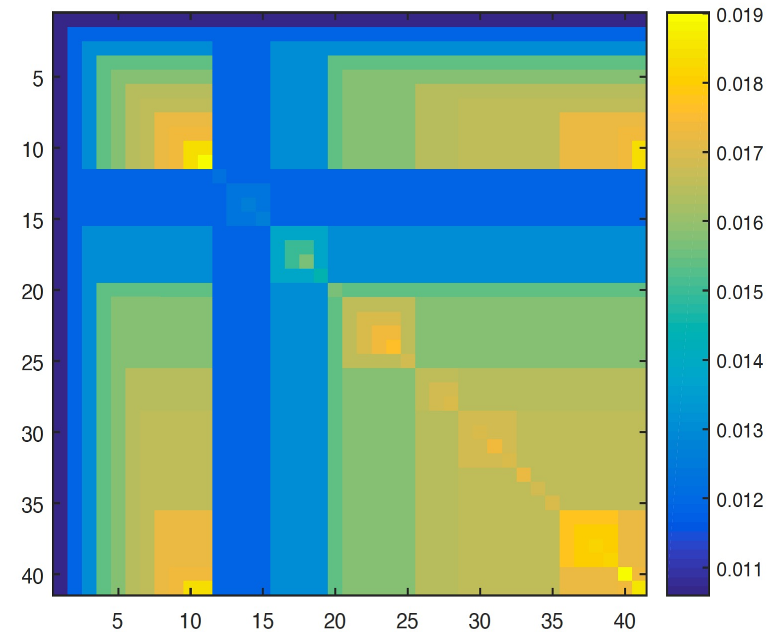


- To find entry $\mathbf{R}_{mn}$ connect buses *n* and *m* to the substation, and add the resistances of the common lines, e.g., $\mathbf{R}_{10,12} = r_{01} + r_{15}$

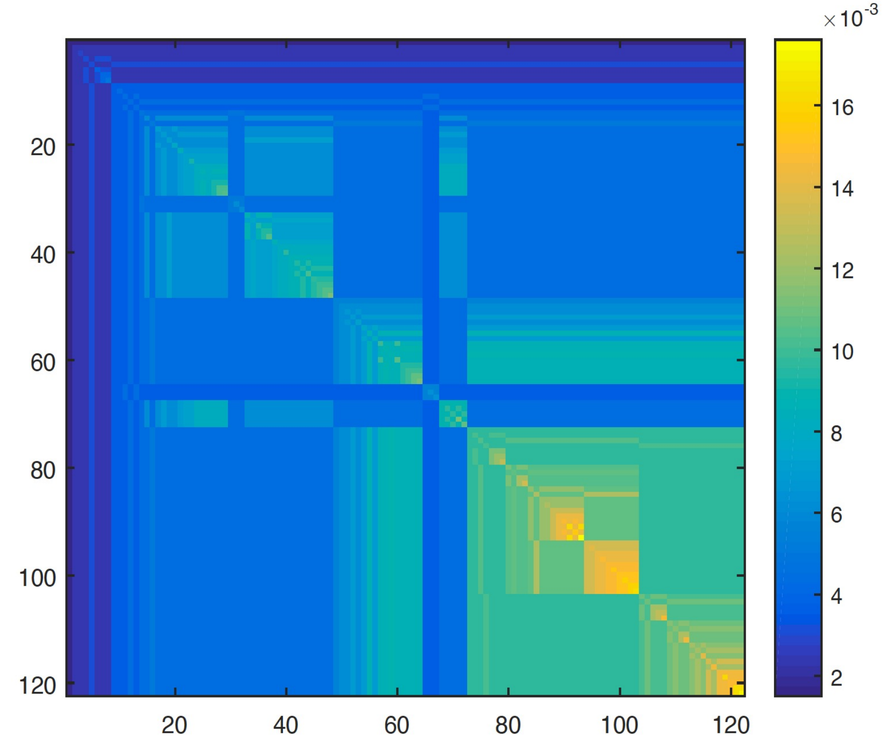
Southern California Edison 47-bus feeder



Matrix $\mathbf{x}$



IEEE 123-bus feeder



Matrix $\mathbf{X}$

LDF approximation error

- Express DistFlow in matrix-vector notation

$$\sum_{k:\ n\to k} P_k = p_n + P_n - r_n\ell_n$$
$$\sum_{k:\ n\to k} Q_k = q_n + Q_n - x_n\ell_n$$
$$v_n = v_{\pi_n} - 2r_nP_n - 2x_nQ_n + (r_n^2 + x_n^2)\ell_n$$

$$\mathbf{p} = \mathbf{A}^\top\mathbf{P} + \mathbf{D}_r\boldsymbol{\ell}$$
$$\mathbf{q} = \mathbf{A}^\top\mathbf{Q} + \mathbf{D}_x\boldsymbol{\ell}$$
$$\mathbf{A}\mathbf{v} + v_0\mathbf{a}_0 = 2\mathbf{D}_r\mathbf{P} + 2\mathbf{D}_x\mathbf{Q} - \left(\mathbf{D}_r^2 + \mathbf{D}_x^2\right)\boldsymbol{\ell}$$

- LDF gives an over-estimator for squared voltage magnitudes

$$\mathbf{v} = \hat{\mathbf{v}} + \underbrace{\mathbf{F}\mathbf{D}_r\left[-\mathbf{I} - 2\mathbf{F}^\top\right]\mathbf{D}_r\boldsymbol{\ell} + \mathbf{F}\mathbf{D}_x\left[-\mathbf{I} - 2\mathbf{F}^\top\right]\mathbf{D}_x\boldsymbol{\ell}}_{\leq \mathbf{0}} \leq \hat{\mathbf{v}}$$

- LDF gives an under-estimator for line flows $\mathbf{P} = \mathbf{F}^\top\mathbf{p} - \mathbf{F}^\top\mathbf{D}_r\boldsymbol{\ell} \geq \hat{\mathbf{P}}$

- Approximation accuracy depends on loading conditions

Linearized power flow models

- Recall *linearized* or so-termed *DC power flow model* in transmission systems $\mathbf{p} = \mathbf{B}\boldsymbol{\theta}$

- It has been derived under three approximations:
  1. Voltage magnitudes close to unity $|V_n| = 1 + \epsilon_n$ with $\epsilon_n \simeq 0$
  2. Voltage angle differences across lines close to zero $\theta_{nm} = \theta_n - \theta_m \simeq 0$
  3. Ignoring line resistances and shunt elements

- Repeat the same analysis for a *meshed grid* without the third assumption [6]
- Consider voltages $V_n = (1+\epsilon_n)e^{j\theta_n}$ and $V_m = (1+\epsilon_m)e^{j\theta_m}$ $\quad \mathbf{x} = \begin{bmatrix} \epsilon_n \\ \epsilon_m \\ \theta_{nm} \end{bmatrix}$

- Consider power flow from bus n to m: $S_{nm} = V_n(V_n^* - V_m^*)y_{nm}^* = f(\mathbf{x})y_{nm}^*$

$$\begin{aligned} f(\mathbf{x}) &= |V_n|^2 - |V_n||V_m|(\cos\theta_{nm} + j\sin\theta_{nm}) \\ &= (1+\epsilon_n)^2 - (1+\epsilon_n)(1+\epsilon_m)(\cos\theta_{nm} + j\sin\theta_{nm}) \end{aligned}$$

- First-order Taylor's series expansion $f(\mathbf{x}) \simeq f(\mathbf{0}) + (\nabla_{\mathbf{x}} f(\mathbf{0}))^\top(\mathbf{x} - \mathbf{0})$

- Observe that $f(\mathbf{0}) = 0$

Linearized power flow models (cont'd)

- Compute gradient of $f(\mathbf{x}) = (1+\epsilon_n)^2 - (1+\epsilon_n)(1+\epsilon_m)(\cos\theta_{nm} + j\sin\theta_{nm})$

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} 2(1+\epsilon_n) - (1+\epsilon_m)(\cos\theta_{nm} + j\sin\theta_{nm}) \\ -(1+\epsilon_n)(\cos\theta_{nm} + j\sin\theta_{nm}) \\ -(1+\epsilon_n)(1+\epsilon_m)(-\sin\theta_{nm} + j\cos\theta_{nm}) \end{bmatrix} \Rightarrow \nabla_{\mathbf{x}} f(\mathbf{0}) = \begin{bmatrix} +1 \\ -1 \\ -j \end{bmatrix}$$

- Linearization $\quad f(\mathbf{x}) \simeq 0 + [+1 \; -1 \; -j][\epsilon_n \; \epsilon_m \; \theta_{nm}]^\top$

- Therefore, power flow on line *(n,m)* can be linearized as

$$S_{nm} \simeq [(\epsilon_n - \epsilon_m) - j(\theta_n - \theta_m)]\,(g_{nm} + jb_{nm}) \Rightarrow \begin{aligned} P_{nm} &\simeq g_{nm}(\epsilon_n - \epsilon_m) + b_{nm}(\theta_n - \theta_m) \\ Q_{nm} &\simeq b_{nm}(\epsilon_n - \epsilon_m) - g_{nm}(\theta_n - \theta_m) \end{aligned}$$

- Stacking line power flows

$$\begin{aligned} \mathbf{P} &= \mathbf{D}_g\mathbf{A}\boldsymbol{\epsilon} + \mathbf{D}_b\mathbf{A}\boldsymbol{\theta} \\ \mathbf{Q} &= \mathbf{D}_b\mathbf{A}\boldsymbol{\epsilon} - \mathbf{D}_g\mathbf{A}\boldsymbol{\theta} \end{aligned}$$

- Converting to power injections

$$\begin{aligned} \mathbf{p} &= \mathbf{A}^\top\mathbf{P} = \mathbf{G}\boldsymbol{\epsilon} + \mathbf{B}\boldsymbol{\theta} \qquad & \mathbf{G} &:= \mathbf{A}^\top\mathbf{D}_g\mathbf{A} \\ \mathbf{q} &= \mathbf{A}^\top\mathbf{Q} = \mathbf{B}\boldsymbol{\epsilon} - \mathbf{G}\boldsymbol{\theta} \qquad & \mathbf{B} &:= \mathbf{A}^\top\mathbf{D}_b\mathbf{A} \end{aligned}$$

compare to 'DC' model for transmission grids

Linearized power flow models (cont'd)

- Solve equations wrt voltage magnitudes and angles

$$\boldsymbol{\epsilon} = \left(\mathbf{G} + \mathbf{B}\mathbf{G}^{-1}\mathbf{B}\right)^{-1}\mathbf{p} + \left(\mathbf{B} + \mathbf{G}\mathbf{B}^{-1}\mathbf{G}\right)^{-1}\mathbf{q}$$
$$\boldsymbol{\theta} = \left(\mathbf{B} + \mathbf{G}\mathbf{B}^{-1}\mathbf{G}\right)^{-1}\mathbf{p} - \left(\mathbf{G} + \mathbf{B}\mathbf{G}^{-1}\mathbf{B}\right)^{-1}\mathbf{q}$$

- Formula is general; holds even for meshed grids

- For radial grids (square and invertible $\mathbf{A}$), equations simplify to

$$\boldsymbol{\epsilon} = \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q}$$
$$\boldsymbol{\theta} = \mathbf{X}\mathbf{p} - \mathbf{R}\mathbf{q}$$

- Compare to LDF; linear approximation for voltage angles too
- Linearization conducted at flat voltage profile
- Another reference state can be used; but (**R**,**X**,**B**,**G**) will depend on that state

Power flow via convex relaxation

- Instead of the BF solver, solve the PF problem as a minimization [3]-[4]

$$
\begin{aligned}
\min \quad & \sum_{n=1}^{N} r_n \ell_n \\
\text{over} \quad & \mathbf{P}, \mathbf{Q}, \mathbf{v}, \boldsymbol{\ell}, p_0, q_0 \\
\text{s.t.} \quad & \sum_{k:\ n \to k} P_k = p_n + P_n - r_n \ell_n \\
& \sum_{k:\ n \to k} Q_k = q_n + Q_n - x_n \ell_n \\
& v_n = v_{\pi_n} - 2 r_n P_n - 2 x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
& \ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \qquad \Longrightarrow \qquad \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n
\end{aligned}
$$

- Non-convex constraint relaxed to second-order cone constraints (SOC)

$$
\left\| \begin{bmatrix} 2P_n \\ 2Q_n \\ \ell_n - v_{\pi_n} \end{bmatrix} \right\|_2 \leq \ell_n + v_{\pi_n}
$$

- It can be solved efficiently as a second-order cone program (SOCP)

- Oftentimes, the relaxation is exact: SOC are satisfied with equality

Optimal power flow via convex relaxation

- OPF has to be solved to perform any meaningful grid optimization task

    1. power loss minimization
    2. voltage regulation
    3. conservation voltage reduction
    4. demand response
    5. electric vehicle charging
    6. optimal coordination of energy storage

- Power injections $\mathbf{s}$ become *control variables* rather than *fixed (inelastic load)*

- Optimally control devices while satisfying the PF equations and network constraints

Optimal power flow via convex relaxation

- Solving OPF in single-phase radial grids through via an SOCP [3]

$$\begin{aligned}
\min \quad & \sum_{n=1}^{N} r_n \ell_n + \sum_{n=1}^{N} c_n p_n^g + \sum_{n=1}^{N} \alpha_n v_n \\
\text{over} \quad & \mathbf{P}, \mathbf{Q}, \mathbf{v}, \boldsymbol{\ell}, p_0, q_0, \color{red}{\mathbf{s}} \\
\text{s.t.} \quad & \sum_{k:\ n \to k} P_k = p_n + P_n - r_n \ell_n \\
& \sum_{k:\ n \to k} Q_k = q_n + Q_n - x_n \ell_n \\
& v_n = v_{\pi_n} - 2 r_n P_n - 2 x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
& \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \color{red}{\leq} \ell_n
\end{aligned}$$

relaxed BFM equations

$$\begin{aligned}
& \mathbf{p} = \mathbf{p}^g - \mathbf{p}^c \\
& \mathbf{q} = \mathbf{q}^g - \mathbf{q}^c \\
& \underline{p}_n^g \leq p_n^g \leq \overline{p}_n^g, \ \forall n \\
& \underline{q}_n^g \leq q_n^g \leq \overline{q}_n^g, \ \forall n \\
& (p_n^g)^2 + (q_n^g)^2 \leq \overline{s}_n^g, \ \forall n
\end{aligned}$$

injection constraints

$$\begin{aligned}
& \underline{v}_n \leq v_n \leq \overline{v}_n, \ \forall n \\
& \ell_n \leq \overline{\ell}_n, \ \forall n
\end{aligned}$$

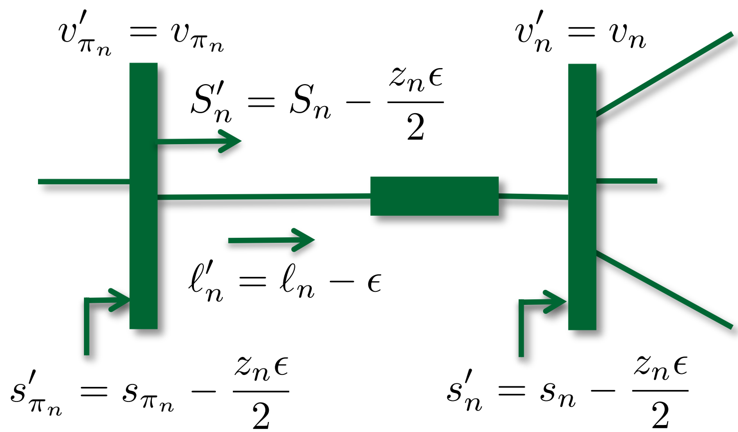
network constraints

- Oftentimes, the relaxation is exact: SOCs are satisfied with equality

Exactness under load over-satisfaction

Theorem ([3]): If power injections are unbounded below, the relaxation is exact

- Assume problem has been solved, but SOC for line *n* is inexact $P_n^2 + Q_n^2 < \ell_n v_{\pi_n}$



injections can be reduced without bound

- Given current solution $(\mathbf{S}, \mathbf{s}, \mathbf{v}, \boldsymbol{\ell}, s_0)$, construct another point $(\mathbf{S}', \mathbf{s}', \mathbf{v}', \boldsymbol{\ell}', s'_0)$ by changing only the quantities related to line *n* as shown above

- Show that new point is feasible; satisfies SOC with equality; and yields lower cost!

$$\sum_{k:\ n \to k} P'_k = p'_n + P'_n - r_n \ell'_n$$

$$\sum_{k:\ n \to k} Q'_k = q'_n + Q'_n - x_n \ell'_n$$

$$v_n = v_{\pi_n} - 2 r_n P'_n - 2 x_n Q'_n + (r_n^2 + x_n^2) \ell'_n$$

$$\ell'_n = \frac{(P'_n)^2 + (Q'_n)^2}{v_{\pi_n}}$$

Exactness of SOCP convex relaxation

- Exactness of SOCP relaxation for OPF in radial grids has been studied extensively [6]

- Different sets of sufficient conditions have been derived:
  - no reverse power flows
  - identical r/x ratios for all lines
  - r/x increase downstream and there are no reverse active power flows
  - r/x decrease downstream and there are no reverse reactive power flows

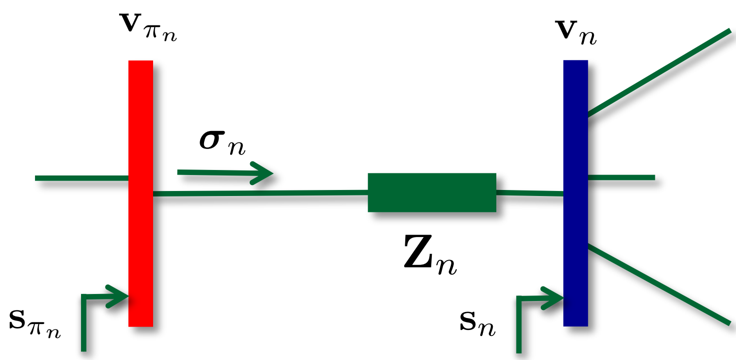
- If the SOCP is exact, the minimizer is *unique*

- To make BFM exact for meshed grids, add phase shifters to implement angle differences [3]

- Otherwise, one can use a semidefinite program relaxation based on the bus injection model (BIM) [4]

- *How do these schemes extend to multiphase grids? [7]*

Multiphase branch flow model



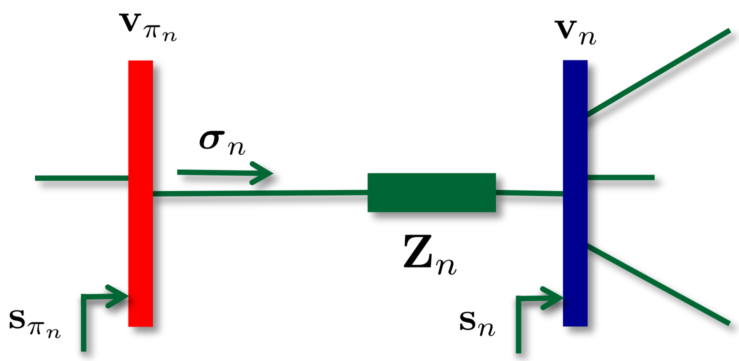
from scalars to vectors and matrices

$$V_{\pi_n} - V_n = z_n I_n$$
$$S_n = V_{\pi_n} I_n^*$$
$$S_n - z_n |I_n|^2 + s_n = \sum_{k:\ n \to k} S_k$$

$$\mathbf{v}_{\pi_n} - \mathbf{v}_n = \mathbf{Z}_n \mathbf{i}_n$$
$$\mathbf{S}_n = \mathbf{v}_{\pi_n} \mathbf{i}_n^H \qquad \text{matrix variable?}$$
$$\mathrm{dg}\left(\mathbf{S}_n - \mathbf{Z}_n \mathbf{i}_n \mathbf{i}_n^H\right) + \mathbf{s}_n = \sum_{k:\ n \to k} \mathrm{dg}\left(\mathbf{S}_k\right)$$

- Power received at node *n* $\quad \mathrm{dg}\left(\mathbf{v}_n \mathbf{i}_n^H\right) = \mathrm{dg}\left[(\mathbf{v}_{\pi_n} - \mathbf{Z}_n \mathbf{i}_n)\mathbf{i}_n^H\right]$
- Actual power sent from parent bus $\quad \boldsymbol{\sigma}_n = \mathrm{dg}(\mathbf{S}_n)$

Relaxed multiphase BFM



- 'Square' (multiply by conjugate transpose) the voltage drop equation

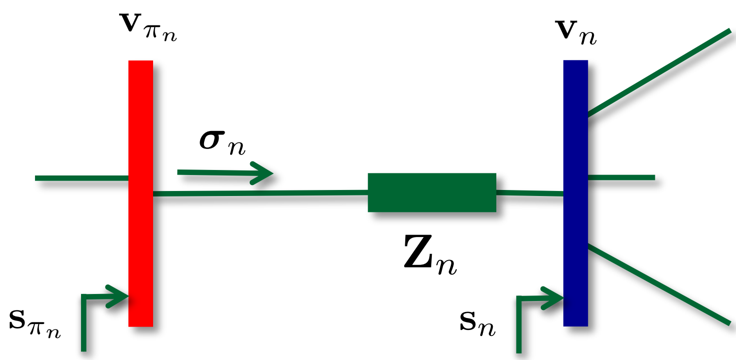
$$\mathbf{v}_n = \mathbf{v}_{\pi_n} - \mathbf{Z}_n \mathbf{i}_n \quad \Longrightarrow \quad \mathbf{v}_n \mathbf{v}_n^H = \mathbf{v}_{\pi_n} \mathbf{v}_{\pi_n}^H + \mathbf{Z}_n \mathbf{i}_n \mathbf{i}_n^H \mathbf{Z}_n^H - \mathbf{v}_{\pi_n} \mathbf{i}_n^H \mathbf{Z}_n^H - \mathbf{Z}_n \mathbf{i}_n \mathbf{v}_{\pi_n}^H$$

- Define 'squared' voltages and currents $\quad \mathbf{V}_n = \mathbf{v}_n \mathbf{v}_n^H \qquad \mathbf{L}_n = \mathbf{i}_n \mathbf{i}_n^H$

- Express 'squared' voltage drop as $\boxed{\mathbf{V}_n = \mathbf{V}_{\pi_n} + \mathbf{Z}_n \mathbf{L}_n \mathbf{Z}_n^H - \mathbf{S}_n \mathbf{Z}_n^H - \mathbf{Z}_n \mathbf{S}_n^H}$

- Linear equation; but complexity is hidden under 'squared' variables $(\mathbf{V}_n, \mathbf{L}_n, \mathbf{S}_n)$

Relaxed multiphase BFM (cont'd)



- In *single-phase* grids $S_n = V_{\pi_n} I_n^* \xrightarrow{\text{'square'}} |S_n|^2 = v_{\pi_n}\ell_n \xrightarrow{\text{relax}} |S_n|^2 \leq v_{\pi_n}\ell_n$

- Relaxation can be also written $\begin{bmatrix} v_{\pi_n} & S_n \\ S_n^* & \ell_n \end{bmatrix} = \begin{bmatrix} V_{\pi_n} \\ I_n \end{bmatrix} \begin{bmatrix} V_{\pi_n} \\ I_n \end{bmatrix}^H \succeq \mathbf{0}$ and ~~rank 1~~

- In *multi-phase* grids, the relaxation becomes

$$\boxed{\begin{bmatrix} \mathbf{V}_{\pi_n} & \mathbf{S}_n \\ \mathbf{S}_n^* & \mathbf{L}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\pi_n} \\ \mathbf{i}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\pi_n} \\ \mathbf{i}_n \end{bmatrix}^H \succeq \mathbf{0}}$$ and ~~rank-1~~

- Semidefinite program (SDP) constraint captures all quadratic relationships

OPF with multiphase BFM

$$\begin{aligned}
\min \quad & \text{losses and/or CVR and/or generation cost} \\
\text{over} \quad & \{\mathbf{S}_n, \mathbf{s}_n, \mathbf{V}_n, \mathbf{L}_n\}_n
\end{aligned}$$

s.t.

$$\mathrm{dg}\left(\mathbf{S}_n - \mathbf{Z}_n \mathbf{L}_n\right) + \mathbf{s}_n = \sum_{k:\ n \to k} \mathrm{dg}\left(\mathbf{S}_k\right)$$

$$\mathbf{V}_n = \mathbf{V}_{\pi_n} + \mathbf{Z}_n \mathbf{L}_n \mathbf{Z}_n^H - \mathbf{S}_n \mathbf{Z}_n^H - \mathbf{Z}_n \mathbf{S}_n^H$$

$$\begin{bmatrix} \mathbf{V}_{\pi_n} & \mathbf{S}_n \\ \mathbf{S}_n^* & \mathbf{L}_n \end{bmatrix} \succeq \mathbf{0}$$

relaxed BFM equations

injection constraints

$$\mathbf{p} = \mathbf{p}^g - \mathbf{p}^c$$

$$\mathbf{q} = \mathbf{q}^g - \mathbf{q}^c$$

$$\underline{p}^g_{n,\phi} \leq p^g_{n,\phi} \leq \overline{p}^g_{n,\phi}, \ \forall n, \phi$$

$$\underline{q}^g_{n,\phi} \leq q^g_{n,\phi} \leq \overline{q}^g_{n,\phi}, \ \forall n, \phi$$

$$(p^g_{n,\phi})^2 + (q^g_{n,\phi})^2 \leq \overline{s}^g_{n,\phi}, \ \forall n, \phi$$

network constraints

$$\underline{v} \leq \mathrm{dg}(\mathbf{V}_n) \leq \overline{v}, \ \forall n$$

$$\mathrm{dg}(\mathbf{L}_n) \leq \overline{\ell}, \ \forall n$$

- Relaxation is exact (constraint satisfied with equality) under practical conditions

Linear approximation for multiphase grids

- Ignore losses to get approximate power conservation

$$\mathrm{dg}\left(\mathbf{S}_n - \cancel{\mathbf{Z}_n\mathbf{L}_n}\right) + \mathbf{s}_n = \sum_{k:\ n\to k} \mathrm{dg}\left(\mathbf{S}_k\right) \quad \Rightarrow \quad \boxed{\boldsymbol{\sigma}_n + \mathbf{s}_n = \sum_{k:\ n\to k} \boldsymbol{\sigma}_k}$$

- Voltage drop requires approximating the full matrix $\mathbf{S}_n$

$$\mathbf{V}_n = \mathbf{V}_{\pi_n} + \cancel{\mathbf{Z}_n\mathbf{L}_n}\mathbf{Z}_n^H - \mathbf{S}_n\mathbf{Z}_n^H - \mathbf{Z}_n\mathbf{S}_n^H$$

- Assuming approximately balanced voltages (and currents)

$$\mathbf{v}_{\pi_n} \simeq V_{\pi_n}\boldsymbol{\alpha},\ \mathbf{i}_n \simeq I_n\boldsymbol{\alpha} \qquad \boldsymbol{\alpha} = \begin{bmatrix} 1 \\ \alpha^* \\ \alpha \end{bmatrix},\ \alpha = e^{j2\pi/3}$$

- Power flow matrix can be approximated as $\mathbf{S}_n = \boldsymbol{\alpha}\boldsymbol{\alpha}^H\mathrm{dg}(\boldsymbol{\sigma}_n)$

$$\boxed{\mathrm{dg}(\mathbf{V}_n) = \mathrm{dg}(\mathbf{V}_{\pi_n}) - \mathrm{dg}(\boldsymbol{\alpha}\boldsymbol{\alpha}^H\mathrm{dg}(\boldsymbol{\sigma}_n)\mathbf{Z}_n^H) - \mathrm{dg}(\mathbf{Z}_n\mathrm{dg}(\boldsymbol{\sigma}_n)^*\boldsymbol{\alpha}\boldsymbol{\alpha}^H)}$$

Inter-phase coupling

- Simplify approximate voltage drop using the property

$$\mathrm{dg}\left(\mathbf{A}\mathrm{dg}(\mathbf{x})\mathbf{B}\right) = (\mathbf{A} \odot \mathbf{B}^\top)\mathbf{x}, \quad \odot : \text{entry-wise (Hadamard) product}$$

- Approximate voltage drop

$$\boxed{\mathbf{v}_{\pi_n} - \mathbf{v}_n \simeq 2\mathrm{Re}\left\{\bar{\mathbf{Z}}_n \boldsymbol{\sigma}_n^*\right\}, \quad \text{where} \quad \bar{\mathbf{Z}}_n = \mathbf{Z}_n \odot \boldsymbol{\alpha}^* \boldsymbol{\alpha}^\top}$$

$$\mathbf{Z}_n = \begin{bmatrix} 0.530 + 1.112i & 0.127 + 0.404i & 0.126 + 0.423i \\ 0.127 + 0.404i & 0.545 + 1.043i & 0.133 + 0.374i \\ 0.126 + 0.423i & 0.133 + 0.374i & 0.542 + 1.056i \end{bmatrix}$$

HW2-Exercise 1

$$\bar{\mathbf{Z}}_n = \begin{bmatrix} 0.530 + 1.112i & 0.286 - 0.312i & -0.430 - 0.103i \\ -0.413 - 0.092i & 0.545 + 1.0429i & 0.258 - 0.303i \\ 0.304 - 0.321i & -0.391 - 0.072i & 0.542 + 1.056i \end{bmatrix}$$

- How do complex injections affect voltage drops?

$$\mathrm{sign}\left[\mathrm{Re}\left\{\bar{\mathbf{Z}}_n\right\}\right] = \begin{bmatrix} + & + & - \\ - & + & + \\ + & - & + \end{bmatrix} \qquad \mathrm{sign}\left[\mathrm{Im}\left\{\bar{\mathbf{Z}}_n\right\}\right] = \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}$$

See Section IV of [8] for an analysis of these patterns

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