ECE 595: Power Distribution System Analysis

Lecture 11: DistFlow and LinDistFlow

Reference: see publications list at the end

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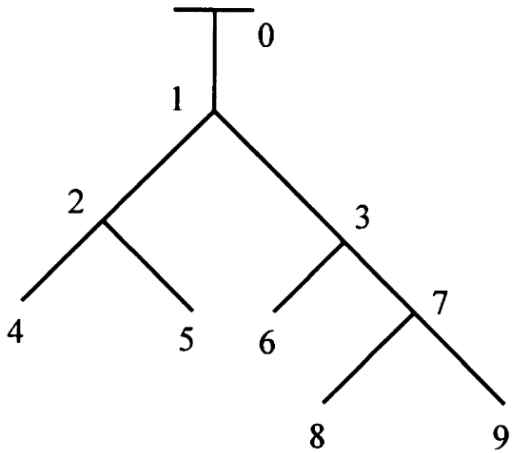
Outline

1. Branch flow model (BFM)
2. DistFlow model
3. DistFlow model for power flow
4. DistFlow model for optimal power flow
5. LinDistFlow model for approximate analysis

Branch-bus incidence matrix

- *Single-phase* and *radial* feeder represented by tree graph

  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with $|\mathcal{N}| = N + 1$ and $|\mathcal{E}| = L = N$



$|\mathcal{N}| = 10$ and $|\mathcal{E}| = L = N = 9$

breadth-first
(vs. depth-first)
numbering

- Branch-bus incidence matrix

buses

lines

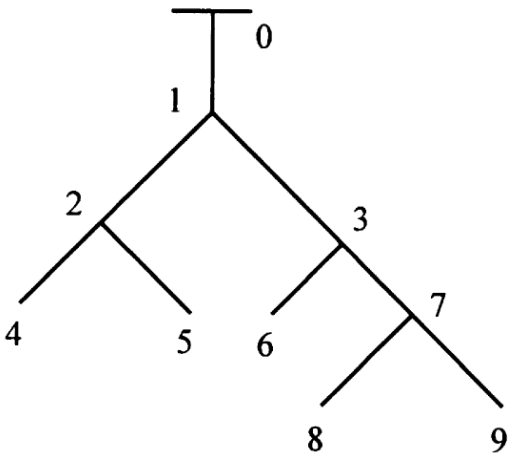
$$\tilde{\mathbf{A}} = \left[\begin{array}{c|ccccccccc}
1 & -1 & & & & & & & & \\
 & 1 & -1 & & & & & & & \\
 & 1 & & -1 & & & & & & \\
 & & 1 & & -1 & & & & & \\
 & & 1 & & & -1 & & & & \\
 & & & 1 & & & -1 & & & \\
 & & & 1 & & & & -1 & & \\
 & & & & & & & 1 & -1 & \\
 & & & & & & & 1 & & -1
\end{array}\right]$$

reduced branch-bus incidence matrix

$$\tilde{\mathbf{A}} = [\mathbf{a}_0 \;\; \mathbf{A}]$$

- The line feeding bus n is indexed as line n-th

Branch-bus incidence matrix (cont'd)



reduced branch-bus incidence matrix

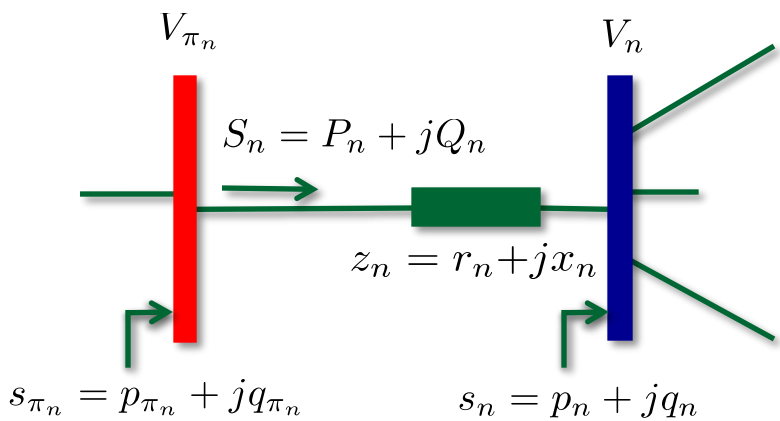
$$\mathbf{A} = \begin{bmatrix} -1 & & & & & & & & \\ 1 & -1 & & & & & & & \\ 1 & & -1 & & & & & & \\ & 1 & & -1 & & & & & \\ & 1 & & & -1 & & & & \\ & & 1 & & & -1 & & & \\ & & 1 & & & & -1 & & \\ & & & & & & 1 & -1 & \\ & & & & & & 1 & & -1 \end{bmatrix}$$

- Branch-bus incidence matrix $\tilde{\mathbf{A}} = [\mathbf{a}_0 \;\; \mathbf{A}]$

$$\begin{aligned} \tilde{\mathbf{A}}\mathbf{1}_{N+1} &= \mathbf{0} \Rightarrow \\ \mathbf{a}_0 + \mathbf{A}\mathbf{1}_N &= \mathbf{0} \\ \mathbf{1}_N &= -\mathbf{A}^{-1}\mathbf{a}_0 \end{aligned}$$

$$\mathbf{F} = \mathbf{A}^{-1} = \begin{vmatrix} -1 & & & & & & & & \\ -1 & -1 & & & & & & & \\ -1 & & -1 & & & & & & \\ -1 & -1 & & -1 & & & & & \\ -1 & -1 & & & -1 & & & & \\ -1 & & -1 & & & -1 & & & \\ -1 & & -1 & & & & -1 & & \\ -1 & & -1 & & & & -1 & -1 & \\ -1 & & -1 & & & & -1 & & -1 \end{vmatrix}$$

Branch flow model (BFM)



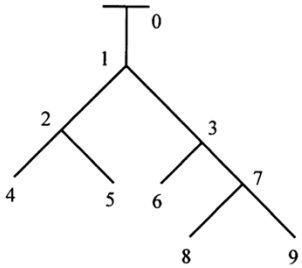
Line n feeding bus n from its parent bus π_n

- Branch flow equations on $\mathbf{x}(\mathbf{s}) = (\mathbf{S}, \mathbf{I}, \mathbf{V}, s_0)$

$$V_{\pi_n} - V_n = z_n I_n \qquad \Rightarrow \qquad V_n I_n^* = (V_{\pi_n} - z_n I_n)\, I_n^*$$

$$S_n = V_{\pi_n} I_n^*$$

$$S_n - z_n |I_n|^2 + s_n = \sum_{k:\ n \to k} S_k$$



- Boundary conditions?

- Given **s**, solve *2L+N+1* equations in *2L+N+1* complex unknowns [3]

- Equivalent with typical bus injection model (BIM); a.k.a. power flow equations

Branch flow model *squared*

- Introduce squared voltage and current magnitudes

$$v_n = |V_n|^2 \quad \text{and} \quad \ell_n = |I_n|^2$$

- Rearrange power injection equations

$$S_n - z_n|I_n|^2 + s_n = \sum_{k:\ n\to k} S_k \quad \Rightarrow \quad \begin{aligned} \sum_{k:\ n\to k} P_k &= p_n + P_n - r_n\ell_n \\ \sum_{k:\ n\to k} Q_k &= q_n + Q_n - x_n\ell_n \end{aligned}$$

- Ohm's law squared (multiply both sides by complex conjugate)

$$V_{\pi_n} - V_n = z_n I_n \quad \Rightarrow \quad \begin{aligned} V_n &= V_{\pi_n} - z_n I_n \Rightarrow \\ V_n V_n^* &= (V_{\pi_n} - z_n I_n)(V_{\pi_n} - z_n I_n)^* \Rightarrow \\ v_n &= v_{\pi_n} - 2\mathrm{Re}\left[z_n^* V_{\pi_n} I_n^*\right] + |z_n|^2 \ell_n \Rightarrow \\ v_n &= v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2)\ell_n \end{aligned}$$

- Definition of complex power flow squared

$$S_n = V_{\pi_n} I_n^* \quad \Rightarrow \quad \ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}$$

Relaxed branch flow model

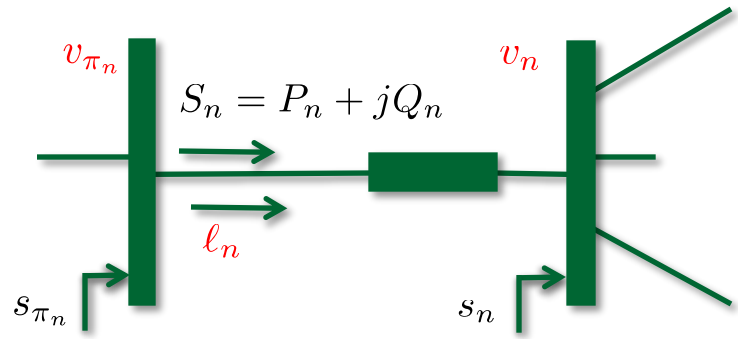
- Relaxed BFM on $\mathbf{y}(\mathbf{s}) := (\mathbf{S}, \boldsymbol{\ell}, \mathbf{v}, p_0, q_0)$

$$\sum_{k:\ n\to k} P_k = p_n + P_n - r_n \ell_n$$

$$\sum_{k:\ n\to k} Q_k = q_n + Q_n - x_n \ell_n$$

$$v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2)\ell_n$$

$$\ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}$$

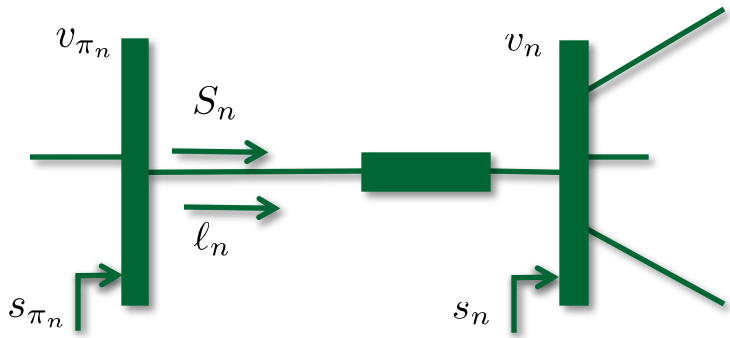


$$v_n = |V_n|^2 \quad \text{and} \quad \ell_n = |I_n|^2$$

current and voltage phase angles have been dropped!

- Boundary conditions?
- Current mags. can be eliminated; equations remain nonlinear
- Given **s**, solve *2(L+N+1)* equations in *3L+N+2* real unknowns [1]-[2]
- In radial grids, we get *4N+2* equations in *4N+2* real unknowns
- Unique solution for practical networks

Recovering phases

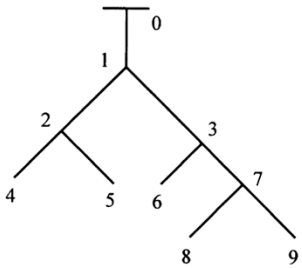


- After the relaxed branch flow equations have been solved [3]

- Recover voltage phases

$$
\begin{aligned}
V_{\pi_n} - V_n &= z_n I_n \Rightarrow \\
V^*_{\pi_n} - V^*_n &= z^*_n I^*_n \Rightarrow \\
V^*_n &= V^*_{\pi_n} - z^*_n I^*_n \Rightarrow \\
V_{\pi_n} V^*_n &= v_{\pi_n} - z^*_n S_n \Rightarrow \\
\theta_{\pi_n} - \theta_n &= \angle(v_{\pi_n} - z^*_n S_n)
\end{aligned}
$$

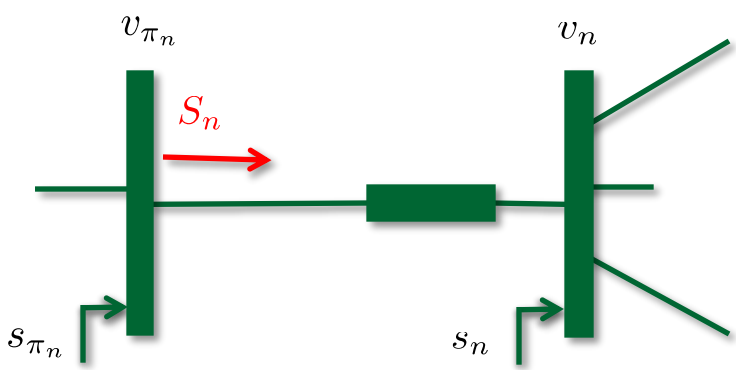
linear system can be inverted only when L=N



- Recover current phasors $\quad I_n = \left(\frac{S_n}{V_{\pi_n}}\right)^*$

Linearized distribution flow (LinDistFlow)

- Approximate model to overcome the complexity of quadratic equations [1]-[2]
- Derived from DistFlow model upon dropping terms related to losses



DistFlow

$$\sum_{k:\ n\to k} P_k = p_n + P_n - r_n \ell_n$$

$$\sum_{k:\ n\to k} Q_k = q_n + Q_n - x_n \ell_n$$

$$v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2)\ell_n$$

$$\ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}$$

LinDistFlow

$$\sum_{k:\ n\to k} P_k \simeq p_n + P_n$$

$$\sum_{k:\ n\to k} Q_k \simeq q_n + Q_n$$

$$v_n \simeq v_{\pi_n} - 2r_n P_n - 2x_n Q_n$$

*Voltage drop and line power flows are approximately **linearly** related to power injections*

Comparison to Lecture 3

- Drop in squared voltage magnitudes from LDF $v_{\pi_n} - v_n \simeq 2r_n P_n + 2x_n Q_n$

- Drop in voltage magnitudes from chapter 3 $|V_{\pi_n}| - |V_n| \simeq \mathrm{Re}\{z_n I_n\}$

- How are these two approximations related?

- Consider first-order Taylor series expansion around $|V_0| = 1$ (in per unit wlog)

$$\begin{aligned} v_n = |V_n|^2 &\simeq |V_0|^2 + 2|V_0|\left(|V_n| - |V_0|\right) \\ &= 1 + 2(|V_n| - 1) = 2|V_n| - 1 \end{aligned} \quad \Rightarrow \quad v_{\pi_n} - v_n \simeq 2\left(|V_{\pi_n}| - |V_n|\right)$$

$$r_n P_n + x_n Q_n = \mathrm{Re}\{z_n S_n^*\} = \mathrm{Re}\{z_n I_n V_{\pi_n}^*\} \simeq \mathrm{Re}\{z_n I_n\}$$

- Equivalent useful approximation $|V_{\pi_n}| - |V_n| \simeq r_n P_n + x_n Q_n$

LDF in compact form

- Express LDF in matrix-vector notation

$$\sum_{k:\ n\to k} \hat{P}_k \simeq p_n + \hat{P}_n$$

$$\sum_{k:\ n\to k} \hat{Q}_k \simeq q_n + \hat{Q}_n$$

$$\hat{v}_n \simeq \hat{v}_{\pi_n} - 2r_n\hat{P}_n - 2x_n\hat{Q}_n$$

$$\mathbf{D}_r = \mathrm{dg}(\mathbf{r}), \mathbf{D}_x = \mathrm{dg}(\mathbf{x})$$

$$\mathbf{p} = \mathbf{A}^\top \hat{\mathbf{P}}$$

$$\mathbf{q} = \mathbf{A}^\top \hat{\mathbf{Q}}$$

$$[\mathbf{a}_0\ \mathbf{A}] \begin{bmatrix} v_0 \\ \hat{\mathbf{v}} \end{bmatrix} = \mathbf{A}\hat{\mathbf{v}} + v_0\mathbf{a}_0 = 2\mathbf{D}_r\hat{\mathbf{P}} + 2\mathbf{D}_x\hat{\mathbf{Q}}$$

$$\begin{aligned} \hat{\mathbf{v}} &= 2\mathbf{A}^{-1}\mathbf{D}_r\mathbf{A}^{-1}\mathbf{p} + 2\mathbf{A}^{-1}\mathbf{D}_x\mathbf{A}^{-1}\mathbf{q} - v_0\mathbf{A}^{-1}\mathbf{a}_0 \\ &= 2\mathbf{F}\mathbf{D}_r\mathbf{F}^\top\mathbf{p} + 2\mathbf{F}\mathbf{D}_x\mathbf{F}^\top\mathbf{q} + v_0\mathbf{1} \\ &= 2\mathbf{R}\mathbf{p} + 2\mathbf{X}\mathbf{q} + v_0\mathbf{1} \end{aligned}$$

$$\mathbf{1} = -\mathbf{A}^{-1}\mathbf{a}_0$$

$$\mathbf{F} = \mathbf{A}^{-1}$$

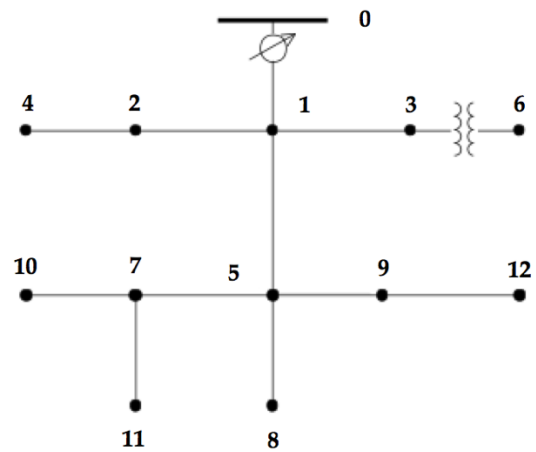
- Matrices $(\mathbf{R}, \mathbf{X})$ are *symmetric positive definite* and have *positive entries*

- Both matrices are almost equally important

DISTRIBUTION LINE RESISTANCE-TO-REACTANCE RATIOS

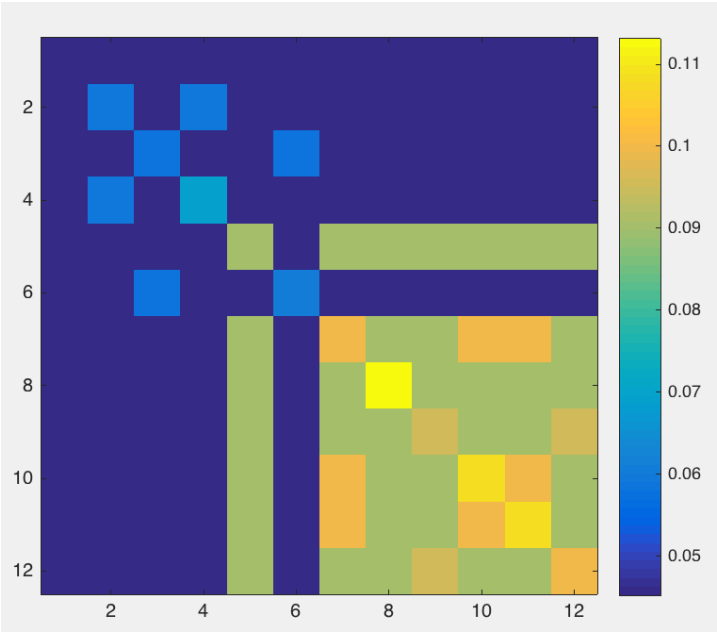
Feeder	$\alpha_{\min}$	$\alpha_{\max}$	mean	std	median
IEEE 34-bus	1.00	1.88	1.41	0.29	1.37
IEEE 37-bus	1.48	2.70	2.72	0.45	1.93
IEEE 123-bus	0.42	2.02	0.74	0.38	0.97

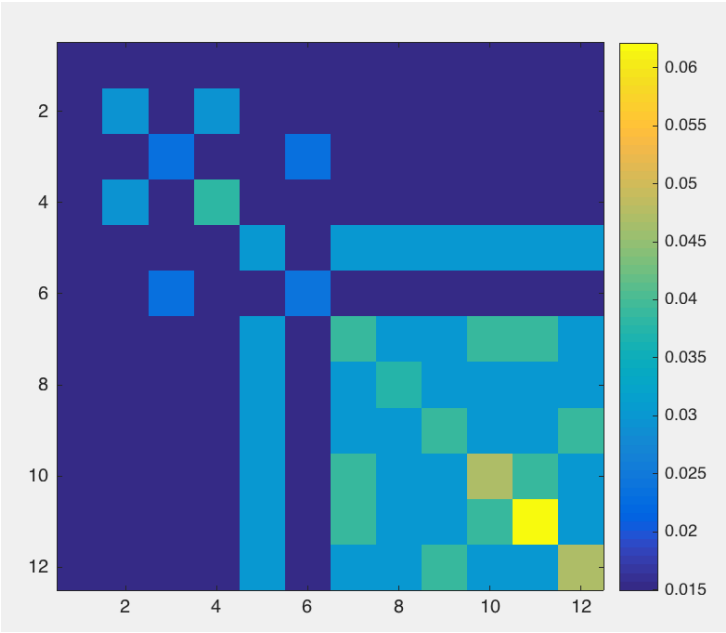
IEEE 13-bus feeder



- Assume transposed lines; average diagonal and off-diagonal entries; take positive-sequence impedance

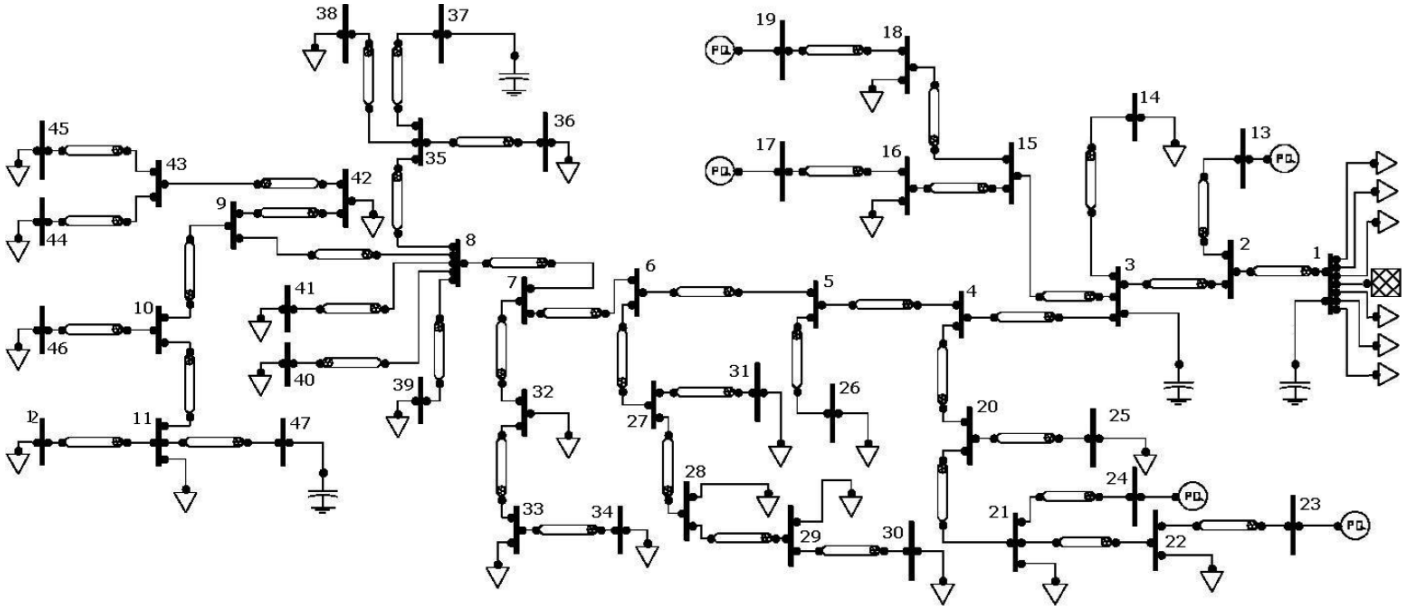
$$\hat{\mathbf{v}} = v_0\mathbf{1} + 2\mathbf{R}\mathbf{p} + 2\mathbf{X}\mathbf{q}$$



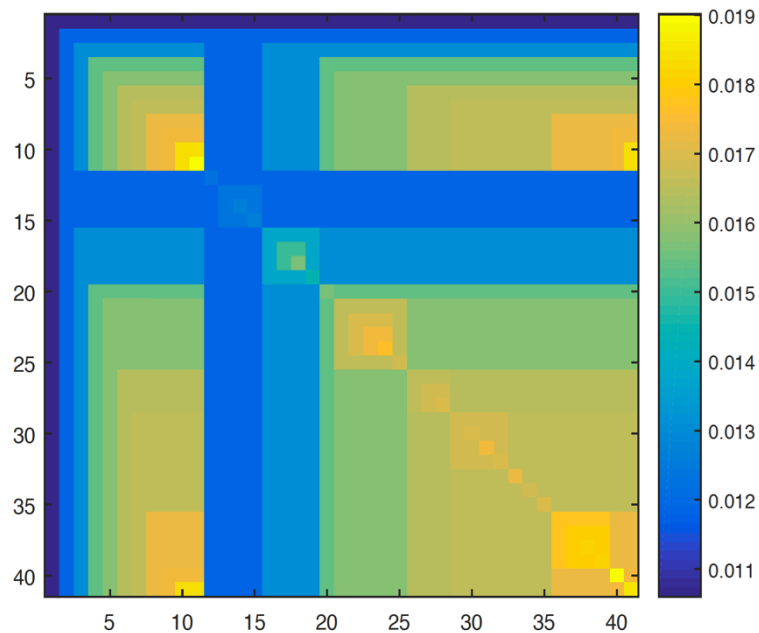


- To find entry $\mathbf{R}_{mn}$ connect buses n and m to the substation, and add the resistances of the common lines, e.g., $\mathbf{R}_{10,12} = r_{01} + r_{15}$

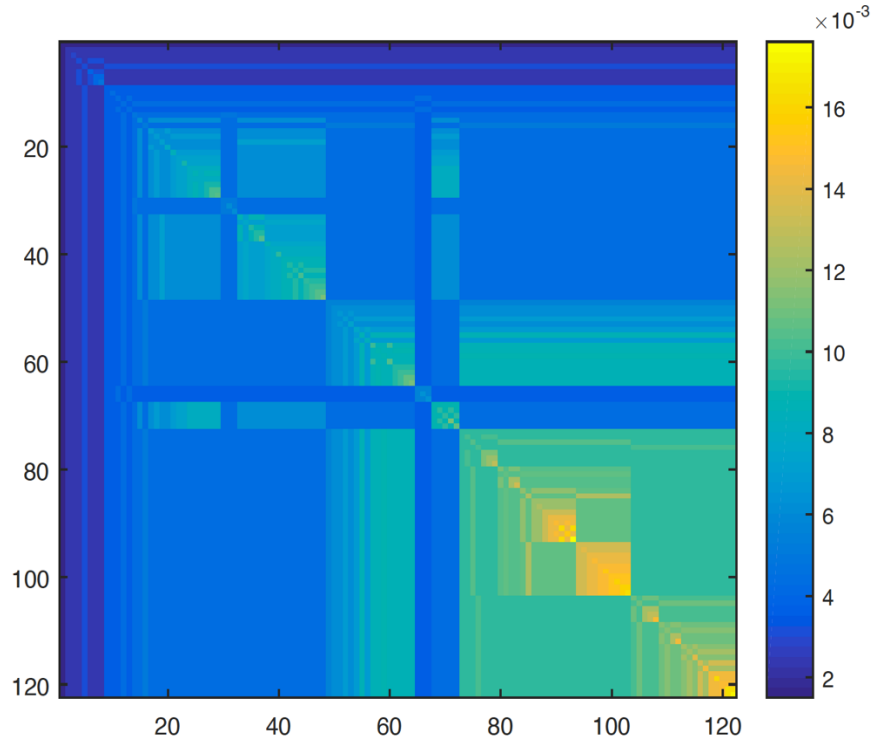
Southern California Edison 47-bus feeder



Matrix $\mathbf{X}$



IEEE 123-bus feeder



Matrix $\mathbf{X}$

LDF approximation error

- Express DistFlow in matrix-vector notation

$$\sum_{k:\ n\to k} P_k = p_n + P_n - r_n \ell_n$$

$$\sum_{k:\ n\to k} Q_k = q_n + Q_n - x_n \ell_n$$

$$v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2)\ell_n$$

$$\mathbf{p} = \mathbf{A}^\top \mathbf{P} + \mathbf{D}_r \boldsymbol{\ell}$$

$$\mathbf{q} = \mathbf{A}^\top \mathbf{Q} + \mathbf{D}_x \boldsymbol{\ell}$$

$$\mathbf{A}\mathbf{v} + v_0 \mathbf{a}_0 = 2\mathbf{D}_r \mathbf{P} + 2\mathbf{D}_x \mathbf{Q} - \left(\mathbf{D}_r^2 + \mathbf{D}_x^2\right) \boldsymbol{\ell}$$

- LDF gives an over-estimator for squared voltage magnitudes

$$\mathbf{v} = \hat{\mathbf{v}} + \underbrace{\mathbf{F}\mathbf{D}_r \left[-\mathbf{I} - 2\mathbf{F}^\top\right] \mathbf{D}_r \boldsymbol{\ell} + \mathbf{F}\mathbf{D}_x \left[-\mathbf{I} - 2\mathbf{F}^\top\right] \mathbf{D}_x \boldsymbol{\ell}}_{\leq \mathbf{0}} \leq \hat{\mathbf{v}}$$

- LDF gives an under-estimator for line flows $\mathbf{P} = \mathbf{F}^\top \mathbf{p} - \mathbf{F}^\top \mathbf{D}_r \boldsymbol{\ell} \geq \hat{\mathbf{P}}$

- Approximation accuracy depends on loading conditions

Linearized power flow models

- Recall *linearized* or so-termed *DC power flow model* in transmission systems $\mathbf{p} = \mathbf{B}\boldsymbol{\theta}$

- It has been derived under three approximations:
  1. Voltage magnitudes close to unity $|V_n| = 1 + \epsilon_n$ with $\epsilon_n \simeq 0$
  2. Voltage angle differences across lines close to zero $\theta_{nm} = \theta_n - \theta_m \simeq 0$
  3. Ignoring line resistances and shunt elements

- Repeat the same analysis for a *meshed grid* without the third assumption [6]

- Consider voltages $V_n = (1+\epsilon_n)e^{j\theta_n}$ and $V_m = (1+\epsilon_m)e^{j\theta_m}$ $\quad \mathbf{x} = \begin{bmatrix} \epsilon_n \\ \epsilon_m \\ \theta_{nm} \end{bmatrix}$

- Consider power flow from bus n to m: $S_{nm} = V_n(V_n^* - V_m^*)y_{nm}^* = f(\mathbf{x})y_{nm}^*$

$$\begin{aligned} f(\mathbf{x}) &= |V_n|^2 - |V_n||V_m|(\cos\theta_{nm} + j\sin\theta_{nm}) \\ &= (1+\epsilon_n)^2 - (1+\epsilon_n)(1+\epsilon_m)(\cos\theta_{nm} + j\sin\theta_{nm}) \end{aligned}$$

- First-order Taylor's series expansion $f(\mathbf{x}) \simeq f(\mathbf{0}) + (\nabla_{\mathbf{x}} f(\mathbf{0}))^\top(\mathbf{x} - \mathbf{0})$

- Observe that $f(\mathbf{0}) = 0$

Linearized power flow models (cont'd)

- Compute gradient of $f(\mathbf{x}) = (1+\epsilon_n)^2 - (1+\epsilon_n)(1+\epsilon_m)(\cos\theta_{nm} + j\sin\theta_{nm})$

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} 2(1+\epsilon_n) - (1+\epsilon_m)(\cos\theta_{nm} + j\sin\theta_{nm}) \\ -(1+\epsilon_n)(\cos\theta_{nm} + j\sin\theta_{nm}) \\ -(1+\epsilon_n)(1+\epsilon_m)(-\sin\theta_{nm} + j\cos\theta_{nm}) \end{bmatrix} \Rightarrow \nabla_{\mathbf{x}} f(\mathbf{0}) = \begin{bmatrix} +1 \\ -1 \\ -j \end{bmatrix}$$

- Linearization $f(\mathbf{x}) \simeq 0 + [+1 \;\; -1 \;\; -j][\epsilon_n \;\; \epsilon_m \;\; \theta_{nm}]^\top$

- Therefore, power flow on line *(n,m)* can be linearized as

$$S_{nm} \simeq [(\epsilon_n - \epsilon_m) - j(\theta_n - \theta_m)]\,(g_{nm} - jb_{nm}) \Rightarrow \begin{aligned} P_{nm} &\simeq g_{nm}(\epsilon_n - \epsilon_m) - b_{nm}(\theta_n - \theta_m) \\ Q_{nm} &\simeq -b_{nm}(\epsilon_n - \epsilon_m) - g_{nm}(\theta_n - \theta_m) \end{aligned}$$

- Stacking line power flows

$$\begin{aligned} \mathbf{P} &= \mathbf{D}_g\mathbf{A}\boldsymbol{\epsilon} - \mathbf{D}_b\mathbf{A}\boldsymbol{\theta} \\ \mathbf{Q} &= -\mathbf{D}_b\mathbf{A}\boldsymbol{\epsilon} - \mathbf{D}_g\mathbf{A}\boldsymbol{\theta} \end{aligned}$$

- Converting to power injections

$$\begin{aligned} \mathbf{p} &= \mathbf{A}^\top\mathbf{P} = \mathbf{G}\boldsymbol{\epsilon} + \mathbf{B}\boldsymbol{\theta} \\ \mathbf{q} &= \mathbf{A}^\top\mathbf{Q} = \mathbf{B}\boldsymbol{\epsilon} - \mathbf{G}\boldsymbol{\theta} \end{aligned} \qquad \begin{aligned} \mathbf{G} &:= \mathbf{A}^\top\mathbf{D}_g\mathbf{A} \\ \mathbf{B} &:= -\mathbf{A}^\top\mathbf{D}_b\mathbf{A} \end{aligned}$$

compare to 'DC' model for transmission grids

Linearized power flow models (cont'd)

- Solve equations wrt voltage magnitudes and angles

$$\boldsymbol{\epsilon} = \left(\mathbf{G} + \mathbf{B}\mathbf{G}^{-1}\mathbf{B}\right)^{-1}\mathbf{p} + \left(\mathbf{B} + \mathbf{G}\mathbf{B}^{-1}\mathbf{G}\right)^{-1}\mathbf{q}$$
$$\boldsymbol{\theta} = \left(\mathbf{B} + \mathbf{G}\mathbf{B}^{-1}\mathbf{G}\right)^{-1}\mathbf{p} - \left(\mathbf{G} + \mathbf{B}\mathbf{G}^{-1}\mathbf{B}\right)^{-1}\mathbf{q}$$

- Formula is general; holds even for meshed grids

- For radial grids (square and invertible $\mathbf{A}$), equations simplify considerably to:

$$\boldsymbol{\epsilon} = \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q}$$
$$\boldsymbol{\theta} = \mathbf{X}\mathbf{p} - \mathbf{R}\mathbf{q}$$

- Compare to LDF; linear approximation for voltage angles too
- Linearization conducted at flat voltage profile
- Another reference voltage can be used; but ($\mathbf{R}$,$\mathbf{X}$,$\mathbf{B}$,$\mathbf{G}$) will depend on that state

Power flow via convex relaxation

- Instead of the BF solver, solve the PF problem as a minimization [3]-[4]

$$
\begin{aligned}
\min \quad & \sum_{n=1}^{N} r_n \ell_n \\
\text{over} \quad & \mathbf{P}, \mathbf{Q}, \mathbf{v}, \boldsymbol{\ell}, p_0, q_0 \\
\text{s.t.} \quad & \sum_{k:\ n \to k} P_k = p_n + P_n - r_n \ell_n \\
& \sum_{k:\ n \to k} Q_k = q_n + Q_n - x_n \ell_n \\
& v_n = v_{\pi_n} - 2 r_n P_n - 2 x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
& \ell_n = \frac{P_n^2 + Q_n^2}{v_{\pi_n}}
\end{aligned}
\quad \Longrightarrow \quad
\frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n
$$

- Non-convex constraint relaxed to second-order cone constraints (SOC)

$$
\left\| \begin{bmatrix} 2P_n \\ 2Q_n \\ \ell_n - v_{\pi_n} \end{bmatrix} \right\|_2 \leq \ell_n + v_{\pi_n} \quad \Rightarrow \quad 4P_n^2 + 4Q_n^2 + (\ell_n - v_{\pi_n})^2 \leq (\ell_n + v_{\pi_n})^2 \quad \Rightarrow \quad P_n^2 + Q_n^2 \leq v_{\pi_n} \ell_n
$$

- It can be solved efficiently as a second-order cone program (SOCP)
- Oftentimes, the relaxation is exact: SOC are satisfied with equality

Optimal power flow via convex relaxation

- OPF needed to solve any meaningful grid optimization task
    1. power loss minimization
    2. voltage regulation
    3. conservation voltage reduction
    4. demand response
    5. electric vehicle charging
    6. optimal coordination of energy storage

- Power injections **s** become *control variables* rather than *fixed (inelastic load)*

- Optimally control devices while satisfying the PF equations and network constraints

Optimal power flow via convex relaxation

- Solving OPF in single-phase radial grids through via an SOCP [3]

$$\min \quad \sum_{n=1}^{N} r_n \ell_n + \sum_{n=1}^{N} c_n p_n^g + \sum_{n=1}^{N} \alpha_n v_n$$

$$\text{over} \quad \mathbf{P}, \mathbf{Q}, \mathbf{v}, \boldsymbol{\ell}, p_0, q_0, \mathbf{s}$$

$$\text{s.t.} \quad \sum_{k:\ n \to k} P_k = p_n + P_n - r_n \ell_n$$

$$\sum_{k:\ n \to k} Q_k = q_n + Q_n - x_n \ell_n$$

$$v_n = v_{\pi_n} - 2 r_n P_n - 2 x_n Q_n + (r_n^2 + x_n^2) \ell_n$$

$$\frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n$$

relaxed BFM equations

$$\mathbf{p} = \mathbf{p}^g - \mathbf{p}^c$$

$$\mathbf{q} = \mathbf{q}^g - \mathbf{q}^c$$

$$\underline{p}_n^g \leq p_n^g \leq \overline{p}_n^g, \ \forall n$$

$$\underline{q}_n^g \leq q_n^g \leq \overline{q}_n^g, \ \forall n$$

$$(p_n^g)^2 + (q_n^g)^2 \leq \overline{s}_n^g, \ \forall n$$

injection constraints

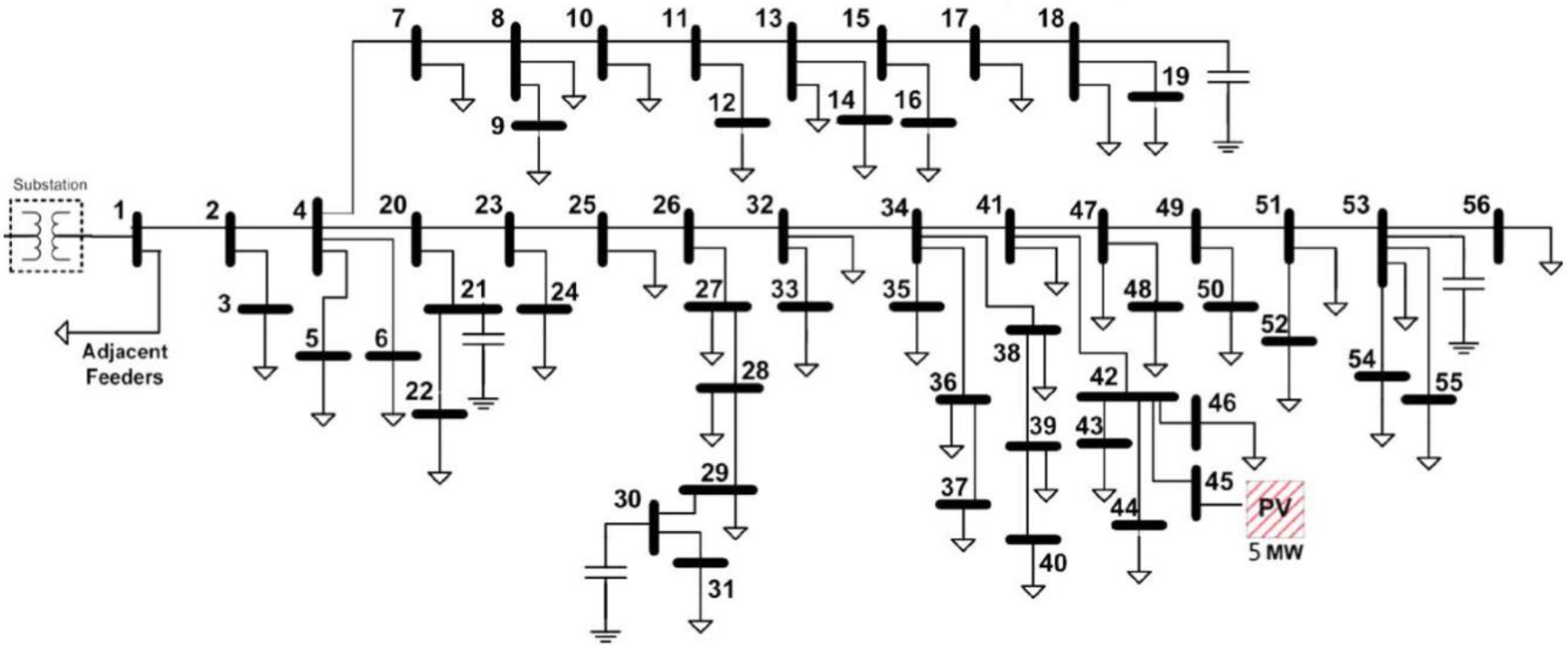
$$\underline{v}_n \leq v_n \leq \overline{v}_n, \ \forall n$$

$$\ell_n \leq \overline{\ell}_n, \ \forall n$$

network constraints

- Oftentimes, the relaxation is exact: SOCs are satisfied with equality

Southern California Edison 56-bus system [9]



Southern California Edison 56-bus system [9]

Network Data																	
Line Data				Line Data				Line Data				Load Data		Load Data		Load Data	
From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	Bus No.	Peak MVA	Bus No.	Peak MVA	Bus No.	Peak MVA
1	2	0.160	0.388	20	21	0.251	0.096	39	40	2.349	0.964	3	0.057	29	0.044	52	0.315
2	3	0.824	0.315	21	22	1.818	0.695	34	41	0.115	0.278	5	0.121	31	0.053	54	0.061
2	4	0.144	0.349	20	23	0.225	0.542	41	42	0.159	0.384	6	0.049	32	0.223	55	0.055
4	5	1.026	0.421	23	24	0.127	0.028	42	43	0.934	0.383	7	0.053	33	0.123	56	0.130
4	6	0.741	0.466	23	25	0.284	0.687	42	44	0.506	0.163	8	0.047	34	0.067	Shunt Cap	
4	7	0.528	0.468	25	26	0.171	0.414	42	45	0.095	0.195	9	0.068	35	0.094	Bus	Mvar
7	8	0.358	0.314	26	27	0.414	0.386	42	46	1.915	0.769	10	0.048	36	0.097	19	0.6
8	9	2.032	0.798	27	28	0.210	0.196	41	47	0.157	0.379	11	0.067	37	0.281	21	0.6
8	10	0.502	0.441	28	29	0.395	0.369	47	48	1.641	0.670	12	0.094	38	0.117	30	0.6
10	11	0.372	0.327	29	30	0.248	0.232	47	49	0.081	0.196	14	0.057	39	0.131	53	0.6
11	12	1.431	0.999	30	31	0.279	0.260	49	50	1.727	0.709	16	0.053	40	0.030	Photovoltaic	
11	13	0.429	0.377	26	32	0.205	0.495	49	51	0.112	0.270	17	0.057	41	0.046	Bus	Capacity
13	14	0.671	0.257	32	33	0.263	0.073	51	52	0.674	0.275	18	0.112	42	0.054		
13	15	0.457	0.401	32	34	0.071	0.171	51	53	0.070	0.170	19	0.087	43	0.083	45	5MW
15	16	1.008	0.385	34	35	0.625	0.273	53	54	2.041	0.780	22	0.063	44	0.057		
15	17	0.153	0.134	34	36	0.510	0.209	53	55	0.813	0.334	24	0.135	46	0.134	V_{base} = 12kV	
17	18	0.971	0.722	36	37	2.018	0.829	53	56	0.141	0.340	25	0.100	47	0.045	S_{base} = 1MVA	
18	19	1.885	0.721	34	38	1.062	0.406					27	0.048	48	0.196	$Z_{\text{base}} = 144\Omega$	
4	20	0.138	0.334	38	39	0.610	0.238					28	0.038	50	0.045		

SCE 56-bus casefile

```
%% system base (all data have been converted to per unit)
mpc.baseMVA = 1;
mpc.basekV = 12;
mpc.baseOhm = 144;

%% bus data
% bus_i type Pd Qd Qcapacitor
mpc.bus = [
1 3 0 0 0;
2 1 0 0 0;
3 1 0.0570 0.0276 0;
4 1 0 0 0;
5 1 0.1210 0.0586 0;
6 1 0.0490 0.0237 0;
...
18 1 0.1120 0.0542 0;
19 1 0.0870 0.0421 0.6000;
...
56 1 0.1300 0.0630 0;
];
```

```
%% branch data
% fbus tbus r x (in Ohms; then converted to pu)
% Given the numbering, row i-th corresponds to the line feeding bus (i+1);
substation is indexed by 1.
% The bus fed by a line is captured in the second column. The first column of
the matrix shows the parent bus.
% For example, row 1 corresponds to the line feeding bus #2, and the parent
% bus of #2 is #1 (the substation).
mpc.branch = [
1 2 0.160 0.388;
2 3 0.824 0.315;
2 4 0.144 0.349;
...
];
% converting impedances in per unit
mpc.branch(:,3) = mpc.branch(:,3)./mpc.baseOhm;
mpc.branch(:,4) = mpc.branch(:,4)./mpc.baseOhm;


%% make branch-bus incidence matrix
N = size(mpc.bus,1); % number of buses including the substation
L = size(mpc.branch,1);
At = zeros(L,N);
for line = 1:L,
  At(line,mpc.branch(line,1)) = +1;
  At(line,mpc.branch(line,2)) = -1;
end;
mpc.At = At;
```

Power flow through SOCP

```
%% loading data from casefile
mpc = casesce56;
At = mpc.At;
N = size(mpc.bus,1); % number of buses including the substation
L = size(mpc.branch,1);

% load demand and line impedance data [pu]
pd = mpc.bus(:,3);
qd = mpc.bus(:,4);
qcap = mpc.bus(:,5);
r = mpc.branch(:,3);
x = mpc.branch(:,4);

%% define solver settings and optimization variables
settings = sdpsettings('solver','sdpt3','verbose',0);
v = sdpvar(N,1); % squared voltage magnitudes (including the
substation)
ell = sdpvar(L,1); % squared line current magnitudes
P = sdpvar(L,1); % active power flows
Q = sdpvar(L,1); % reactive power flows
pg = sdpvar(N,1); % active power injections
qg = sdpvar(N,1); % reactive power injections

%% cost and constraints
cost = r'*ell; % total ohmic losses
```

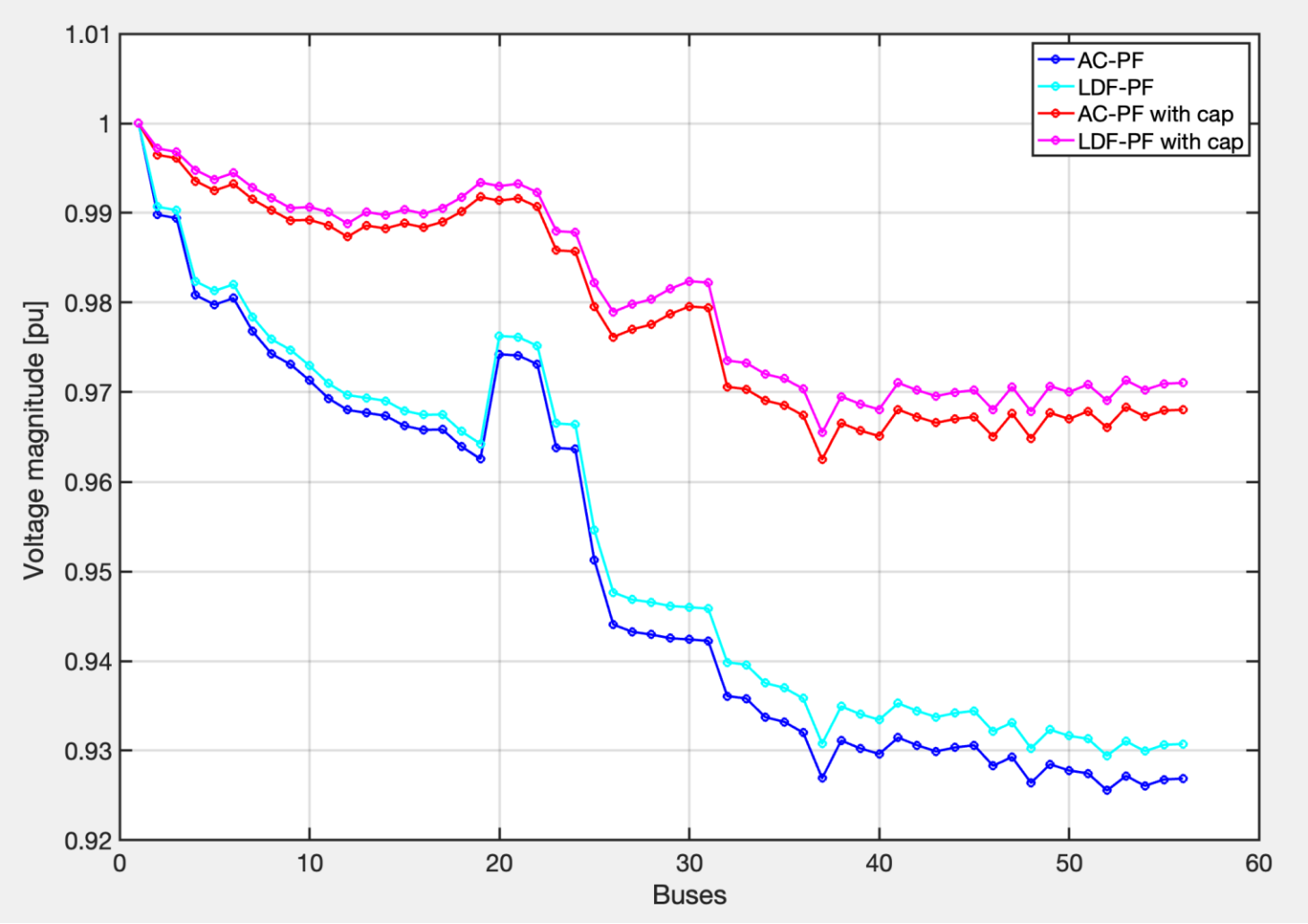
```
% constraints imposing relaxed BFM model (Slide 15 of Lecture 15)
% substation power balance has been included by appending a zero on (r,x,ell)
con1 = (pg-pd==At'*P+diag([0; r])*[0; ell]);
con2 = (qg-qd+qcap==At'*Q+diag([0; x])*[0; ell]);
con3 = (At*v == 2*diag(r)*P + 2*diag(x)*Q -diag(r.^2+x.^2)*ell);
con4 = [];
for line = 1:L,
    % find the parent bus of line
    % in general: [value,parent] = find(A(line,:)==1);
    parent = mpc.branch(line,1);
    % For second-order cone constraints, see:
    % see https://yalmip.github.io/command/cone/
    con4 = con4 + cone([2*P(line); 2*Q(line); ell(line)-
    v(parent)],ell(line)+v(parent));
end;

% injection constraints
pgmin = zeros(N-1,1);
pgmax = zeros(N-1,1);
qgmin = zeros(N-1,1);
qgmax = zeros(N-1,1);
con5 = (pgmin<=pg(2:end)<=pgmax);
con6 = (qgmin<=qg(2:end)<=qgmax);

% (SQUARED) voltage constraints
v0 = 1;
con7 = (v(1)==v0); % fixing substation voltage
vmin = 0.97.^2*ones(N-1,1);
vmax = 1.03.^2*ones(N-1,1);

con8 = (vmin<=v(2:end)<=vmax);
% con9 = (ell<=ellmax); line ampacity ratings not provided
```

Power flow through SOCP and LDF

```
% network constraints are dropped (solving PF rather than OPF)
con = con1+con2+con3+con4+con5+con6+con7;
sol = optimize(con,cost,settings);


% check if optimization terminated succesfully
if sol.problem==0,
  disp('SOCP successfully solved')
else
  error('Optimization did not terminate successfully!');
end;

% check if the relaxation is exact
exact = max(abs(double(P.^2+Q.^2-ell.*v(mpc.branch(:,1)))));
disp(['Maximum SOC violation is ' num2str(exact)])

% read out variables
pg = double(pg);
qg = double(qg);
v = sqrt(double(v));
```

```
%% LDF approximate model
% create R and X matrices (Slide 11 of Lecture 11)
F = inv(At(:,2:end));
R = F*diag(r)*F';
X = F*diag(x)*F';
p = pg-pd;
p = p(2:end); % excluding substation
q = qg-qd;
q = q(2:end); % excluding substation
% loads alone (significant voltage drop)
v1hat = 2*R*p + 2*X*q + v0*ones(N-1,1);
v1hat = [v0; sqrt(v1hat)];
% loads and capacitors
v2hat = 2*R*p + 2*X*(q+qcap(2:end)) + v0*ones(N-1,1);
v2hat = [v0; sqrt(v2hat)];
```

Voltage profiles



SOCP-OPF for CVR

- *Example 1:* Find substation voltage to minimize ohmic losses while keeping voltages within limits

$$\min \quad \sum_{n=1}^{N} r_n \ell_n$$

$$\text{over} \quad \mathbf{P}, \mathbf{Q}, \mathbf{v}, \boldsymbol{\ell}, p_0, q_0, \mathbf{s}$$

$$\text{s.t.} \quad \sum_{k:\ n \to k} P_k = p_n + P_n - r_n \ell_n$$

$$\sum_{k:\ n \to k} Q_k = q_n + Q_n - x_n \ell_n$$

$$v_n = v_{\pi_n} - 2 r_n P_n - 2 x_n Q_n + (r_n^2 + x_n^2) \ell_n$$

$$\frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n$$

relaxed BFM equations

$$\underline{v}_n \leq v_n \leq \overline{v}_n, \ \forall n$$

including substation voltage

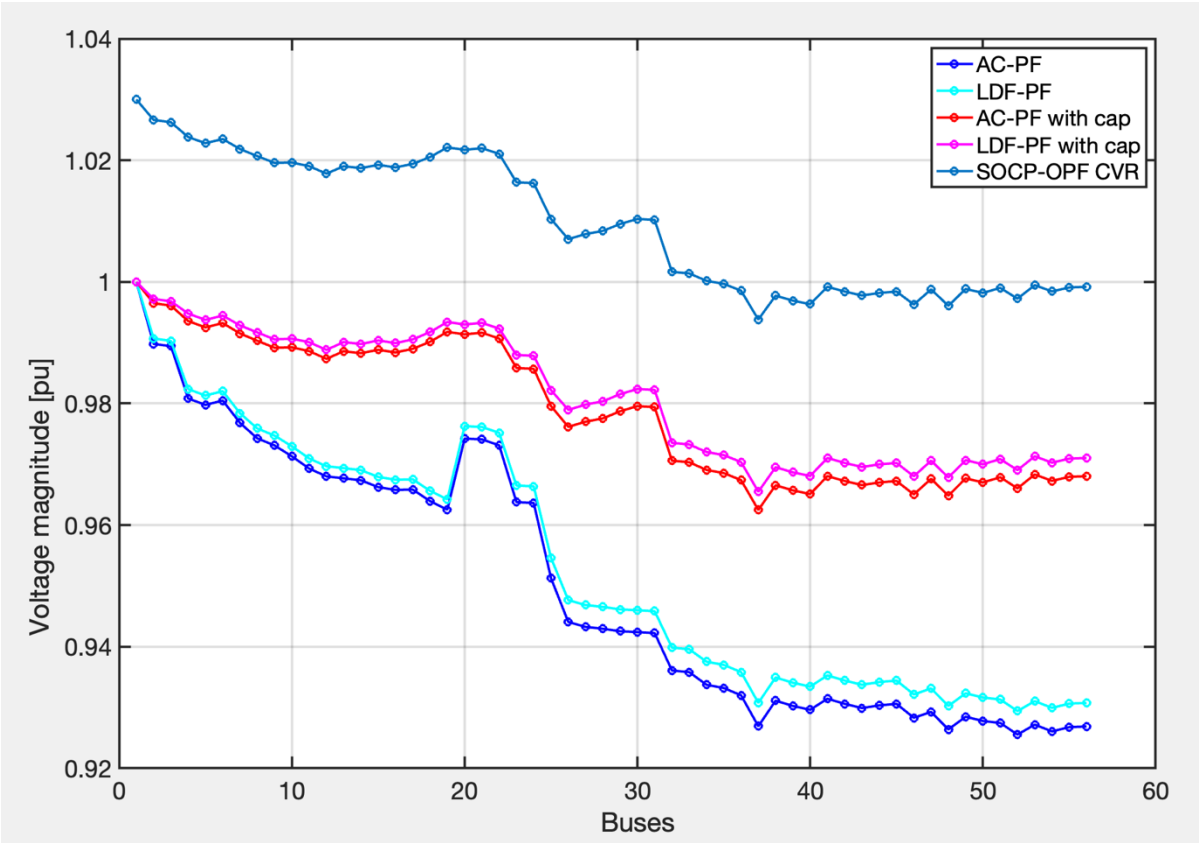
- Modifications to main code

```
% (SQUARED) voltage constraints
vmin = 0.97.^2*ones(N,1);
vmax = 1.03.^2*ones(N,1);
con8 = (vmin<=v<=vmax);
% con9 = (ell<=ellmax); line ampacity ratings not provided by SCE


% include voltage constraint
% only true control variable is substation voltage
con = con1+con2+con3+con4+con5+con6+con8;
sol = optimize(con,cost,settings);
```

- *Assumptions:* Capacitors and loads are modeled as constant-power injections (only benefit comes from lowering losses)

OPF-CVR results

- *Example 1:* Find substation voltage to minimize ohmic losses while keeping voltages within limits



- Total load 3.8350 MW
- Ohmic losses for $v_1 = 1$

$$\mathbf{r}^\top \boldsymbol{\ell} = 0.1159 \text{ MW } (3\%)$$

- Ohmic losses for $v_1^* = 1.03$

$$\mathbf{r}^\top \boldsymbol{\ell} = 0.1088 \text{ MW } (2.84\%)$$

SOCP-OPF for DER control

- *Example 2:* Control p/q setpoints for solar PV DERs to minimize power bought from main grid

$$\begin{aligned}
\min \quad & p_0 \\
\text{over} \quad & \mathbf{P}, \mathbf{Q}, \mathbf{v}, \boldsymbol{\ell}, p_0, q_0, \mathbf{s} \\
\text{s.t.} \quad & \sum_{k:\ n \to k} P_k = p_n + P_n - r_n \ell_n \\
& \sum_{k:\ n \to k} Q_k = q_n + Q_n - x_n \ell_n \\
& v_n = v_{\pi_n} - 2 r_n P_n - 2 x_n Q_n + (r_n^2 + x_n^2) \ell_n \\
& \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n
\end{aligned}$$

relaxed BFM equations

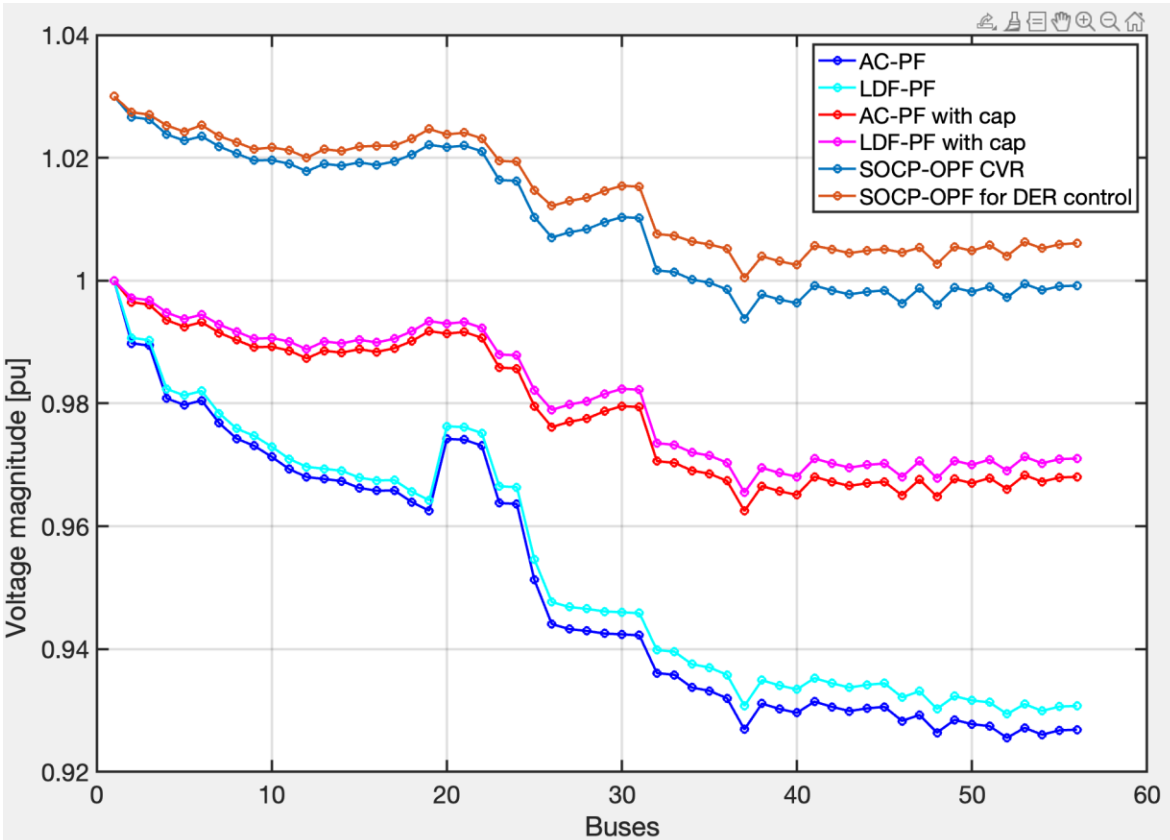
$$\underline{v}_n \leq v_n \leq \overline{v}_n, \ \forall n$$

$$\begin{aligned}
& \mathbf{p} = \mathbf{p}^g - \mathbf{p}^c \\
& \mathbf{q} = \mathbf{q}^g - \mathbf{q}^c \\
& 0 \leq p_n^g \leq \overline{p}_n^g, \ \forall n \in \mathcal{PV} \\
& (p_n^g)^2 + (q_n^g)^2 \leq \overline{s}_n^g, \ \forall n \in \mathcal{PV}
\end{aligned}$$

- Modifications to main code

```
% injection constraints
% injection constraints
pgmin = zeros(N-1,1);
pgmax = zeros(N-1,1);
% installing PVs at buses (5,15,25,35,45,55) with available solar of 0.1 pu
pgmax(5:10:55) = 0.1*ones(6,1);
sgmax = zeros(N-1,1);
sgmax(5:10:55) = 1.1*0.1*ones(6,1); % kVA capacity is oversized by 10%
compared to kW capacity of PVs
con5 = (pgmin<=pg(2:end)<=pgmax);
con6 = (pg(2:end).^2+qg(2:end).^2<=sgmax.^2);
```

OPF results for DER control



- Total load 3.8350 MW
- Total solar 0.6 MW
- Ohmic losses

$$\mathbf{r}^\top \boldsymbol{\ell} = 0.0817 \text{ MW}$$

- Substation voltage

$$v_1^* = 1.03$$

- Net injection and DER setpoints

```
>> pg(1)

ans =

   3.3167

>> [pg(6:10:end) qg(6:10:end)]

ans =

   0.1000   -0.0458
   0.1000   -0.0458
   0.1000   -0.0223
   0.1000    0.0458
   0.1000    0.0458
   0.1000    0.0061
```

SOCP-OPF for DER control

- *Example 3:* In the previous example, we increase solar by 10 times

$$\begin{aligned}
\min \quad & p_0 \\
\text{over} \quad & \mathbf{P}, \mathbf{Q}, \mathbf{v}, \boldsymbol{\ell}, p_0, q_0, \mathbf{s}
\end{aligned}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{k:\ n\to k} P_k = p_n + P_n - r_n \ell_n \\
& \sum_{k:\ n\to k} Q_k = q_n + Q_n - x_n \ell_n \\
& v_n = v_{\pi_n} - 2r_n P_n - 2x_n Q_n + (r_n^2 + x_n^2)\ell_n \\
& \frac{P_n^2 + Q_n^2}{v_{\pi_n}} \leq \ell_n
\end{aligned}$$

relaxed BFM equations

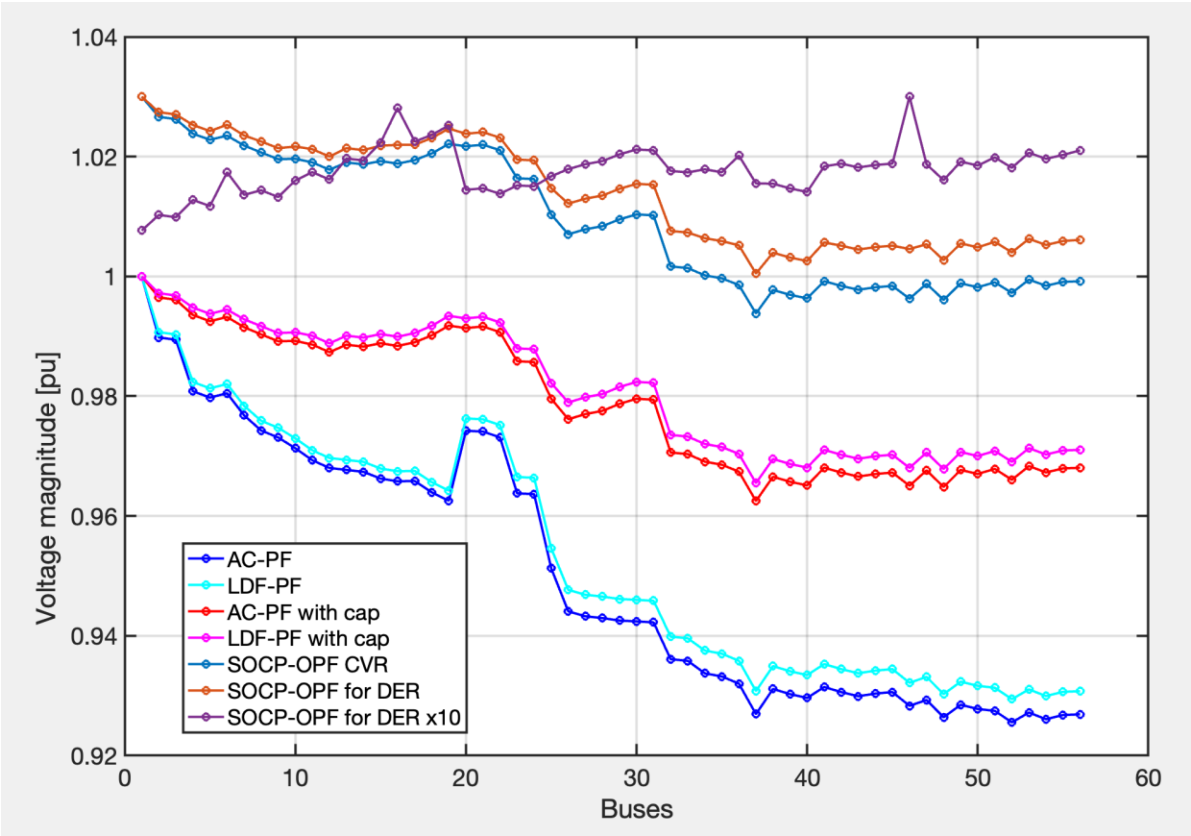
$$\underline{v}_n \leq v_n \leq \overline{v}_n, \ \forall n$$

$$\begin{aligned}
& \mathbf{p} = \mathbf{p}^g - \mathbf{p}^c \\
& \mathbf{q} = \mathbf{q}^g - \mathbf{q}^c \\
& 0 \leq p_n^g \leq \overline{p}_n^g, \ \forall n \in \mathcal{PV} \\
& (p_n^g)^2 + (q_n^g)^2 \leq \overline{s}_n^g, \ \forall n \in \mathcal{PV}
\end{aligned}$$

- Modifications to main code

```
% injection constraints
% injection constraints
pgmin = zeros(N-1,1);
pgmax = zeros(N-1,1);
% installing PVs at buses (6,16,26,36,46,56) with available solar of 0.1 pu
pgmax(5:10:55) = 1.0*ones(6,1);
sgmax = zeros(N-1,1);
sgmax(5:10:55) = 1.1*1.0*ones(6,1); % increasing kVA PV capacity by 10%
con5 = (pgmin<=pg(2:end)<=pgmax);
con6 = (pg(2:end).^2+qg(2:end).^2<=sgmax.^2);
```

OPF results for DER control



- Total load 3.8350 MW
- Total solar 6 MW
- Ohmic losses

$$\mathbf{r}^\top \boldsymbol{\ell} = 0.0817 \text{ MW}$$

- Substation voltage

$$v_1^* = 1.0076$$

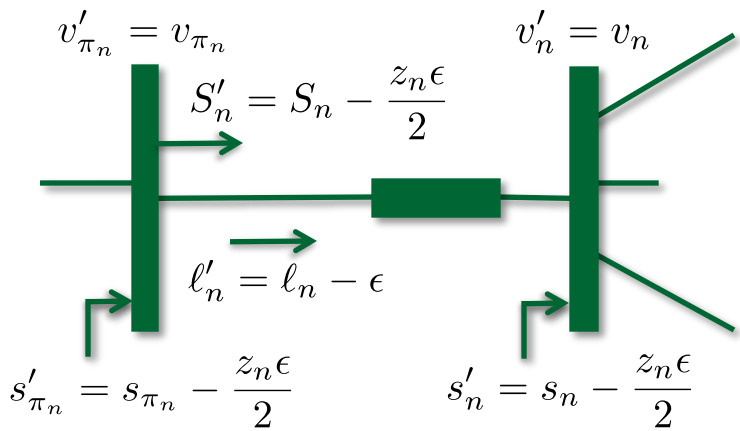
- Net injection and DER setpoints

```
>> pg(1)

ans =

  -2.1039

>> [pg(6:10:end) qg(6:10:end)]

ans =

    1.0000   -0.0318
    1.0000   -0.2362
    1.0000   -0.3278
    1.0000    0.2866
    1.0000    0.0651
    1.0000   -0.1209
```

Exactness under load over-satisfaction

Theorem ([3]): If power injections are unbounded below, the relaxation is exact

- Assume problem has been solved, but SOC for line n is inexact $P_n^2 + Q_n^2 < \ell_n v_{\pi_n}$



injections can be reduced without bound

- Given current solution $(\mathbf{S}, \mathbf{s}, \mathbf{v}, \boldsymbol{\ell}, s_0)$, construct another point $(\mathbf{S}', \mathbf{s}', \mathbf{v}', \boldsymbol{\ell}', s'_0)$ by changing only the quantities related to line n as shown above

- Show that new point is feasible; satisfies SOC with equality; and yields lower cost!

$$\sum_{k:\ n \to k} P'_k = p'_n + P'_n - r_n \ell'_n$$

$$\sum_{k:\ n \to k} Q'_k = q'_n + Q'_n - x_n \ell'_n$$

$$v_n = v_{\pi_n} - 2 r_n P'_n - 2 x_n Q'_n + (r_n^2 + x_n^2) \ell'_n$$

$$\ell'_n = \frac{(P'_n)^2 + (Q'_n)^2}{v_{\pi_n}}$$

Exactness of SOCP convex relaxation

- Exactness of SOCP relaxation for OPF in radial grids has been studied extensively [6]

- Different sets of sufficient conditions have been derived:
  - no reverse power flows
  - identical r/x ratios for all lines
  - r/x increase downstream and there are no reverse active power flows
  - r/x decrease downstream and there are no reverse reactive power flows

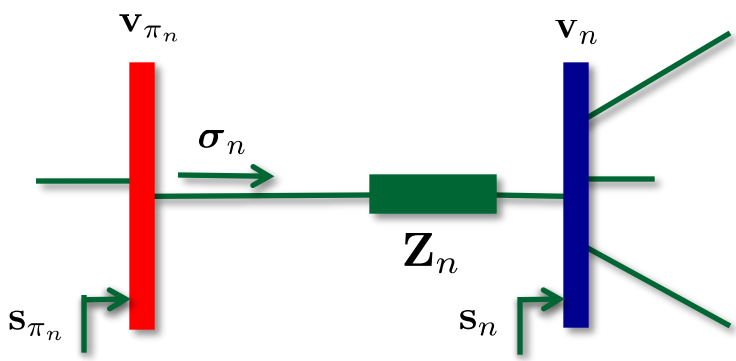
- If the SOCP is exact, the minimizer is *unique*

- To make BFM exact for meshed grids, add phase shifters to implement angle differences [3]

- Otherwise, use a semidefinite program relaxation based on bus injection model (BIM) [4]

- *How do these schemes extend to multiphase grids? [7]*

Relaxed multiphase BFM



- 'Square' (multiply by conjugate transpose) the voltage drop equation

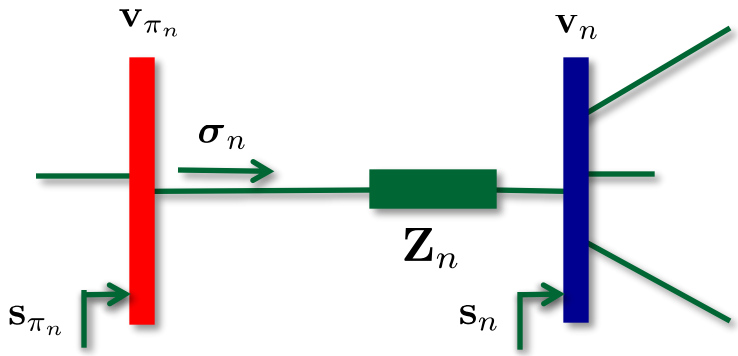
$$\mathbf{v}_n = \mathbf{v}_{\pi_n} - \mathbf{Z}_n \mathbf{i}_n \quad \Rightarrow \quad \mathbf{v}_n \mathbf{v}_n^H = \mathbf{v}_{\pi_n} \mathbf{v}_{\pi_n}^H + \mathbf{Z}_n \mathbf{i}_n \mathbf{i}_n^H \mathbf{Z}_n^H - \mathbf{v}_{\pi_n} \mathbf{i}_n^H \mathbf{Z}_n^H - \mathbf{Z}_n \mathbf{i}_n \mathbf{v}_{\pi_n}^H$$

- Define 'squared' voltages and currents $\mathbf{V}_n = \mathbf{v}_n \mathbf{v}_n^H \qquad \mathbf{L}_n = \mathbf{i}_n \mathbf{i}_n^H$

- Express 'squared' voltage drop as $\boxed{\mathbf{V}_n = \mathbf{V}_{\pi_n} + \mathbf{Z}_n \mathbf{L}_n \mathbf{Z}_n^H - \mathbf{S}_n \mathbf{Z}_n^H - \mathbf{Z}_n \mathbf{S}_n^H}$

- Linear equation; but complexity is hidden under 'squared' variables $(\mathbf{V}_n, \mathbf{L}_n, \mathbf{S}_n)$

Relaxed multiphase BFM (cont'd)



- In *single-phase* grids $\quad S_n = V_{\pi_n} I_n^* \xrightarrow{\text{'square'}} |S_n|^2 = v_{\pi_n} \ell_n \xrightarrow{\text{relax}} |S_n|^2 \leq v_{\pi_n} \ell_n$

- Relaxation can be also written $\begin{bmatrix} v_{\pi_n} & S_n \\ S_n^* & \ell_n \end{bmatrix} = \begin{bmatrix} V_{\pi_n} \\ I_n \end{bmatrix} \begin{bmatrix} V_{\pi_n} \\ I_n \end{bmatrix}^H \succeq \mathbf{0}$ and ~~rank 1~~

- In *multi-phase* grids, the relaxation becomes

$$\boxed{\begin{bmatrix} \mathbf{V}_{\pi_n} & \mathbf{S}_n \\ \mathbf{S}_n^* & \mathbf{L}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\pi_n} \\ \mathbf{i}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_{\pi_n} \\ \mathbf{i}_n \end{bmatrix}^H \succeq \mathbf{0}} \text{ and } \cancel{\text{rank-1}}$$

- Semidefinite program (SDP) constraint captures all quadratic relationships

OPF with multiphase BFM

$$\begin{aligned} \min \quad & \text{losses and/or CVR and/or generation cost} \\ \text{over} \quad & \{\mathbf{S}_n, \mathbf{s}_n, \mathbf{V}_n, \mathbf{L}_n\}_n \end{aligned}$$

$$\begin{aligned} \text{s.t.} \quad & \mathrm{dg}\left(\mathbf{S}_n - \mathbf{Z}_n \mathbf{L}_n\right) + \mathbf{s}_n = \sum_{k:\ n \to k} \mathrm{dg}\left(\mathbf{S}_k\right) \\ & \mathbf{V}_n = \mathbf{V}_{\pi_n} + \mathbf{Z}_n \mathbf{L}_n \mathbf{Z}_n^H - \mathbf{S}_n \mathbf{Z}_n^H - \mathbf{Z}_n \mathbf{S}_n^H \\ & \begin{bmatrix} \mathbf{V}_{\pi_n} & \mathbf{S}_n \\ \mathbf{S}_n^* & \mathbf{L}_n \end{bmatrix} \succeq \mathbf{0} \end{aligned}$$

relaxed BFM equations

$$\begin{aligned} & \mathbf{p} = \mathbf{p}^g - \mathbf{p}^c \\ & \mathbf{q} = \mathbf{q}^g - \mathbf{q}^c \\ & \underline{p}^g_{n,\phi} \le p^g_{n,\phi} \le \overline{p}^g_{n,\phi}, \ \forall n, \phi \\ & \underline{q}^g_{n,\phi} \le q^g_{n,\phi} \le \overline{q}^g_{n,\phi}, \ \forall n, \phi \\ & (p^g_{n,\phi})^2 + (q^g_{n,\phi})^2 \le \overline{s}^g_{n,\phi}, \ \forall n, \phi \end{aligned}$$

injection constraints

$$\begin{aligned} & \underline{v} \le \mathrm{dg}(\mathbf{V}_n) \le \overline{v}, \ \forall n \\ & \mathrm{dg}(\mathbf{L}_n) \le \overline{\ell}, \ \forall n \end{aligned}$$

network constraints

- Relaxation is exact (constraint satisfied with equality) under practical conditions

Linear approximation for multiphase grids

- Ignore losses to get approximate power conservation

$$\mathrm{dg}\left(\mathbf{S}_n - \cancel{\mathbf{Z}_n\mathbf{L}_n}\right) + \mathbf{s}_n = \sum_{k:\ n\to k} \mathrm{dg}\left(\mathbf{S}_k\right) \quad \Rightarrow \quad \boxed{\boldsymbol{\sigma}_n + \mathbf{s}_n = \sum_{k:\ n\to k} \boldsymbol{\sigma}_k}$$

- Voltage drop requires approximating the full matrix $\mathbf{S}_n$

$$\mathbf{V}_n = \mathbf{V}_{\pi_n} + \cancel{\mathbf{Z}_n\mathbf{L}_n\mathbf{Z}_n^H} - \mathbf{S}_n\mathbf{Z}_n^H - \mathbf{Z}_n\mathbf{S}_n^H$$

- Power flow matrix can be approximated as $\mathbf{S}_n \simeq \boldsymbol{\alpha}\boldsymbol{\alpha}^H \mathrm{dg}(\boldsymbol{\sigma}_n)$ (see next slide)

$$\boxed{\mathrm{dg}(\mathbf{V}_n) = \mathrm{dg}(\mathbf{V}_{\pi_n}) - \mathrm{dg}(\boldsymbol{\alpha}\boldsymbol{\alpha}^H \mathrm{dg}(\boldsymbol{\sigma}_n)\mathbf{Z}_n^H) - \mathrm{dg}(\mathbf{Z}_n \mathrm{dg}(\boldsymbol{\sigma}_n)^* \boldsymbol{\alpha}\boldsymbol{\alpha}^H)}$$

Approximating power flow matrix

- By definition (drop index bus from subscript)

$$
\begin{aligned}
\mathbf{S} = \mathbf{v}\mathbf{i}^H &= \begin{bmatrix} V_a I_a^* & V_a I_b^* & V_a I_c^* \\ V_b I_a^* & V_b I_b^* & V_b I_c^* \\ V_c I_a^* & V_c I_b^* & V_c I_c^* \end{bmatrix} \\
&\simeq \begin{bmatrix} V_a I_a^* & \alpha V_b I_b^* & \alpha^* V_c I_c^* \\ \alpha^* V_a I_a^* & V_b I_b^* & \alpha V_c I_c^* \\ \alpha V_a I_a^* & \alpha^* V_b I_b^* & V_c I_c^* \end{bmatrix} \\
&= \begin{bmatrix} 1 & \alpha & \alpha^* \\ \alpha^* & 1 & \alpha \\ \alpha & \alpha^* & 1 \end{bmatrix} \begin{bmatrix} V_a I_a^* & & \\ & V_b I_b^* & \\ & & V_c I_c^* \end{bmatrix} \\
&= \boldsymbol{\alpha}\boldsymbol{\alpha}^H \mathrm{dg}(\boldsymbol{\sigma})
\end{aligned}
$$

Inter-phase coupling

- Simplify approximate voltage drop using the property

$$\mathrm{dg}\left(\mathbf{A}\mathrm{dg}(\mathbf{x})\mathbf{B}\right) = (\mathbf{A} \odot \mathbf{B}^\top)\mathbf{x}, \ \odot : \text{entry-wise (Hadamard) product}$$

- Approximate voltage drop

$$\boxed{\mathbf{v}_{\pi_n} - \mathbf{v}_n \simeq 2\mathrm{Re}\left\{\bar{\mathbf{Z}}_n \boldsymbol{\sigma}_n^*\right\}, \quad \text{where} \quad \bar{\mathbf{Z}}_n = \mathbf{Z}_n \odot \boldsymbol{\alpha}^* \boldsymbol{\alpha}^\top}$$

$$\mathbf{Z}_n = \begin{bmatrix} 0.530 + 1.112i & 0.127 + 0.404i & 0.126 + 0.423i \\ 0.127 + 0.404i & 0.545 + 1.043i & 0.133 + 0.374i \\ 0.126 + 0.423i & 0.133 + 0.374i & 0.542 + 1.056i \end{bmatrix}$$

$$\bar{\mathbf{Z}}_n = \begin{bmatrix} 0.530 + 1.112i & 0.286 - 0.312i & -0.430 - 0.103i \\ -0.413 - 0.092i & 0.545 + 1.0429i & 0.258 - 0.303i \\ 0.304 - 0.321i & -0.391 - 0.072i & 0.542 + 1.056i \end{bmatrix}$$

- How do complex injections affect voltage drops?

$$\mathrm{sign}\left[\mathrm{Re}\left\{\bar{\mathbf{Z}}_n\right\}\right] = \begin{bmatrix} + & + & - \\ - & + & + \\ + & - & + \end{bmatrix} \qquad \mathrm{sign}\left[\mathrm{Im}\left\{\bar{\mathbf{Z}}_n\right\}\right] = \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}$$

See Section IV of [8] for an analysis of these patterns

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