Solving the Natural Gas Flow Problem using Semidefinite Program Relaxation

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***Abstract*—Decreasing gas prices and the pressing need for fast-responding electric power generators are currently transforming natural gas networks. The intermittent operation of gas-fired power plants to balance wind energy generation introduces spatiotemporal fluctuations of increasing volumes in gas demand. At the heart of modeling, monitoring, and control of gas networks is a set of nonlinear equations relating nodal gas injections and pressures to flows over pipelines. Given gas demands at all points of the network, the gas flow task aims at finding the rest of the physical quantities involved. For a tree-like network, the problem enjoys a closed-form solution; yet solving the equations for practical meshed networks is non-trivial. This problem is posed here as a feasibility problem involving quadratic equalities and inequalities, and is further relaxed to a convex semidefinite program (SDP) minimization. Drawing parallels to the power flow problem, the relaxation is shown to be exact if the cost function is judiciously designed using a set of frequently occurring network states. Numerical tests on a Belgian network corroborate the superiority of the novel method in recovering the actual gas system state over a Newton-Raphson solver.**

***Index Terms*—Gas flow equations, semidefinite programming**

I. Introduction

Natural gas has currently emerged at the focus of the energy industry since substantial supplies of natural gas are being discovered [1]. Its role is likely to expand further in carbon-constrained electric power systems with higher penetration of renewables. Most gas distribution companies withdraw gas with small intra-day variation, whereas gas-fired power plants consume gas with high intra-day variability and usually purchase it via short-term contracts [2], [3]. Natural gas is transported from its source sites to consumers through a complex continent-wide network of pipelines. Similar to water networks, gas pressure decreases along a pipeline due to friction, and compressors are used to boost it up to desirable levels. Since congestion in the gas network can incur supply interruptions in electric grids, the interdependency between the two infrastructures has to be carefully considered [3].

This increasing variability in gas demand across time and space calls for advanced control and monitoring of the underlying physical network. This work targets solving efficiently the *gas flow (GF)* equations, a set of nonlinear equations governing the distribution of gas flows and pressures across a gas network. Given gas injections and withdrawals across all nodes, the *gas flow* problem aims at finding flows across pipelines together with the pressure at all nodes. Even under steady-state and balanced conditions, solving the GF problem is hard to solve for non-tree networks [4].

For a network without compressors, the flows and pressures are the optimal primal-dual solutions of a convex minimization problem [5]. For general meshed networks with compressors, the GF problem is typically solved using the Newton-Raphson scheme, but its convergence is conditioned on proper initialization [6]. Under slow time-varying injections, the Newton-Raphson scheme could be initialized at the previous state; nevertheless, this may not be applicable for reliability studies. Building on the theory of monotone operators, preprint [4] shows that a GF solution can be found by solving a set of variational inequalities. The latter are derived from the GF equations after applying a carefully designed linear transformation. The approach is proved successful for tree networks. Otherwise, a quasi-convex optimization problem should be solved to design the linear transformation under the restricting assumption that flow directions are known a priori.

The contribution of this work is twofold: After reviewing a gas network model via a convenient matrix-vector form in Section II, it is first recognized that the GF task can be tackled using a semidefinite program (SDP) relaxation in Section III. The relaxation is shown to be exact for a judiciously selected cost of the related SDP problem. Secondly, a methodology for designing this cost is devised in Section IV. Numerical tests conducted on a Belgian gas network demonstrate the success of the relaxation over a range of system conditions, and its superiority over the Newton-Raphson scheme.

Regarding *notation*, lower- (upper-) case boldface letters denote column vectors (matrices), and $(\cdot)^\top$ stands for transposition. Calligraphic symbols are reserved for sets, and $|\mathcal{X}|$ is the cardinality of set $\mathcal{X}$. Vectors $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{e}_n$, are the all-zeros, all-ones, and the n-th canonical vectors, respectively. Operator $\mathrm{dg}(\mathbf{x})$ defines a diagonal matrix having $\mathbf{x}$ on its main diagonal. A symmetric positive (semi)definite matrix is denoted by $\mathbf{X} \succ \mathbf{0}$ ($\mathbf{X} \succeq \mathbf{0}$). Finally, symbols $\mathbb{S}^N$ and $\mathbb{S}^N_+$ ($\mathbb{S}^N_{++}$) denote respectively the sets of $N \times N$ symmetric and symmetric positive (semi)definite matrices.

II. Natural Gas Network Modeling

Consider a natural gas network modeled by a directed graph $\mathcal{G} := (\mathcal{N}_0, \mathcal{L})$. The graph vertices $\mathcal{N}_0 = \{0, \cdots, N\}$ model nodes where gas is injected or withdrawn from the network, or

simple junctions. The graph edges $\mathcal{L} = \{1, \ldots, L\}$ correspond to gas pipelines connecting two network nodes. Let $p_i > 0$ be the gas pressure at node i for all $i \in \mathcal{N}_0$. One of the nodes (conventionally one hosting a large gas producer) is selected as the reference node, it is indexed by 0, and its gas pressure is fixed to a known value p_0. The remaining nodes form the set $\mathcal{N} := \{1, \ldots, N\}$. The gas injection q_i at node $i \in \mathcal{N}_0$ is positive for an injection node, negative for a withdrawal node, and zero for network junctions. Without loss of generality, edges are assigned an arbitrary direction denoted by $\ell : (i, j) \in \mathcal{L}$ for $i, j \in \mathcal{N}_0$. The gas flow ϕ_ℓ on pipeline $\ell : (i, j) \in \mathcal{L}$ is positive when gas flows from node i to node j, and negative, otherwise. Conservation of mass implies that

$$q_i = \sum_{\ell:(i,j)\in\mathcal{L}} \phi_\ell - \sum_{\ell:(j,i)\in\mathcal{L}} \phi_\ell \tag{1}$$

for every $i \in \mathcal{N}_0$. Moreover, summing injections over all nodes should yield zero in steady state, that is $q_0 = -\sum_{i\in\mathcal{N}} q_i$. Therefore, given balanced $\{q_i\}_{i\in\mathcal{N}_0}$, model (1) provides N rather than $N + 1$ independent linear equations on $\{\phi_\ell\}_{\ell\in\mathcal{L}}$.

For high- and medium-pressure networks, the pressure drop and energy loss over pipelines are captured by a partial differential equation involving one spatial dimension along the pipeline length and the time dimension [7], [8]. After ignoring friction, any possible pipeline tilt, and assuming time-invariant gas injections, the partial differential equation simplifies to the *Weymouth equation* [9]:

$$p_i^2 - p_j^2 = a_\ell \phi_\ell |\phi_\ell| \tag{2}$$

characterizing the pressure difference across the endpoints of pipeline $\ell : (i, j) \in \mathcal{L}$. The parameter $a_\ell > 0$ in (2) depends on the physical properties of the pipeline [7]. The Weymouth equation asserts that gas pressure drops across a pipeline in the direction of gas flow. To be precise, the difference of squared pressures is proportional to the squared gas flow.

To avoid unacceptably low or high pressures, network operators install compressors at selected pipelines, henceforth referred to as *active pipelines* comprising the set $\mathcal{L}_a \subseteq \mathcal{L}$ with $L_a = |\mathcal{L}_a|$. A compressor amplifies the squared pressure between its input and output by a ratio α_ℓ. Suppose $\phi_\ell > 0$ and that the compressor along pipeline $\ell : (i, j) \in \mathcal{L}_a$ is located at normalized distances r_ℓ with $r_\ell \in [0, 1]$ from node i and $(1 - r_\ell)$ from node j. From the definition of the compression ratio, and upon applying (2) between node i and the compressor input, as well as between the compressor output and node j, it can be shown that [9]

$$\alpha_\ell p_i^2 - p_j^2 = c_\ell \phi_\ell |\phi_\ell| \tag{3}$$

where $c_\ell := a_\ell[1 - (1 - \alpha_\ell)r_\ell)] > 0$. The pressure drop described by (3) generalizes the Weymouth equation in (2), since it applies to active and non-active pipelines alike: simply set $r_\ell = 0$ and $\alpha_\ell = 1$ for non-active pipelines $\ell \notin \mathcal{L}_a$.

Given the reference pressure p_0, balanced nodal injections $\{q_i\}_{i\in\mathcal{N}_0}$, and the pipeline parameters $\{\alpha_\ell, c_\ell\}_{\ell\in\mathcal{L}}$, the *gas flow (GF) problem* aims at finding nodal pressures $\{p_i\}_{i\in\mathcal{N}}$ and pipeline flows $\{\phi_\ell\}_{\ell\in\mathcal{L}}$ satisfying the GF equations (1), (3), and $\{\phi_\ell \geq 0\}_{\ell\in\mathcal{L}_a}$. It hence implies solving $N + L$ equations over $N + L$ unknowns. Albeit (1) is linear, the generalized Weymouth equation in (3) is piecewise quadratic and not everywhere differentiable; the requirement $\{\phi_\ell \geq 0\}_{\ell\in\mathcal{L}_a}$ further complicates the task. The GF task is typically solved using the Newton-Raphson's scheme, which converges only if initialized sufficiently close to the solution [6].

To handle the non-differentiability of the absolute value in (3), introduce variable $\psi_\ell = |\phi_\ell|$ for all $\ell \in \mathcal{L}$; see also [4]. The latter equation can be equivalently written as $\psi_\ell^2 = \phi_\ell^2$ and $\psi_\ell \geq 0$ for all ℓ. To express the gas flow equations in a matrix-vector form, collect all nodal quantities excluding the reference bus in $\mathbf{p} := [p_1 \ \cdots \ p_N]^\top$ and $\mathbf{q} := [q_1 \ \cdots \ q_N]^\top$. Similarly, stack edge quantities in vectors $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, $\mathbf{c}$, and $\boldsymbol{\alpha}$.

The connectivity of the gas network graph is captured by the $L \times (N + 1)$ incidence matrix $\tilde{\mathbf{A}}$ with entries

$$\tilde{A}_{\ell,n} = \begin{cases} +1, & n = i \\ -1, & n = j \\ 0, & \text{otherwise} \end{cases} \qquad \forall\ \ell : (i, j) \in \mathcal{L}. \tag{4}$$

Isolating the first column corresponding to the reference node, matrix $\tilde{\mathbf{A}}$ can be partitioned as $\tilde{\mathbf{A}} = [\mathbf{a}_0 \ \mathbf{A}]$. Equation (1) is equivalent to $q_0 = \mathbf{a}_0^\top \boldsymbol{\phi}$ and $\mathbf{q} = \mathbf{A}^\top \boldsymbol{\phi}$. Because $\tilde{\mathbf{A}}\mathbf{1} = \mathbf{0}$, equation $q_0 = \mathbf{a}_0^\top \boldsymbol{\phi}$ can be ignored assuming balanced steady-state injections satisfying $q_0 = -\mathbf{1}^\top \mathbf{q}$. The GF problem can now be compactly expressed as [4]

$$\mathbf{A}^\top \boldsymbol{\phi} = \mathbf{q} \tag{5a}$$

$$\mathbf{B}(\mathbf{p} \odot \mathbf{p}) = \mathbf{c} \odot \boldsymbol{\phi} \odot \boldsymbol{\psi} - p_0^2 \mathbf{b}_0 \tag{5b}$$

$$\boldsymbol{\phi} \odot \boldsymbol{\phi} = \boldsymbol{\psi} \odot \boldsymbol{\psi} \tag{5c}$$

$$\boldsymbol{\psi} \geq \mathbf{0},\ \{\phi_\ell \geq 0\}_{\ell\in\mathcal{L}_a} \tag{5d}$$

where $\odot$ is the entry-wise product, while $\mathbf{B}$ and $\mathbf{b}_0$ are [4]

$$\mathbf{B} := \mathrm{dg}(\boldsymbol{\alpha})[\mathbf{A}]_+ - [-\mathbf{A}]_+ \tag{6a}$$

$$\mathbf{b}_0 := \mathrm{dg}(\boldsymbol{\alpha})[\mathbf{a}_0]_+ - [-\mathbf{a}_0]_+ \tag{6b}$$

where the operator $[x]_+ := \max\{0, x\}$ is applied entry-wise.

If the gas network is a tree ($L = N$), matrices $\mathbf{A}$ and $\mathbf{B}$ are square and invertible. In this case, the gas flows $\boldsymbol{\phi}$ can be readily found from (5a) and the nodal pressures $\boldsymbol{\beta}$ can be subsequently calculated through (5b). In practice though, natural gas networks exhibit a non-radial structure [9].

For a gas network without compressors, matrix $\mathrm{dg}(\boldsymbol{\alpha})$ becomes the identity matrix $\mathbf{I}_L$ and thus $\mathbf{B} = \mathbf{A}$ and $\mathbf{b}_0 = \mathbf{a}_0$. Under this setup, flows can be found as the minimizers of the convex optimization problem [5]

$$\min_{\boldsymbol{\phi}} \quad \sum_{\ell=1}^{L} \frac{a_\ell}{3} |\phi_\ell|^3 \tag{7a}$$

$$\text{s.to} \quad \mathbf{A}^\top \boldsymbol{\phi} = \mathbf{q}. \tag{7b}$$

Moreover, the related nodal pressures $\mathbf{p}$ can be recovered through the $(N + 1)$-length vector $\boldsymbol{\xi}$ of the optimal Lagrange multipliers corresponding to (7b). In detail, vector $\boldsymbol{\xi}$ can be shifted by a constant without loss of optimality. If this

constant is selected such that the first entry of $\boldsymbol{\xi}$ is p_0^2, the remaining entries of $\boldsymbol{\xi}$ are equal to the squared nodal pressures. This approach can be extended to meshed networks with compressors only if there are no active pipelines in loops [9].

III. Semidefinite Program Relaxation (SDR)

The GF problem entails finding the $2N + L$ unknowns $(\boldsymbol{\phi}, \boldsymbol{\psi}, \mathbf{p})$ through the N linear equations of (5a) and the $2L$ quadratic equations of (5b)–(5c) under the $L + L_a$ linear inequalities of (5d). Since the system involves non-homogeneous quadratic equations, it is hard to solve in general. Spurred by its success in tackling computational tasks involving quadratic constraints [10], we apply the powerful tool of semidefinite programming relaxation (SDR) to solve the GF problem.

To convert all functions to homogeneous quadratic ones [10], augment the unknown variables into the *system state vector* $\mathbf{x} := [\boldsymbol{\phi}^\top \ \boldsymbol{\psi}^\top \ \mathbf{p}^\top \ 1]^\top$ of length $K := 2L + N + 1$. Each equation in (5a)–(5c) is expressible as a homogeneous quadratic equality constraint on $\mathbf{x}$ as

$$\mathbf{x}^\top \mathbf{M}_k \mathbf{x} = s_k \tag{8}$$

where $\mathbf{M}_k \in \mathbb{S}^K$ and $s_k \in \mathbb{R}$ for $k = 1, \ldots, K-1$. Precisely, if equality k in (8) corresponds to:
(a) the i-th linear equality in (5a), then $s_k := q_i$ and

$$\mathbf{M}_k := \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{a}_i^\top & \mathbf{0}^\top & \mathbf{0}^\top & 0 \end{bmatrix}$$

where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$;
(b) the ℓ-th entry of (5b), then $s_k := -p_0^2 b_{0,\ell}$ and

$$\mathbf{M}_k := \frac{1}{2}\begin{bmatrix} \mathbf{0} & -c_\ell \operatorname{dg}(\mathbf{e}_\ell) & \mathbf{0} & \mathbf{0} \\ -c_\ell \operatorname{dg}(\mathbf{e}_\ell) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2\operatorname{dg}(\mathbf{b}_\ell) & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & 0 \end{bmatrix}$$

where $\mathbf{b}_\ell$ is the ℓ-th row of $\mathbf{B}$;
(c) the ℓ-th entry of (5c), then $s_k := 0$ and

$$\mathbf{M}_k := \frac{1}{2}\begin{bmatrix} \operatorname{dg}(\mathbf{e}_l) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\operatorname{dg}(\mathbf{e}_l) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & 0 \end{bmatrix}.$$

(d) To guarantee that the last entry of $\mathbf{x}$ is unity, introduce the additional constraint $x_K^2 = 1$. The latter can also be posed as in (8) by selecting $s_K := 1$ and $\mathbf{M}_K := \mathbf{e}_K \mathbf{e}_K^\top$. If x_K turns out to be -1, then vector $-\mathbf{x}$ is a GF solution in lieu of $\mathbf{x}$. Likewise, the linear inequalities in (5d) are written as $\mathbf{x}^\top \mathbf{N}_m \mathbf{x} \leq 0$ for appropriately defined matrices $\mathbf{N}_m \in \mathbb{S}^K$ for $m = 1, \ldots, L + L_a$.

Solving (5) can be now expressed as the feasibility problem:

$$\begin{aligned} \text{find} \quad & \mathbf{x} \\ \text{s.to} \quad & \mathbf{x}^\top \mathbf{M}_k \mathbf{x} = s_k, \quad k = 1, \ldots, K \\ & \mathbf{x}^\top \mathbf{N}_m \mathbf{x} \leq 0, \quad m = 1, \ldots, L + L_a. \end{aligned} \tag{9}$$

Nonetheless, solving (9) remains computationally hard since it entails quadratic (in)equalities. To tackle the non-convexity, we leverage the technique of semidefinite program relaxation (SDR) [10]. To that end, introduce the matrix variable $\mathbf{X} \in \mathbb{S}^K$ and upon enforcing $\mathbf{X} = \mathbf{x}\mathbf{x}^\top$, rewrite (9) as

$$\begin{aligned} \underset{\mathbf{X}=\mathbf{x}\mathbf{x}^\top}{\text{find}} \quad & (\mathbf{X}, \mathbf{x}) \\ \text{s.to} \quad & \operatorname{Tr}(\mathbf{M}_k \mathbf{X}) = s_k, \quad k = 1, \ldots, K \\ & \operatorname{Tr}(\mathbf{N}_m \mathbf{X}) \leq 0, \quad m = 1, \ldots, L + L_a. \end{aligned} \tag{10}$$

The constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^\top$ can be equivalently expressed as $\mathbf{X} \succeq \mathbf{0}$ and $\operatorname{rank}(\mathbf{X}) = 1$. By introducing these two constraints in (10), the original variable $\mathbf{x}$ can be eliminated. Also, the resultant feasibility problem can be transformed to a minimization by assigning the cost $\operatorname{Tr}(\mathbf{M}\mathbf{X})$ for some $\mathbf{M} \succeq \mathbf{0}$:

$$\begin{aligned} \min_{\mathbf{X}\succeq\mathbf{0}, \operatorname{rank}(\mathbf{X})=1} \quad & \operatorname{Tr}(\mathbf{M}\mathbf{X}) \\ \text{s.to} \quad & \operatorname{Tr}(\mathbf{M}_k \mathbf{X}) = s_k, \quad k = 1, \ldots, K \\ & \operatorname{Tr}(\mathbf{N}_m \mathbf{X}) \leq 0, \quad m = 1, \ldots L + L_a. \end{aligned} \tag{11}$$

Enforcing the rank constraint in (11) though is NP hard in general [10]. The SDR technique suggests relaxing the feasible set of (11) by dropping its rank constraint to get:

$$\min_{\mathbf{X}\succeq\mathbf{0}} \quad \operatorname{Tr}(\mathbf{M}\mathbf{X}) \tag{12a}$$

$$\text{s.to} \quad \operatorname{Tr}(\mathbf{M}_k \mathbf{X}) = s_k, \quad k = 1, \ldots, K \tag{12b}$$

$$\operatorname{Tr}(\mathbf{N}_m \mathbf{X}) \leq 0, \quad m = 1, \ldots, L + L_a \tag{12c}$$

that is a convex SDP problem. Due to the relaxation, the optimal value of (12) serves as a lower bound on the optimal value of (11). Moreover, if the minimizer $\hat{\mathbf{X}}$ of (12) turns out to be rank-1, then $\hat{\mathbf{X}}$ is feasible for the problem in (11) as well, the optimal values in (11) and (12) coincide, and therefore $\hat{\mathbf{X}}$ is a minimizer of the non-convex SDP in (11). In this case, the relaxation is deemed *exact*, and the sought solution $\hat{\mathbf{x}}$ to (10) is obtained by simply decomposing $\hat{\mathbf{X}}$ as $\hat{\mathbf{X}} = \hat{\mathbf{x}}\hat{\mathbf{x}}^\top$.

The existence of a rank-1 solution for (12) depends on the network parameters along with the specification vector $\mathbf{s} := [s_1 \ \cdots \ s_K]^\top$. On the other hand, the matrix $\mathbf{M}$ appearing in the cost of (12) offers multiple degrees of freedom. Inspired by the SDR in solving the power problem in [11], the next section selects $\mathbf{M}$ to favor a rank-1 minimizer of (12).

IV. Objective Function Design

To design matrix $\mathbf{M}$, consider the mapping $\mathbf{s}(\mathbf{x}) : \mathbb{R}^K \to \mathbb{R}^K$ whose k-th entry is $s_k(\mathbf{x}) := \mathbf{x}^\top \mathbf{M}_k \mathbf{x}$ for $k = 1, \ldots, K$. The associated Jacobian matrix evaluated at $\mathbf{x}$ is [11]

$$\mathbf{J}(\mathbf{x}) = 2[\mathbf{M}_1 \mathbf{x} \ \ldots \ \mathbf{M}_K \mathbf{x}]. \tag{13}$$

Since the mapping $\mathbf{s}(\mathbf{x})$ is continuous, the inverse function theorem asserts that $\mathbf{s}(\mathbf{x})$ is invertible close to $\mathbf{x}$ if $\mathbf{J}(\mathbf{x})$ is invertible. By the definition of $\mathbf{s}(\mathbf{x})$, the Jacobian matrix $\mathbf{J}(\mathbf{x})$ is sparse for all $\mathbf{x}$. However, the invertibility of $\mathbf{J}(\mathbf{x})$ depends on $\mathbf{x}$, and it thus hard to characterize. For this reason, we resort to studying the generic rank of $\mathbf{J}(\mathbf{x})$, which is the maximal rank over all possible values for the non-zero entries of $\mathbf{J}(\mathbf{x})$.

Proposition 1. *The Jacobian matrix* $\mathbf{J}(\mathbf{x})$ *associated with the mapping* $\mathbf{s}(\mathbf{x})$ *is full-rank in general.*

Proof of Prop. 1: The generic rank of $\mathbf{J}(\mathbf{x})$ is characterized leveraging a result from [12]: Given an $N \times N$ matrix $\mathbf{C}$, construct a graph with $2N$ nodes. Nodes $\{v_i\}_{i=1}^N$ correspond to the rows of $\mathbf{C}$ and nodes $\{u_j\}_{j=1}^N$ to its columns. An edge is drawn between nodes v_i and u_j only if $C_{ij} \neq 0$. If each node u_j can be matched to a different node v_i, then $\mathbf{C}$ is full-rank in general. To apply this result to matrix $\mathbf{J}(\mathbf{x})$, let us capture its sparsity pattern by the binary matrix

$$\mathbf{J}_b = \begin{bmatrix} |\mathbf{A}| & \mathbf{I}_L & \mathbf{I}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_L & \mathbf{I}_L & \mathbf{0} \\ \mathbf{0} & |\mathbf{A}|^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & 1 \end{bmatrix}$$

where $|\mathbf{A}|$ contains the absolute values of $\mathbf{A}$. The last column of $\mathbf{J}_b$ can be matched to its last row by the entry 1. Its third block column can be matched to its second block row due to $\mathbf{I}_L$. The remaining entries of $\mathbf{J}_b$ form the matrix

$$\tilde{\mathbf{J}}_b = \begin{bmatrix} |\mathbf{A}| & \mathbf{I}_L \\ \mathbf{0} & |\mathbf{A}|^\top \end{bmatrix}.$$

Consider a spanning tree $\mathcal{T}$ on the graph $\mathcal{G}$ representing the gas network. Without loss of generality, pipelines can be renumbered so that $\mathbf{A}$ is partitioned as $\mathbf{A} = [\mathbf{A}_\mathcal{T}^\top \ \mathbf{A}_{\overline{\mathcal{T}}}^\top]^\top$, where the $N \times N$ submatrix $\mathbf{A}_\mathcal{T}$ corresponds to the pipelines comprising $\mathcal{T}$, and the $(L-N) \times N$ submatrix $\mathbf{A}_{\overline{\mathcal{T}}}$ to the remaining ones. Matrix $\tilde{\mathbf{J}}_b$ can be then partitioned as

$$\tilde{\mathbf{J}}_b = \begin{bmatrix} |\mathbf{A}_\mathcal{T}| & \mathbf{I}_N & \mathbf{0} \\ |\mathbf{A}_{\overline{\mathcal{T}}}| & \mathbf{0} & \mathbf{I}_{L-N} \\ \mathbf{0} & |\mathbf{A}_\mathcal{T}|^\top & |\mathbf{A}_{\overline{\mathcal{T}}}|^\top \end{bmatrix}.$$

The third block column of $\tilde{\mathbf{J}}_b$ can be matched to its second block row due to $\mathbf{I}_{L-N}$. Each column of $|\mathbf{A}_\mathcal{T}|$ corresponds to a node in $\mathcal{N}$. Since every such node is the destination end of a pipeline in $\mathcal{T}$, every column of $|\mathbf{A}_\mathcal{T}|$ is guaranteed to have an 1 entry at a different row. Hence, the first block column of $\hat{\mathbf{J}}_b$ is matched to its first block row. Likewise, the third block row of $\hat{\mathbf{J}}_b$ is matched to its second block column. Therefore, a perfect matching between the rows and columns of $\mathbf{J}(\mathbf{x})$ has been obtained. ∎

Proposition 1, proved in the Appendix, guarantees only the generic invertibility of $\mathbf{J}(\mathbf{x})$. Nevertheless, all the Jacobian matrices evaluated during our numerical tests were invertible.

Let us next derive the dual problem of (12): let $\boldsymbol{\lambda} \in \mathbb{R}^K$ and $\boldsymbol{\mu} \in \mathbb{R}^{2L}$ be the Lagrange multipliers corresponding to (12b) and (12c), respectively. The dual problem of (12) is

$$\max_{\boldsymbol{\lambda}, \boldsymbol{\mu} \geq \mathbf{0}} \quad -\boldsymbol{\lambda}^\top \mathbf{s} \tag{14}$$
$$\text{s.to} \quad \mathbf{Z}(\boldsymbol{\lambda}, \boldsymbol{\mu}) := \mathbf{M} + \sum_{k=1}^K \lambda_k \mathbf{M}_k + \sum_{m=1}^{2L} \mu_m \mathbf{N}_m \succeq \mathbf{0}.$$

Let function $h_2(\mathbf{Z})$ be the sum of the two smallest eigenvalues of $\mathbf{Z}$, which is known to be concave over $\mathbb{S}_+^K$ [13]. The next result provides conditions for $\mathbf{M}$ to yield an exact relaxation for a state vector $\hat{\mathbf{x}}$ [11]:

Proposition 2. *Consider a gas network state* $\hat{\mathbf{x}}$ *with specifications* $\hat{\mathbf{s}} := \mathbf{s}(\hat{\mathbf{x}})$ *and invertible Jacobian matrix* $\mathbf{J}(\hat{\mathbf{x}})$*. If there exists a vector* $\hat{\boldsymbol{\lambda}}$ *such that:*
(c1) $\mathbf{M}\hat{\mathbf{x}} + \frac{1}{2}\mathbf{J}(\hat{\mathbf{x}})\hat{\boldsymbol{\lambda}} = \mathbf{0}$*;*
(c2) $\hat{\mathbf{Z}} := \mathbf{Z}(\hat{\boldsymbol{\lambda}}, \mathbf{0}) \succeq \mathbf{0}$*; and*
(c3) $h_2(\hat{\mathbf{Z}}) \geq \epsilon$ *for an* $\epsilon > 0$*;*
then $\hat{\mathbf{X}} := \hat{\mathbf{x}}\hat{\mathbf{x}}^\top$ *is a minimizer of* (11) *for specifications* $\hat{\mathbf{s}}$*.*

Proof of Prop. 2: The cost in (12) is bounded below by zero. Assuming there exists feasible $\mathbf{X} \succ \mathbf{0}$, strong duality holds between (12) and its dual problem in (14), and the Karush-Kuhn-Tucker (KKT) conditions apply [14, Lemma 1]. Let $\hat{s}_k$ be the k-th entry of $\hat{\mathbf{s}}$. Since $\hat{\mathbf{x}}$ is a realizable state, $\hat{\mathbf{x}}^\top \mathbf{M}_k \hat{\mathbf{x}} = \hat{s}_k$ for $k = 1, \ldots, K$, and $\hat{\mathbf{x}}^\top \mathbf{N}_m \hat{\mathbf{x}} \leq 0$ for $m = 1, \ldots, 2L$. It follows that $\mathrm{Tr}(\mathbf{M}_k \hat{\mathbf{X}}) = \hat{s}_k$ and $\mathrm{Tr}(\mathbf{N}_m \hat{\mathbf{X}}) \leq 0$ for all k and m, i.e., $\hat{\mathbf{X}} \succeq \mathbf{0}$ is primal feasible.

By (c2), matrix $\hat{\mathbf{Z}}$ is dual feasible. It further holds that:

$$\hat{\mathbf{Z}}\hat{\mathbf{x}} = \mathbf{M}\hat{\mathbf{x}} + \sum_{k=1}^K \lambda_k \mathbf{M}_k \hat{\mathbf{x}} = \mathbf{M}\hat{\mathbf{x}} + \mathbf{J}(\hat{\mathbf{x}})\hat{\boldsymbol{\lambda}} = \mathbf{0}$$

where the second equality follows from (13) and the third one from (c1). Then, it holds that $\hat{\mathbf{Z}}\hat{\mathbf{X}} = \hat{\mathbf{Z}}\hat{\mathbf{x}}\hat{\mathbf{x}}^\top = \mathbf{0}$. Because the $\{\hat{\mu}_m\}_{m=1}^{2L}$ related to $\hat{\boldsymbol{\lambda}}$ and $\hat{\mathbf{Z}}$ have been set to zero, $\hat{\mu}_m \mathrm{Tr}(\mathbf{N}_m \hat{\mathbf{X}}) = 0$ for all m. Since $(\hat{\mathbf{X}}, \hat{\mathbf{Z}})$ satisfy complementary slackness and all KKT conditions, they are primal-dual optimal, and $\hat{\mathbf{x}}$ is a minimizer of (12).

Because $\hat{\mathbf{Z}}\hat{\mathbf{x}} = \mathbf{0}$ and $\hat{\mathbf{Z}} \succeq \mathbf{0}$, the smallest eigenvalue of $\hat{\mathbf{Z}}$ is zero. From (c3), the second smallest eigenvalue of $\hat{\mathbf{Z}}$ is strictly positive, and therefore $\mathrm{rank}(\hat{\mathbf{Z}}) = K - 1$. Complementary slackness asserts that every minimizer $\tilde{\mathbf{X}}$ of (12) satisfies $\hat{\mathbf{Z}}\tilde{\mathbf{X}} = \mathbf{0}$. The latter implies that $\mathrm{rank}(\tilde{\mathbf{X}}) \leq 1$; see [14]. Since $\tilde{\mathbf{X}} = \mathbf{0}$ is not feasible, all minimizers of (12) have to be rank-one, i.e., they are of the form $\tilde{\mathbf{X}} = \tilde{\mathbf{x}}\tilde{\mathbf{x}}^\top$ for some $\tilde{\mathbf{x}}$. Because $\mathbf{J}(\hat{\mathbf{x}})$ is invertible, (c1) defines a single $\hat{\boldsymbol{\lambda}}$; hence $\hat{\mathbf{Z}}$ is unique and its nullspace is spanned solely by $\hat{\mathbf{x}}$ and therefore $\hat{\mathbf{X}}$ is the unique rank-one minimizer of (11). ∎

Proposition 2 asserts that if $\mathbf{M}$ satisfies (c1)–(c3), then the relaxation in (12) is exact for a given system state $\hat{\mathbf{x}}$. This fact may seem to be of limited interest, since it guarantees the success of SDR only for this particular $\hat{\mathbf{x}}$. Nevertheless, the continuity argument adopted in [11, Th. 2] on the exactness of SDR for the power flow problem applies here too. Therefore, it follows that the relaxation in (12) is exact for all realizable states $\mathbf{x}$ with an invertible $\mathbf{J}(\mathbf{x})$ lying in a ball around $\hat{\mathbf{x}}$.

To design $\mathbf{M}$ such that the SDR is exact over a wider range of system states, conditions (c1)–(c3) can be enforced for multiple $\mathbf{x}_i$'s. These states could reflect representative gas flow patterns selected upon historical data. Designing $\mathbf{M}$ satisfying the conditions of Proposition 2 for $\{\mathbf{x}_i\}_{i=1}^R$ can be posed as the feasibility problem for some $\epsilon > 0$ [11]

$$\text{find} \quad (\mathbf{M}, \{\boldsymbol{\lambda}_i\}) \tag{15a}$$
$$\text{s.to} \quad \mathbf{M} \succeq \mathbf{0} \tag{15b}$$

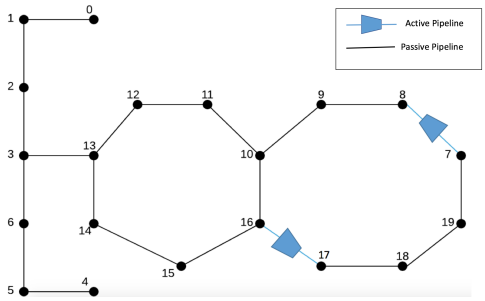

Fig. 1. Modified Belgian gas network [15].

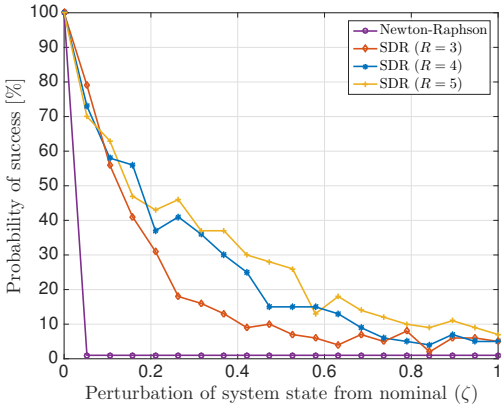

Fig. 2. Probability of success for Newton-Raphson and the SDR schemes.

$$\mathbf{M}\mathbf{x}_i + \tfrac{1}{2}\mathbf{J}(\mathbf{x}_i)\boldsymbol{\lambda}_i = \mathbf{0}, \qquad \forall\, i \qquad \text{(15c)}$$

$$\mathbf{Z}(\boldsymbol{\lambda}_i, \mathbf{0}) \succeq \mathbf{0}, \qquad \forall\, i \qquad \text{(15d)}$$

$$h_2(\mathbf{Z}(\boldsymbol{\lambda}_i, \mathbf{0})) \geq \epsilon, \qquad \forall\, i. \qquad \text{(15e)}$$

Problem (15) can actually be solved as an SDP minimization [13, pp. 67–68].

V. Numerical Tests

The developed SDR approach was tested on a modified version of the Belgian natural gas network [15]. Since the original network has been simplified to a tree in [15], we added two pipelines between 15-16 and 7-19. Matrix $\mathbf{M}$ was first designed by solving (15) for $\epsilon = 0.1$. The R system states $\{\mathbf{x}_i\}_{i=1}^{R}$ were generated as follows: Upon selecting nominal pressures, gas injections and flows were calculated from (5a)–(5b). Nominal pressures were then perturbed by a zero-mean Gaussian variable with standard deviation of 0.25% the nominal pressure to generate $(R-1)$ additional pressure and state vectors $\{\mathbf{p}_i\}_{i=2}^{R}$. The efficacy of the SDR-based GF solver was tested over randomly generated gas states. These validation states were constructed via a process similar to the previous one; yet now the perturbation was set to $\zeta 0.25\%$ the nominal pressures with ζ ranging from 0 to 1 in increments of 0.05. For each ζ, 200 system states were generated.

The same GF problems were solved using the Newton-Raphson iterates initialized at the actual state perturbed by ζ. The probability of the Newton-Raphson iterates converging to the actual state and the probability of exact SDR are plotted in Fig. 2. As evidenced, the developed SDR scheme operates over a wider range of network conditions and its performance improves for increasing R.

VI. Conclusions

Solving the GF equations governing routinely the operation of natural gas networks has been posed as a set of quadratic equalities and inequalities. The latter has been reformulated to a feasibility problem that is also computationally intractable, and it later relaxed to an SDP minimization upon dropping the rank constraints. The objective function has been judiciously designed so that the relaxation is successful around prespecified states of possible interest. The advantages of the method over the classic Newton-Rapshon approach were demonstrated using a Belgian natural gas network. Spurred by these promising results, our current work targets GF solvers scalable to thousands of nodes and pipelines by exploiting the sparse problem structure and distributed implementations.

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