Minimax Hypothesis Testing

$H_0 : Y \sim f_0$

vs.

$H_1 : Y \sim f_1$

$[c_{i,j}]$

$$R_0(\delta) = c_{00} P_0(\Gamma_0) + c_{10} P_0(\Gamma_1)$$
$$= c_{00}[1 - P_{FA}] + c_{10} P_{FA}$$

$$R_1(\delta) = c_{11} P_1(\Gamma_1) + c_{01} P_1(\Gamma_0)$$
$$= c_{11} P_D + c_{01}[1 - P_D]$$

Suppose π_0, π_1 unknown.

- Bayes risk no longer an acceptable criterion.
- Instead seek a d.r. that minimizes over all δ the maximum of conditional risks

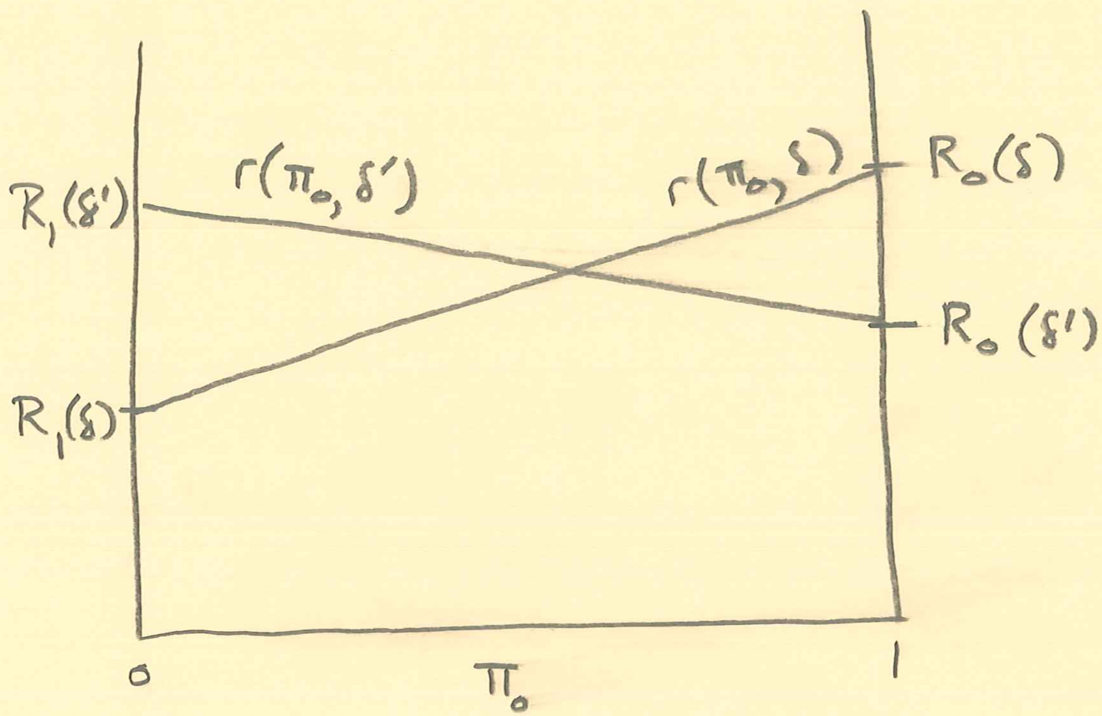
$$\min_{\delta} \max\{R_0(\delta), R_1(\delta)\}$$

Solution to this is called a minimax test.

For an arbitrary d.r. δ <u>define</u>

$$r(\pi_0, \delta) = \pi_0 R_0(\delta) + (1-\pi_0) R_1(\delta) \qquad 0 \le \pi_0 \le 1$$

This is a straight line as a function of π_0.



Notes:

For fixed δ the max of $r(\pi_0, \delta)$ is at the end...

$$\max_{0 \le \pi_0 \le 1} r(\pi_0, \delta) = \max\{R_0(\delta), R_1(\delta)\}.$$

So the minimax prob. becomes

$$\min_{\delta} \max_{0 \le \pi_0 \le 1} r(\pi_0, \delta)$$

which is more convenient.

Definitions

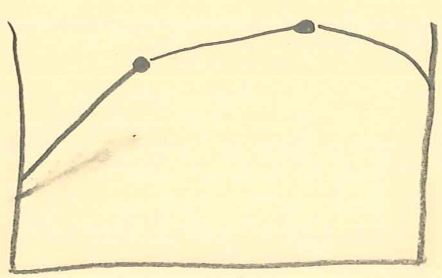
For each π_0 in $[0,1]$ let δ_{π_0} be a Bayes rule.

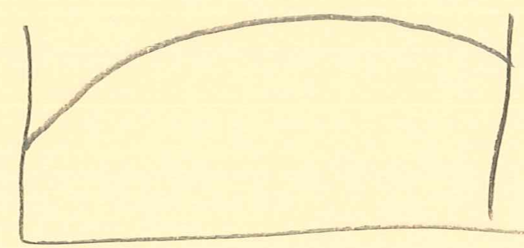
and let

$$V(\pi_0) = r(\pi_0, \delta_{\pi_0})$$ .... the minimum Bayes risk for π_0.

ie $$V(\pi_0) = \min_{\delta} r(\pi_0, \delta)$$

Fact The min Bayes risk function $V(\pi)$ is concave & continuous on $[0,1]$ and $V(0) = c_{11}$ & $V(1) = c_{00}$.



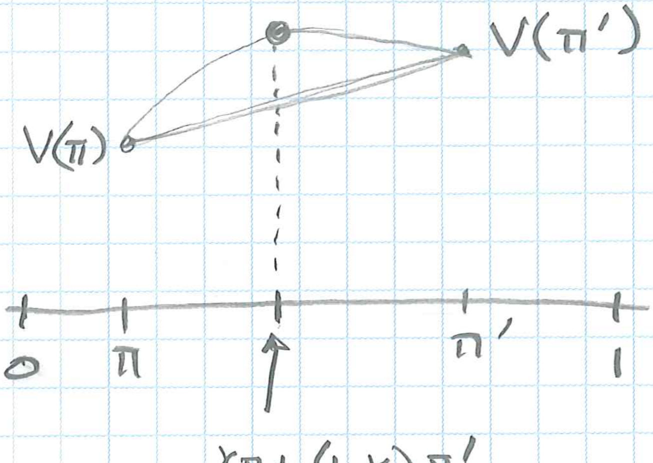


Useful Concave/Convex Facts

V concave on $[0, 1]$ ie for any 2 pts $\pi, \pi' \in [0,1]$

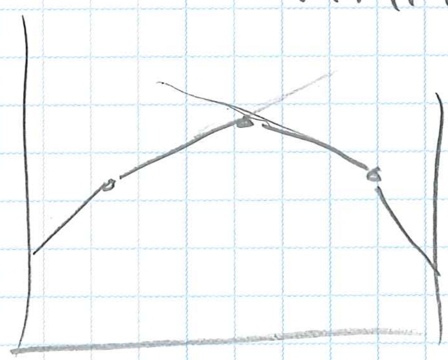
$$V(\gamma\pi + (1-\gamma)\pi') \geq \gamma V(\pi) + (1-\gamma)V(\pi')$$

where $0 \leq \gamma \leq 1$.



For such functions:

1) V is continuous on $(0,1)$
2) Left and right derivatives exist at each point in $(0,1)$.
3) The derivative of V exists except at a countable set.



Summarizing ...

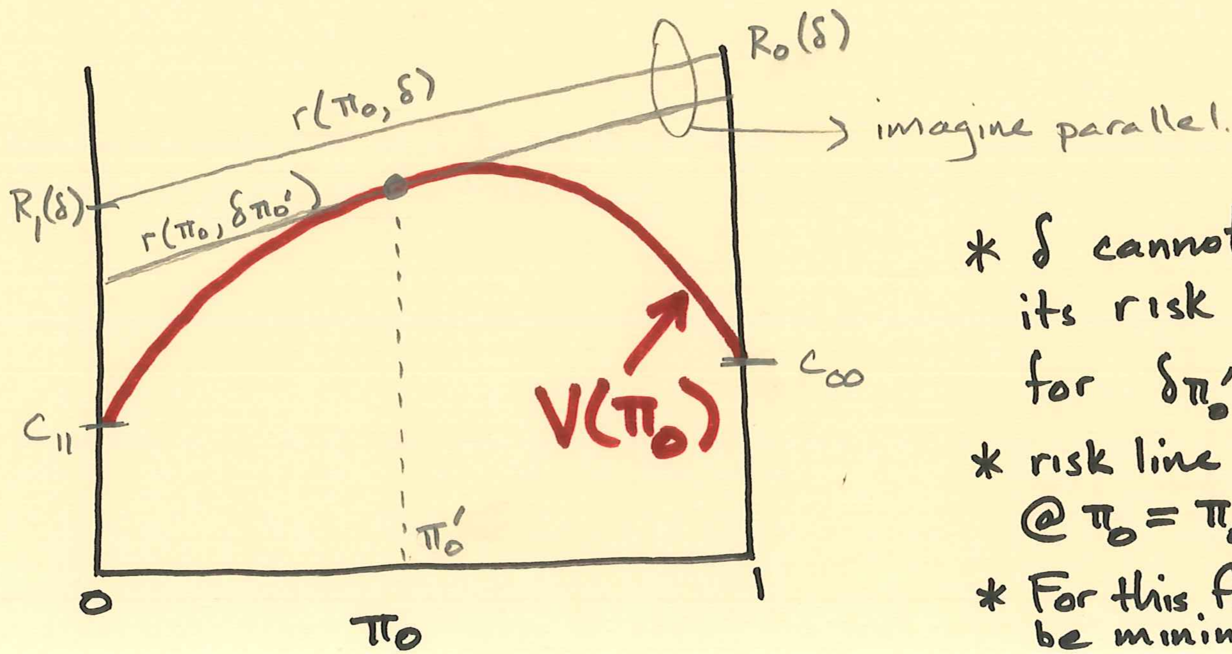
A minimax d.r. satisfies

$$\min_{\delta} \max \{R_0(\delta), R_1(\delta)\} = \min_{\delta} \max_{0 \le \pi_0 \le 1} r(\pi_0, \delta)$$

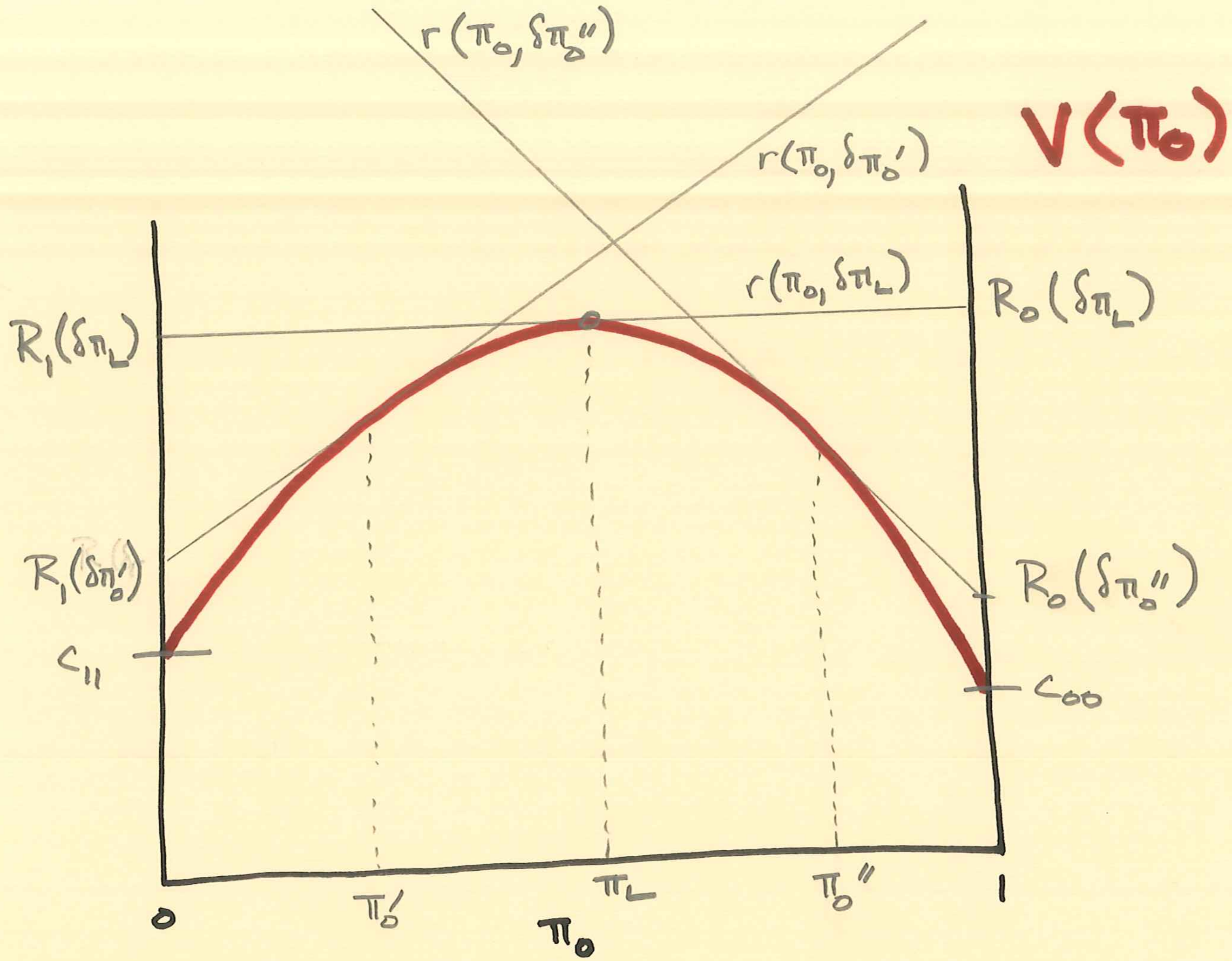
Minimum Bayes Risk Function

$$V(\pi_0) = \min_{\delta} r(\pi_0, \delta) = r(\pi_0, \delta_{\pi_0})$$

δ_{π_0} denotes the Bayes rule for π_0.



* δ cannot be a minimax rule because its risk line is completely above that for $\delta_{\pi_0'}$
* risk line for $\delta_{\pi_0'}$ must touch $V(\pi_0)$ @ $\pi_0 = \pi_0'$.
* For this figure only Bayes rules can be minimax.



The Minimax Test

Suppose π_L is a solution to $V(\pi_L) = \max_{0 \le \pi \le 1} V(\pi)$.

Suppose also that either $\pi_L = 0$ or $\pi_L = 1$ or

$R_1(\delta\pi_L) = R_0(\delta\pi_L)$

Then $\delta\pi_L$ is a minimax rule.

$\pi_L \triangleq$ least favorable prior

a d.r. with equal cond. risks $\triangleq$ an equalizer rule.

Example $H_0 : Y \sim N(\mu_0, \sigma^2)$ vs $H_1 : Y \sim N(\mu_1, \sigma^2)$ $\qquad \mu_1 > \mu_0$

Want $V(\pi_0)$. Do this for unif. costs $c_{00} = c_{11} = 0$, $c_{01} = c_{10} = 1$.

$$\Gamma_1 = \left\{ y \in \Gamma : y \geq \tau' = \frac{\sigma^2}{\mu_1 - \mu_0} \log\left(\frac{\pi_0}{1-\pi_0}\right) + \frac{\mu_0 + \mu_1}{2} \right\}$$

$$V(\pi_0) = \pi_0 P_0(\Gamma_1) + \pi_1 P_1(\Gamma_0)$$

$$= \pi_0 \left[1 - \Phi\left(\frac{\tau' - \mu_0}{\sigma}\right) \right] + (1 - \pi_0)\, \Phi\left(\frac{\tau' - \mu_1}{\sigma}\right)$$

Φ cdf of $N(0,1)$

For this problem $V(0) = c_{11} = 0$, $V(1) = c_{00} = 0$.

$\Longrightarrow$ $V(\pi_0)$ must have its max at some π_L aka least fav. prior in $(0,1)$.

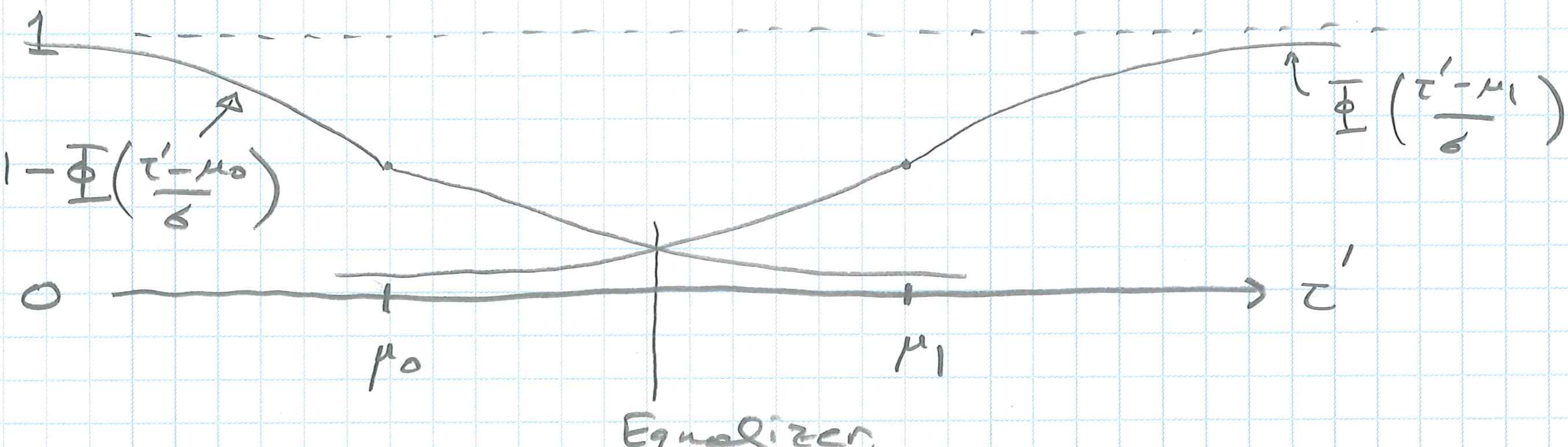
$\Longrightarrow$ $V(\pi_0)$ is nice & differentiable in $(0,1) \Longrightarrow$ No randomization needed.

All need is to find equalizer

$$R_0(\delta_{\pi_L}) = R_1(\delta_{\pi_L})$$

$$1 - \Phi\left(\frac{\tau' - \mu_0}{\sigma}\right) = \Phi\left(\frac{\tau' - \mu_1}{\sigma}\right).$$

→ Solve for τ'.



$$\tau' = \frac{\mu_0 + \mu_1}{2}. \longrightarrow \pi_L = 1/2$$

$$V(1/2) = 1 - \Phi\left(\frac{\mu_1 - \mu_0}{2\sigma}\right).$$

1 Composite Hypothesis Testing

- Previously had the case where under each hypothesis there was only one possible distribution for the observation:

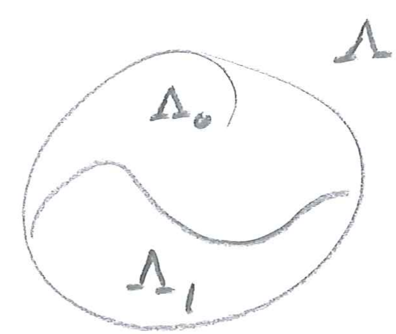
$$\begin{array}{l} H_0: \; Y \sim P_0 \\ \text{vs.} \\ H_1: \; Y \sim P_1 \end{array}.$$

This is said to be a simple hypothesis test.

- Now consider the case where under each hypothesis there are many possible distributions for the observation Y. Hypotheses of this type are known as composite hypotheses.

- A family $\{P_\theta : \theta \in \Lambda\}$ of probability distributions on the observation space Γ. Here parameter set is a disjoint union $\Lambda = \Lambda_0 \cup \Lambda_1$ and the two hypotheses are

$$\begin{array}{l} H_0: \; Y \sim P_\theta, \; \theta \in \Lambda_0 \\ \text{vs.} \\ H_1: \; Y \sim P_\theta, \; \theta \in \Lambda_1 \end{array}.$$



- Consider two approaches: Bayesian and non-Bayesian.

Start with this.