(Feb 1, 2021)

Randomized Decision Rules

δ_{NP} solves NP if $\delta_{NP} = \arg\max_{\delta} P_D(\delta)$ subject to $P_F(\delta) \le \alpha$

called signif. level.

δ is a randomized d.r.

$$\delta : \Gamma \longrightarrow [0, 1] \qquad \delta(y) \triangleq P(\to H_1 \mid Y=y)$$

$\Longrightarrow$

$$P_F(\delta) \triangleq E_0\{\delta(Y)\} = \int_\Gamma \delta(y) f_0(y)\, dy$$

$$P_D(\delta) \triangleq E_1\{\delta(Y)\} = \int_\Gamma \delta(y) f_1(y)\, dy.$$

Neyman-Pearson Lemma

Consider the hypothesis pair with indicated pdfs

$$H_0 : \quad Y \sim f_0$$
$$\text{vs.}$$
$$H_1 : \quad Y \sim f_1$$

and suppose that $\alpha > 0$. Then the following statements are true.

1. (Optimality) Let δ be any decision rule satisfying $P_F(\delta) \leq \alpha$ and let δ' be any decision rule of the form

$$\delta'(y) = \begin{cases} 1 & \text{if } f_1(y) > \eta f_0(y) \\ \gamma(y) & \text{if } f_1(y) = \eta f_0(y) \\ 0 & \text{if } f_1(y) < \eta f_0(y) \end{cases} \tag{1}$$

   where $\eta \geq 0$ and $0 \leq \gamma(y) \leq 1$ are such that $P_F(\delta') = \alpha$. Then

$$P_D(\delta') \geq P_D(\delta).$$

   That is, any size α decision rule of the form of Eq. (1) is a Neyman-Pearson rule.

2. (Existence) For every α s.t. $0 < \alpha < 1$ there is a decision rule δ_{NP} of the form of Eq. (1) with $\gamma(y) = \gamma_0$ (a constant) for which $P_F(\delta_{NP}) = \alpha$.

3. (Uniqueness) Suppose that δ'' is any α-level Neyman-Pearson decision rule for H_0 vs. H_1. Then δ'' must be of the form of Eq. (1) except possibly on a subset of Γ having zero probability under H_0 and H_1.

Proof: NP Lemma #1 Optimality

δ is any decision rule st $P_F(\delta) \le \alpha$ and δ' a decision rule of the form of Eq. (1) with $P_F(\delta') = \alpha$.

Claim $\quad [\delta'(y) - \delta(y)][f_1(y) - \eta f_0(y)] \geq 0 \quad \forall\, y \in \Gamma$

Proof of Claim

Let $\Gamma = \Gamma_+ \cup \Gamma_{zero} \cup \Gamma_-$ a disjoint union st

$$\Gamma_+ = \{y \in \Gamma : f_1(y) > \eta f_0(y)\}$$

$$\Gamma_{zero} = \{y \in \Gamma : f_1(y) = \eta f_0(y)\}$$

$$\Gamma_- = \{y \in \Gamma : f_1(y) < \eta f_0(y)\}$$

Easy to see that Claim is true for $y \in \Gamma_{zero}$ since the form is equal to zero on that set.

Now say $y \in \Gamma_+$ whence $f_1(y) > \eta f_0(y)$ ie. $f_1(y) - \eta f_0(y) > 0$. On this set

$$\delta'(y) = 1 \geq \delta(y)$$ for any legitimate decision rule. (because drs must sat $0 \le \delta \le 1$)

Therefore,

$$y \in \Gamma_+ \implies [\delta'(y) - \delta(y)][f_1(y) - \eta f_0(y)] \geq 0$$

Now say $y \in \Gamma_-$ whence $f_1(y) - \eta f_0(y) < 0$. On this set

$\delta'(y) = 0 \leq \delta(y)$ for any legitimate decision rule (since a dr satisfies $0 \leq \delta \leq 1$).

Therefore $\delta'(y) - \delta(y) \leq 0$ and

$$y \in \Gamma_- \implies [\delta'(y) - \delta(y)][f_1(y) - \eta f_0(y)] \geq 0$$

(Hence claim is true).

Then integrating the expression in the "claim" we have

$$\int_\Gamma [\delta'(y) - \delta(y)][f_1(y) - \eta f_0(y)]\, dy \geq 0$$

$$\implies \underbrace{\int_\Gamma \delta'(y) f_1(y)\, dy}_{P_D(\delta')} - \underbrace{\int_\Gamma \delta(y) f_1(y)\, dy}_{P_D(\delta)} \geq \eta \underbrace{\int_\Gamma \delta'(y) f_0(y)\, dy}_{P_F(\delta')} - \eta \underbrace{\int_\Gamma \delta(y) f_0(y)\, dy}_{P_F(\delta)}$$

$$\Rightarrow P_D(\delta') - P_D(\delta) \geq \eta\left[P_F(\delta') - P_F(\delta)\right] = \eta\left[\alpha - P_F(\delta)\right] \geq 0$$

since $P_F(\delta) \leq \alpha$

$\Rightarrow P_D(\delta') \geq P_D(\delta)$ as was to be shown.

Proof: NP Lemma #2 Existence

Here the desired test is constructed by examination of the complementary cdf of the r.v.

$$L(Y) = f_1(Y) / f_0(Y)$$

However, since it is constructed as a ratio of pdfs we need to do a little work to make sure it is well-defined.

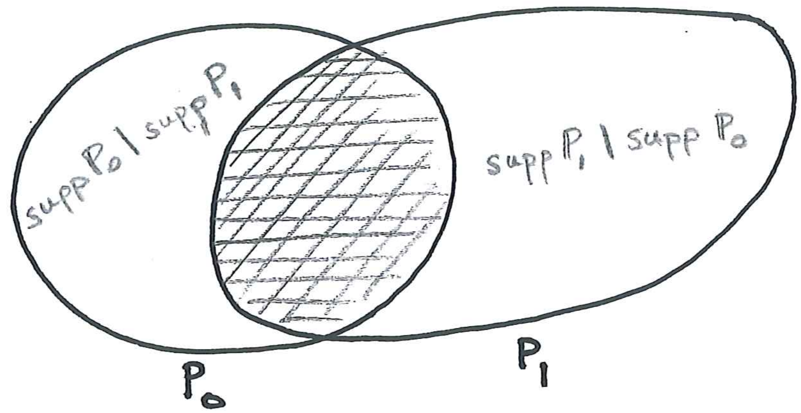
Define $\text{supp } P_i = \{ y \in \Gamma : f_i(y) > 0 \}$

Note that it is no loss of generality to assume

$$\Gamma = \text{supp } P_0 \cup \text{supp } P_1$$

Let the level satisfy $0 < \alpha < 1$. Examine the points $y \in \Gamma$ by

$$\Gamma = \left(\operatorname{supp} P_1 \setminus \operatorname{supp} P_0\right) \cup \left(\operatorname{supp} P_1 \cap \operatorname{supp} P_0\right) \cup \left(\operatorname{supp} P_0 \setminus \operatorname{supp} P_1\right)$$



If

$$y \in \operatorname{supp} P_1 \setminus \operatorname{supp} P_0 \implies L(y) \triangleq +\infty$$

$$y \in \operatorname{supp} P_1 \cap \operatorname{supp} P_0 \implies L(y) = f_1(y)/f_0(y)$$

$$y \in \operatorname{supp} P_0 \setminus \operatorname{supp} P_1 \implies L(y) = 0$$

$\therefore$ $L(Y)$ is a well-defined r.v. so long as we accept the possibility of such extended values.

We need to construct a test δ_{NP} of the required form and st. $P_F(\delta_{NP}) = \alpha$.

If $y \in \text{supp}\, P_1 \setminus \text{supp}\, P_0$ then define $\delta_{NP}(y) = 1$. On this set

$$f_1(y) > \eta f_0(y) = 0$$

for any value of η.

If $y \in \text{supp}\, P_0 \setminus \text{supp}\, P_1$ then define $\delta_{NP}(y) = 0$. On this set

$$0 = f_1(y) < \eta f_0(y)$$

for any positive value of η.

Finally we consider the points $y \in \text{supp}\, P_0 \supset (\text{supp} P_0 \cap \text{supp}\, P_1)$ where we need to show how to pick the parameters in Eq(1), i.e., how to pick $\gamma(y)$ and $\eta \geq 0$ st. $P_F(\delta_{NP}) = \alpha$.

Note: Under hypoth. $H_0 \Longleftrightarrow$ dist. P_0 holds, the r.v. $L(Y)$ is finite with probability one (hence don't have to worry about extended values)

Therefore its cdf and complementary cdf wrt P_0 have the usual properties (right continuous, left-hand limits exist, limits for large and small values behave). Say

$$g(\eta) = P_0\big(L(Y) > \eta\big) = P_0\big(f_1(Y) > \eta f_0(Y)\big)$$

be the complementary cdf.

And cdf is strictly monotone increasing.

$\Updownarrow$

complementary cdf is strictly monotone decreasing

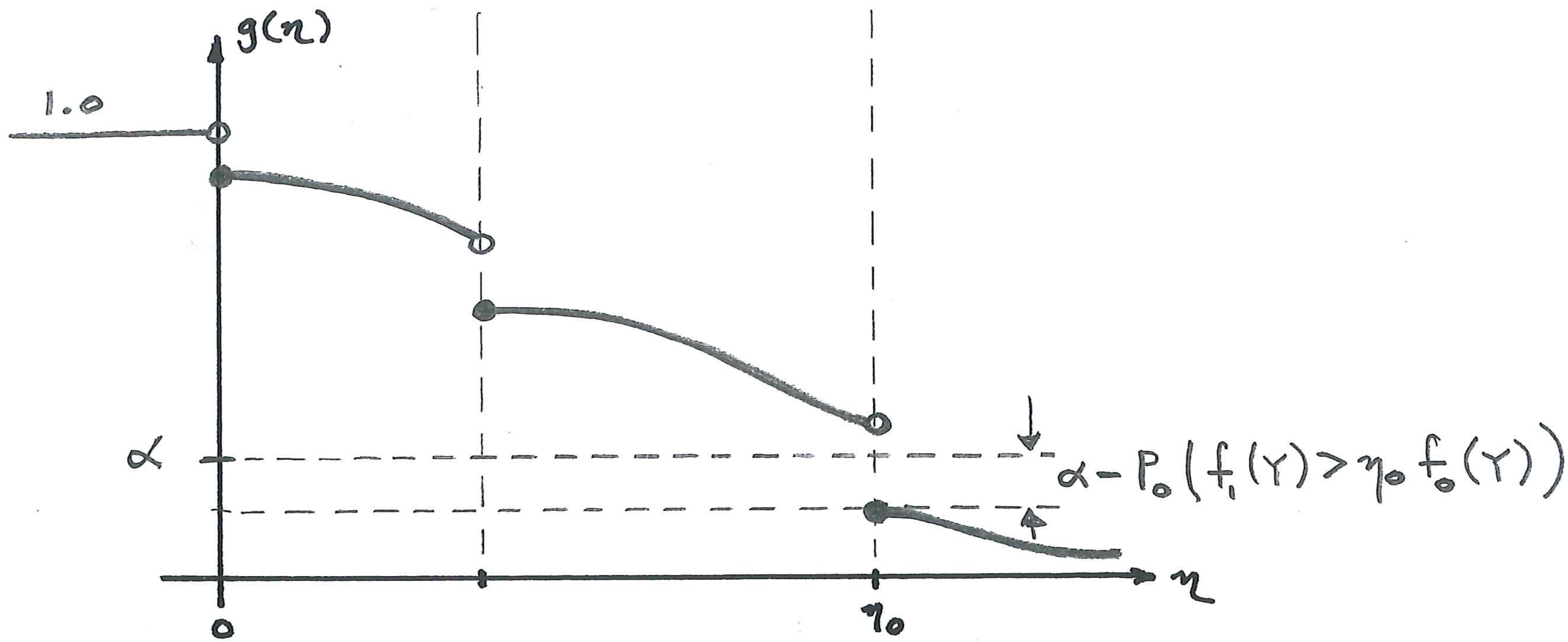
$g(\eta)$ right continuous, left-hand limits exist

$$\lim_{\eta \to \infty} g(\eta) = 0$$

$$\lim_{\eta \to 0^-} g(\eta) = 1 \quad (\text{because } L(Y) \text{ is a } \geq 0 \text{ r.v.})$$

Note: It is possible that $g(0) < 1$.

A generic complementary cdf under P_0 might look like:



Two cases to consider for an α s.t. $0 < \alpha < 1$. Case 1: $\exists!\, \eta_\alpha$ s.t. $g(\eta_\alpha) = \alpha$ and Case 2: $\nexists\, \eta_\alpha$ s.t. $g(\eta_\alpha) = \alpha$.

Case 1

Let η_α be the ! threshold s.t. $g(\eta_\alpha) = \alpha = P_0\big(f_1(Y) > \eta_\alpha f_0(Y)\big)$ and let δ_{NP} be

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } f_1(y) \geq \eta_\alpha f_0(y) \\ 0 & \phantom{\text{if }} f_1(y) < \eta_\alpha f_0(y) \end{cases}$$

which is of the correct form.

Case 2 on Wed.