January 29

Re: Bayes Risk $M=2$ $[c_{ij}]$ π_0, π_1

$$r(\delta) = E_{YJ}\{C(\delta(Y), J)\}$$

$J = 0, 1$
$P(J=0) = \pi_0$ ...

We can iterate this expectation in either order ...

$$r(\delta) = E_Y\{E_{J|Y}\{C(\delta(Y), J) | Y\}\} \longrightarrow \cdots \text{(A)}$$

$$= E_J\{E_{Y|J}\{C(\delta(Y), J) | J\}\} \longrightarrow \cdots \text{(B)}$$

Re: (A) $E_{J|Y}\{C(\delta(Y), J) \mid Y=y\} = C(\delta(y), 0)\pi_0(y) + C(\delta(y), 1)\pi_1(y)$

$$E_Y\{\text{above}\} = \int_\Gamma [C(\delta(y), 0)\pi_0(y) + C(\delta(y), 1)\pi_1(y)] f(y)\, dy.$$

$\pi_0(y) f(y) = \pi_0 f_0(y)$

$\pi_1(y) f(y) = \pi_1 f_1(y)$

where $f(y) = \pi_0 f_0(y) + \pi_1 f_1(y)$

Plug into the above formula and separate integral over Γ into regions Γ_1 and Γ_0 $\Longrightarrow$ Get back to formula from last time.

(B) $$E_{Y|J}\{C(\delta(Y),j) \mid J=j\} = \int_{\Gamma} C(\delta(y),j) f_j(y)\, dy \quad j=0,1$$

Then

$$E_J\{\text{above}\}$$

$$= \int_{\Gamma} C(\delta(y),0) f_0(y)\, dy \; \pi_0 + \int_{\Gamma} C(\delta(y),1) f_1(y)\, dy \; \pi_1$$

$\Longrightarrow$ Arrive at same eqn.

Then:

$$r(\delta) = \pi_0 \underbrace{\int_\Gamma C(\delta(y),0) f_0(y)\,dy}_{R_0(\delta)} + \pi_1 \underbrace{\int_\Gamma C(\delta(y),1) f_1(y)\,dy}_{R_1(\delta)}$$

$\Gamma_0 = \{y \in \Gamma : \delta(y) = 0\}$ $\Gamma_1 = \{y : \delta(y) = 1\}$ $\Gamma_0 \cap \Gamma_1 = \phi$ $\Gamma_0 \cup \Gamma_1 = \Gamma$

$$R_0(\delta) = \int_\Gamma \cdots = \int_{\Gamma_0} \cdots + \int_{\Gamma_1} \cdots$$

$$= \int_{\Gamma_1} C(1,0) f_0(y)\,dy + \int_{\Gamma_0} C(0,0) f_0(y)\,dy.$$

$$= C(1,0) \underbrace{\int_{\Gamma_1} f_0(y)\,dy}_{P_0(\Gamma_1) = P(\to H_1 | H_0)} + C(0,0) \underbrace{\int_{\Gamma_0} f_0(y)\,dy}_{P_0(\Gamma_0) = P(\to H_0 | H_0)}.$$

Similarly:

$$R_1(\delta) = c_{01} P(\rightarrow H_0 | H_1) + c_{11} P(\rightarrow H_1 | H_1)$$

False Alarm

$$P_F \triangleq P(\rightarrow H_1 | H_0)$$

Miss

$$P_M \triangleq P(\rightarrow H_0 | H_1)$$

Detection

$$P_D = P(\rightarrow H_1 | H_1) = 1 - P_M$$

$r(\delta)$ as function of P_F, P_M

Have

$$r(\delta) = \pi_0 \left\{ c_{10} P_F + c_{00} (1 - P_F) \right\} + \pi_1 \left\{ c_{01} P_M + c_{11} (1 - P_M) \right\}.$$

Properties of $\mathcal{D} \triangleq \{(P_F(\delta), P_D(\delta)) : \delta \text{ is a d.r.}\}$.

1) $(0,0) \in \mathcal{D}$ δ s.t. $\Gamma_0 = \Gamma$, $\Gamma_1 = \phi$ always say H_0.

2) $(1,1) \in \mathcal{D}$ δ s.t. $\Gamma_0 = \phi$, $\Gamma_1 = \Gamma$ always say H_1.

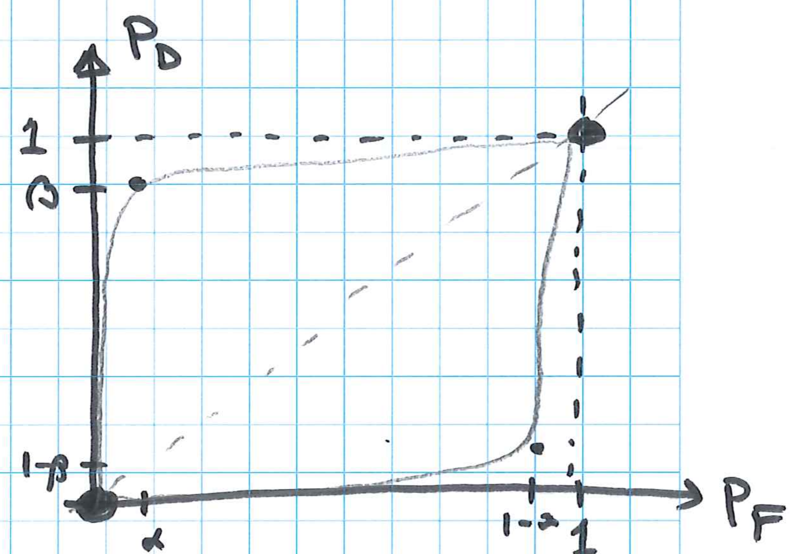
3) If $(\alpha, \beta) \in \mathcal{D} \Longrightarrow (1-\alpha, 1-\beta) \in \mathcal{D}$

Suppose δ is a d.r. corresp. to Γ_0, Γ_1 st. $P_F = \alpha = P_0(\Gamma_1)$
$P_D = \beta = P_1(\Gamma_1)$

Let δ^* be a d.r. where Γ_0^*, Γ_1^* be

$$\Gamma_0^* = \Gamma_1 \qquad \Gamma_1^* = \Gamma_0$$

$$P_F^* = P_0(\Gamma_1^*) = P_0(\Gamma_0) = 1-\alpha$$
$$P_D^* = P_1(\Gamma_1^*) = P_1(\Gamma_0) = 1-\beta$$



Properties $D \triangleq \{(P_F(\delta), P_D(\delta)) : \delta \text{ a d.r.}\}$ (cont'd)

D is convex. To show

$$\forall\ (\alpha, \beta) \in D \quad \text{and} \quad (\alpha', \beta') \in D$$

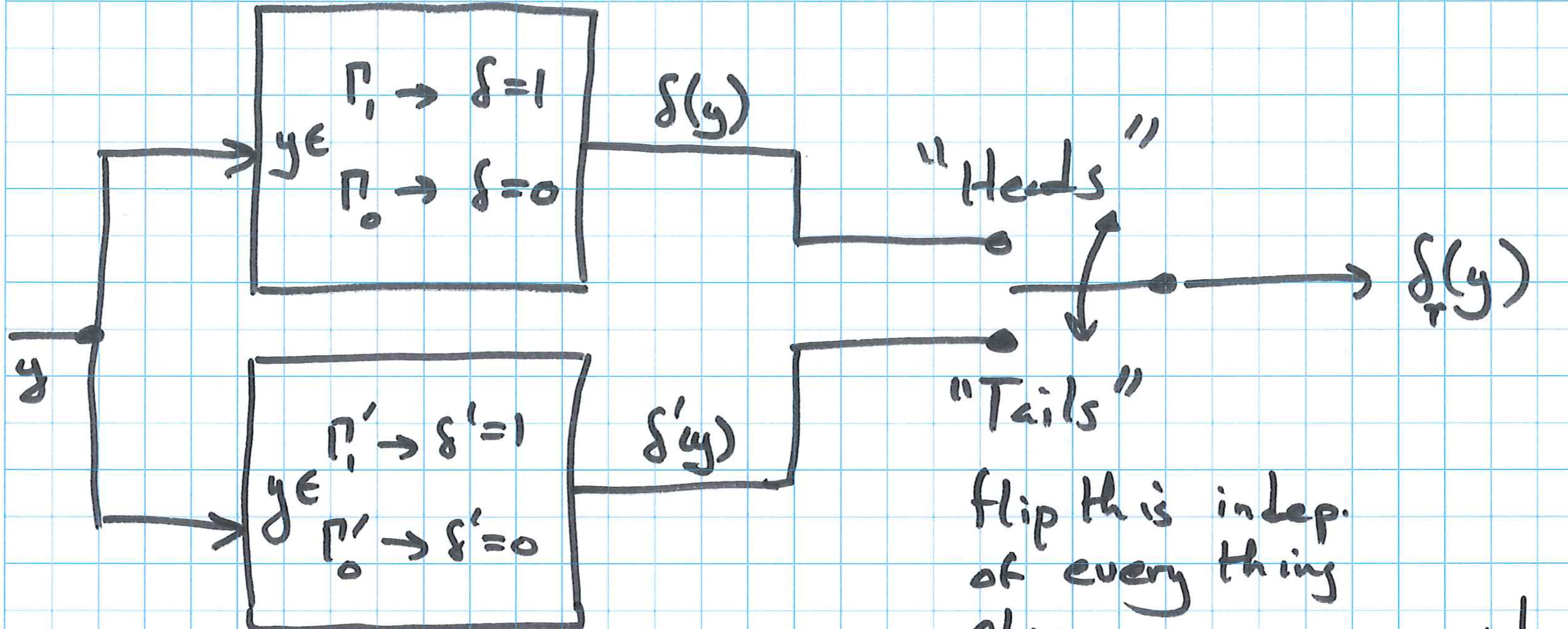
then the convex combination of these two points must be in D

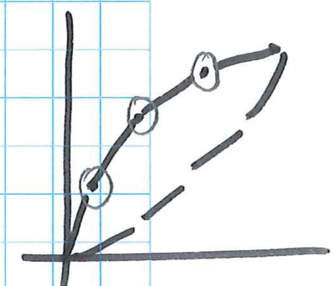
$$\left(\lambda\alpha + (1-\lambda)\alpha',\ \lambda\beta + (1-\lambda)\beta'\right) \in D$$

To prove this (and in order to be true) we need to generalize the idea of binary d.r. to include randomization.

One Type of Randomized D.R.

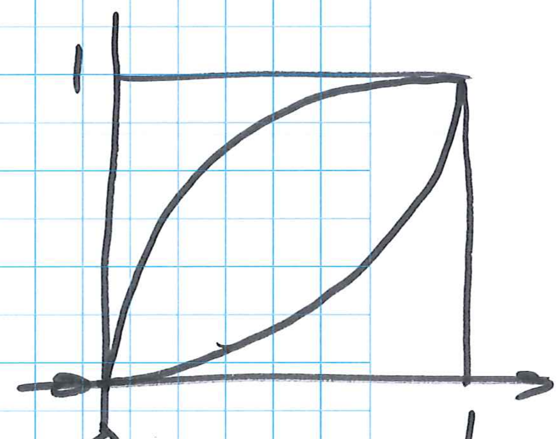
Take 2 of our old drs and randomly choose which to use.





flip this is indep. of everything else

$P(\text{Heads}) = \lambda$



$$P_F(\delta_r) = \lambda P_F(\delta) + (1-\lambda) P_F(\delta')$$

$$P_D(\delta_r) = \lambda P_D(\delta) + (1-\lambda) P_D(\delta').$$

NP criterion is a particular way to design a test for H_0 vs. H_1 by trading off $P_F(\delta)$ and $P_M(\delta)$. Bayes is just another way to tradeoff. Minimax is another. The NP criterion

$$\max_{\delta} P_D(\delta) \quad \text{subject to} \quad P_F(\delta) \leq \alpha$$

where α is a fixed constant $(0 \leq \alpha \leq 1)$ called the level or significance level of the test. One says that

The design goal of NP is to find the most powerful α-level test of H_0 vs. H_1.

- NP criterion recognizes a basic asymmetry in the importance of the two hypotheses (in contrast to Bayes and minimax).
- For a general solution to NP problem it is necessary to consider randomized tests.

Define a randomized decision rule δ for H_0 vs. H_1 as a function mapping Γ to the unit interval $[0, 1]$ with the interpretation that $\delta(y)$ is the conditional probability with which we accept H_1 given that we observe $Y = y$.

Can of course view non-randomized decision rules as special cases of randomized rules and therefore we won't use a different notation for randomized and non-randomized decision rules.

7.

For a randomized decision rule δ

$$
\begin{aligned}
P_F(\delta) &= \mathrm{E}_0\{\delta(Y)\} = \int_\Gamma \delta(y) f_0(y) \mathrm{d}y \\
P_D(\delta) &= \mathrm{E}_1\{\delta(Y)\} = \int_\Gamma \delta(y) f_1(y) \mathrm{d}y
\end{aligned}
$$

which, of course, reduces to the previous definition when the randomized decision rule is actually non-randomized.

3.1 Example: A common sense argument for NP solution

See following page.

Can think of NP testing as the setting up of a "preference order" for points y in the observation space. Equivalently, there is an analogy: *Smart student shopper*.

Suppose that a student makes a trip to ~~Marsh~~ Payless intending to spend at most α dollars. The student has two functions in mind

- $P_0(y)$ = cost of item y.
- $P_1(y)$ = "value" placed on item y (this is subjective, after all, who is to say that Cheetos are more valuable than Fruitloops).

Let S be the contents of the shopping cart. Then the smart shopping objective

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Analogy to NP   Smart Student Shopper.

You go to Payless. and you have α dollars to spend.
Have 2 functions in mind.

$P_0(y) =$ cost of item y.

$P_1(y) =$ happiness you get from item y.

$S =$ contents of the shopping cart. $\Gamma =$ inventory of payless

$$\sum_{y \in S} P_0(y) \le \alpha \quad \text{and} \quad \sum_{y \in S} P_1(y) \text{ is max.}$$

Alg: Rate (Rank) all items in store by $L(y) = P_1(y) / P_0(y)$.
Go down list from top to bottom in $L(y)$ meas. until all money is gone.