(January 27)

Previously on ECE 645

$$H_0 : Y \sim f_0(y)$$
vs.
$$H_1 : Y \sim f_1(y)$$

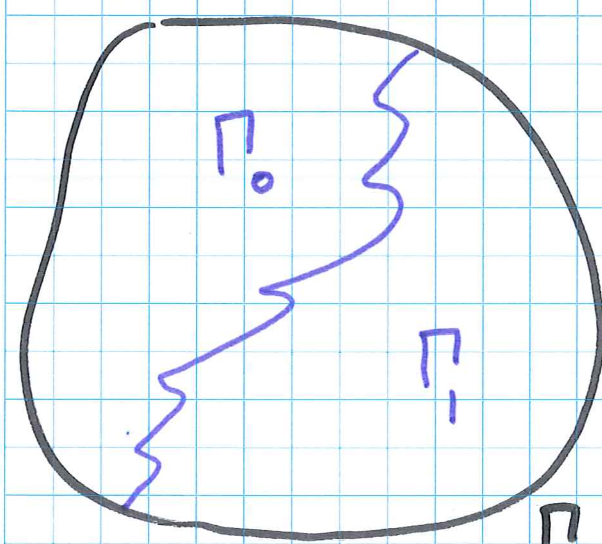
Costs: $\begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix}$

$$\pi_0 = P(H_0 \text{ true})$$
$$\pi_1 = P(H_1 \text{ true})$$
$$\pi_0 + \pi_1 = 1$$

$$\delta_B = \underset{\delta}{\operatorname{argmin}}\, r(\delta) = \underset{\delta}{\operatorname{argmin}}\, E\{E\{C(\delta(Y), J) \mid J\}\}$$

δ_B → Bayes decision rule

$$J \text{ r.v.} = \begin{cases} 0 & \text{if } H_0 \text{ true} \\ 1 & \text{if } H_1 \text{ true} \end{cases}$$



$$\Gamma_1 = \{y \in \Gamma : \pi_1(c_{11} - c_{01}) f_1(y) \leq \pi_0 (c_{00} - c_{10}) f_0(y)\}$$

$$= \left\{ y \in \Gamma : \underbrace{\frac{f_1(y)}{f_0(y)}}_{\text{A Likelihood Ratio Test.}} \geq \tau = \frac{\pi_0 (c_{00} - c_{10})}{\pi_1 (c_{11} - c_{01})} \right\}$$

Minimum Probability of Error

Unif Costs

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Notation $\rightarrow H_0 \triangleq$ dec. H_0

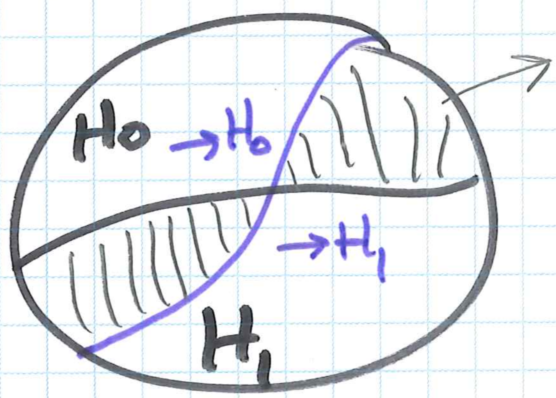
$H_0 \triangleq H_0$ is true

$$r(\delta) = \pi_0 \underbrace{P_0(\Gamma_1)}_{P(\rightarrow H_1 | H_0)} + \pi_1 \underbrace{P_1(\Gamma_0)}_{P(\rightarrow H_0 | H_1)}$$

$$= P(\rightarrow H_1 | H_0) P(H_0) + P(\rightarrow H_0 | H_1) P(H_1)$$

$$= P(\rightarrow H_1, H_0) + P(\rightarrow H_0, H_1)$$

The entire space of outcomes is included in



$\Longrightarrow r(\delta) =$ Prob. of Error.

Posterior Costs

Remember Bayes Formula:

$F_1 \triangleq \{H_0 \text{ is true}\}$

$F_2 \triangleq \{H_1 \text{ is true}\}$.

$E = \{Y = y\}$ assume disc. obs. st $P(E) > 0$.

$F_1, F_2, \ldots F_n$ events s.t.

$\underset{i \neq j}{F_i \cap F_j = \phi} \quad \bigcup_{k=1}^{n} F_k = \Omega$

E another event ....

$$P(F_1 | E) = \frac{P(F_1 \cap E)}{P(E)}$$

$$= \frac{P(E|F_1)P(F_1)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

$P(H_0 \text{ true} | Y=y)$

$$= \frac{\overbrace{P(Y=y | H_0 \text{ true})}^{p_0(y)}\overbrace{P(H_0 \text{ true})}^{\pi_0}}{\underbrace{P(Y=y | H_0 \text{ true})}_{p_0(y)}\overbrace{P(H_0 \text{ true})}^{\pi_0} + \underbrace{P(Y=y | H_1 \text{ true})}_{p_1(y)}\underbrace{P(H_1 \text{ true})}_{\pi_1}}$$

$$= \frac{p_0(y)\,\pi_0}{p_0(y)\,\pi_0 + p_1(y)\,\pi_1} \triangleq \pi_0(y)$$

Posterior prob that H_0 is true given $Y=y$.

Also $\pi_1(y)$

Rewrite $\Gamma_1 = \left\{ y \in \Gamma : \pi_1 (c_{11} - c_{01}) p_1(y) \le \pi_0 (c_{00} - c_{10}) p_0(y) \right\}$

$$\frac{\pi_1 (c_{11} - c_{01}) p_1(y)}{\pi_0 p_0(y) + \pi_1 p_1(y)} \le \frac{\pi_0 (c_{00} - c_{10}) p_0(y)}{\pi_0 p_0(y) + \pi_1 p_1(y)}$$

$$\underbrace{\frac{\pi_1 p_1(y)}{\pi_0 p_0(y) + \pi_1 p_1(y)}}_{}, \qquad \underbrace{\frac{\pi_0 p_0(y)}{\pi_0 p_0(y) + \pi_1 p_1(y)}}_{}$$

c_{ij} = cost of saying i true when j is actually true.

$$(c_{11} - c_{01}) \pi_1(y) \qquad (c_{00} - c_{10}) \pi_0(y)$$

$$\Gamma_1 = \left\{ y \in \Gamma : \cancel{\ldots} \; c_{11} \pi_1(y) + c_{10} \pi_0(y) \le c_{00} \pi_0(y) + c_{01} \pi_1(y) \right\}$$

expected cost of choosing H_1 given $Y = y$

expected cost of choosing H_0 given $Y = y$.

i.e. put y in Γ_1 if the posterior cost of doing so is smaller than alternative.

Maximum a Posteriori (MAP)

For unif. costs $\quad c_{11} = c_{00} = 0$
$\qquad\qquad\qquad\quad c_{01} = c_{10} = 1$

$$\Gamma_1 = \left\{ y \in \Gamma : \pi_1(y) \geq \pi_0(y) \right\}$$

$\therefore$ The minimum probability of error decision rule is also the MAP decision rule.

Consider a Bayesian hypothesis testing problem with equal prior probabilities and uniform costs. Under H_0 suppose that $Y \sim p_0(y)$ and under H_1 suppose that $Y \sim p_1(y)$ where

$\pi_0 = \pi_1 = 1/2$

$$p_0(y) = \begin{cases} \sqrt{2/\pi} e^{-y^2/2} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$p_1(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}.$$

(a) Find the minimum probability of error hypothesis test for H_0 vs. H_1. Specify the likelihood ratio, threshold, and randomization if needed.

(b) Simplify the test from (a) to the extent possible. Draw a picture to illustrate the test.

(c) Find the minimum achievable probability of error for testing H_0 vs. H_1.

All Bayes $M=2$ are L.R.T. $\quad L(y) \underset{\text{say } H_0}{\overset{\text{say } H_1}{\gtrless}} \tau$

Note unif costs, equal priors $\tau = 1$.

$$L(y) = \frac{f_1(y)}{f_0(y)} = \frac{e^{-y}}{\sqrt{2/\pi}\, e^{-y^2/2}} \qquad y > 0.$$

$$= \sqrt{\frac{\pi}{2}}\, e^{+y^2/2 - y} \gtrless 1 \quad \rightarrow \text{(a)}$$

(b) Simplify.

$$e^{+\frac{y^2}{2} - y} \gtrless \sqrt{2/\pi}$$

Take ln $\quad \frac{y^2}{2} - y \gtrless \ln\sqrt{2/\pi}$

$\times 2 \qquad \times 2$

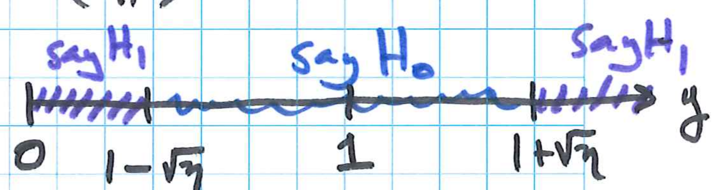
$$y^2 - 2y \gtrless 2\ln\sqrt{2/\pi}$$

$$y^2 - 2y + 1 \gtrless 2\ln\sqrt{\frac{2}{\pi}} + 1$$

$$(y-1)^2 \gtrless \ln\frac{2}{\pi} + \ln e = \overbrace{\ln\left(\frac{2e}{\pi}\right)}^{\eta}$$

$$(y-1)^2 \underset{\text{" } H_0}{\overset{\text{say } H_1}{\gtrless}} \eta$$

$$|y-1| \underset{\text{say } H_0}{\overset{\text{say } H_1}{\gtrless}} \sqrt{\eta}$$



$$\text{Bayes risk} = P_e = \frac{1}{2}\underbrace{P_0(\text{say } H_1)}_{P_F} + \frac{1}{2}\underbrace{P_1(\text{say } H_0)}_{P_M}$$

$$f_0(y) = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}$$

$$P_F = P_0\left(0 \le Y \le 1-\sqrt{\eta}\right) + P_0\left(Y > 1+\sqrt{\eta}\right).$$

cdf of unit normal rv. $N(0,1)$ $\frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ $y \in \mathbb{R}$

$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy.$$

Also look up erf, erfc

$$P_F = 2\left[\Phi(1-\sqrt{\eta}) - \Phi(0)\right] + 2\left[1 - \Phi(1+\sqrt{\eta})\right]$$

$$P_M = P_1(\text{say } H_0) = \int_{1-\sqrt{\eta}}^{1+\sqrt{\eta}} e^{-y}\, dy = 2e^{-1}\sinh(\sqrt{\eta}).$$

$$\Longrightarrow P_e = \frac{1}{2} + \Phi(1-\sqrt{\eta}) - \Phi(1+\sqrt{\eta}) + e^{-1}\sinh(\sqrt{\eta})$$