Monday April 12 Lecture 35

* HW due Fri.
* THE 2 → JVK desire Wed 8am Sun 8am
* Proj. Rep + Vid. → Fri 23rd.

Example: Signal + Noise Model

$$Y_k = \mu s_k + N_k \qquad k = 1, 2, \ldots n$$

$\mu > 0$ amplitude parameter

$s_1\ s_2 \ldots s_n$ a known signal

$N_k \sim N(0, \sigma^2)$ iid, $\sigma^2 > 0$ is noise variance

Can look @ this model in several ways —

$\mu$ known or unknown

$\sigma^2$ known or unknown

MVUE or ML ?

CRLB

Recall Session 28A : Case .... $\mu$ unknown, $\sigma^2$ known
found MVUE for $\mu$

Probability density function of $Y = [Y_1 \; Y_2 \ldots Y_n]^T$ was ...

$$f_\theta(y) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k - \mu s_k)^2\right\}$$

For this situation defined $\theta_1 = \mu/\sigma^2$, $T_1(y) = \sum_{k=1}^{n} s_k y_k$ whence pdf written ...

$$f_{\theta_1}(y) = \underbrace{\left[\frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{\frac{-\sigma^2\theta_1^2}{2}\sum_{k=1}^{n} s_k^2\right\}\right]}_{c(\theta_1)} \exp\left\{\theta_1 T_1(y)\right\} \underbrace{\left[\exp\left\{-\frac{1}{2\sigma^2} \cdot \sum_{k=1}^{n} y_k^2\right\}\right]}_{h(y)}$$

This is form of one param. exp. family where

$$\Lambda = \{\theta_1 : -\infty < \theta_1 < \infty\}$$

contains a one-dim. rectangle $\Longrightarrow T_1(y)$ is a complete suff. stat. for $\theta_1$

Estimation: $\mu = g(\theta_1) = \sigma^2 \cdot \theta_1$

With any unbiased estimator of $\mu$ can get MVUE using Rao-Blackwell ...

$$\tilde{g}(T_1(y)) = E_\theta\{\hat{g}(Y) \mid T_1(Y) = T_1(y)\}$$

we used $Y_1/s_1$

and then found MVUE $\tilde{g}(T_1(y)) = \frac{1}{n\overline{s^2}} \sum_{k=1}^{n} s_k y_k$

The variance was ... $Var_\theta\{\tilde{g}(T_1(Y)\} = \frac{\sigma^2}{n\overline{s^2}}$

$$\overline{s^2} = \frac{1}{n}\sum_{k=1}^{n} s_k^2$$

Suppose both $\mu$ & $\sigma^2$ are unknown. Say $\mu \in \mathbb{R}$ $\sigma^2 > 0$. Want to est. both. Need to re-examine the form for exp. family ...

$$\frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k - \mu s_k)^2\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum y_k^2 + \frac{\mu}{\sigma^2}\sum s_k y_k - \frac{\mu^2}{\sigma^2}\sum s_k^2\right\}$$

Define: $T_1(y) = \sum_{k=1}^{n} s_k y_k$ $\qquad T_2(y) = \sum_{k=1}^{n} y_k^2$

$\theta_1 = \mu/\sigma^2 \qquad \theta_2 = -\frac{1}{2\sigma^2}$

$$\Rightarrow \text{pdf} \quad \underbrace{C(\theta)}_{\hookrightarrow \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{\frac{-\mu^2}{2\sigma^2}\sum_{k=1}^{n} s_k^2\right\}} \exp\left\{\theta_1 T_1(y) + \theta_2 T_2(y)\right\} \underbrace{h(y)}_{1} \qquad \theta = (\theta_1, \theta_2)^T$$

Can show: $C(\theta) = \dfrac{1}{(-\pi/\theta_2)^{n/2}} \exp\left\{ \dfrac{\theta_1^2}{4\theta_2} \sum_{k=1}^{n} s_k^2 \right\}$

Re: C.S.S.

$\{(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}$

Corresp. to

$\{(\theta_1, \theta_2) : \theta_1 \in \mathbb{R}, \theta_2 < 0\}$ ... does contain a 2D rectangle.

$\Rightarrow$ Thus $T = (T_1, T_2)^T$ is a complete s.s. for $\theta = (\theta_1, \theta_2)^T$.

Wish to est.

$$\mu = g_1(\theta) = -\frac{1}{2}\frac{\theta_1}{\theta_2} \quad \& \quad \sigma^2 = g_2(\theta) = \frac{-1}{2\theta_2}$$

From the 1st example

$$\tilde{g}_1(T_1(y)) = \frac{1}{n\overline{s^2}} \sum_{k=1}^{n} s_k y_k$$

is an unbiased est. of $\mu = g_1(\theta)$ and a funct of $T_1(y)$, therefore, a funct. of $T(y) = (T_1(y), T_2(y))$

$\therefore$ It is MVUE of $\mu$ even in case where $\sigma^2$ is unknown.

Now just need MVUE of $\sigma^2$. Here we will look if there is an easy way to find an unbiased est. of $\sigma^2$ which is already a funct. of $T = (T_1, T_2)$...

<u>Start with $T_1$</u> $\quad T_1(Y) = \sum_{k=1}^{n} s_k Y_k \sim N(n\mu\overline{s^2}, n\sigma^2\overline{s^2})$

Thus $$E_\theta\{T_1^2(Y)\} = Var_\theta\{T_1(Y)\} + (E_\theta\{T_1(Y)\})^2$$
$$= n\sigma^2\overline{s^2} + (n\mu\overline{s^2})^2$$

$$E_\theta\left\{T_2(Y)\right\} = E_\theta\left\{\sum_{k=1}^{n} Y_k^2\right\} = \sum_{k=1}^{n}\left(\sigma^2 + \mu^2 s_k^2\right)$$

$$= n\sigma^2 + n\mu^2\,\overline{s^2}$$

Then notice

$$E_\theta\left\{T_2(Y) - \frac{1}{n\overline{s^2}}T_1^2(Y)\right\} = n\sigma^2 + n\mu^2\overline{s^2} - \sigma^2 - \frac{(n\mu\overline{s^2})^2}{n\overline{s^2}}$$

$$= (n-1)\sigma^2$$

$$\therefore\ \tilde{g}_2(T(y)) = \left[T_2(y) - \frac{T_1^2(y)}{n\overline{s^2}}\right]\frac{1}{n-1}$$

is an unbiased est. of $\sigma^2$ ... and must be MVUE since a funct. of Comp. SS.

Rewrite as:

$$\tilde{g}_2(T(y)) = \frac{1}{n-1}\sum_{k=1}^{n}\left(y_k - \hat{\mu}s_k\right)^2 \triangleq \hat{\sigma}^2 \longrightarrow \text{MVUE of } \sigma^2.$$

$$\hat{\mu} = \frac{1}{n\overline{s^2}}\sum_{k=1}^{n} s_k y_k \longrightarrow \text{MVUE of } \mu$$

**Now for ML Est** $\quad Y_k = N_k + \mu s_k$ ... as before.

Case: $\mu$ is to be estimated, $\sigma^2$ known. $\Rightarrow \theta = \mu$

$$\log f_\mu(y) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k - \mu s_k)^2$$

and find likelihood eqn:

$$\left.\frac{\partial}{\partial\mu}\log f_\mu(y)\right|_{\mu=\hat{\mu}_{ML}(y)} = 0$$

$$\left.\frac{\partial}{\partial\mu}\left\{\text{above}\right\}\right|_{\mu=\hat{\mu}_{ML}(y)} = \frac{1}{\sigma^2}\sum_{k=1}^{n} s_k(y_k - \mu s_k) = 0$$

solve for $\mu = \hat{\mu}_{ML}(y)$

$$\Rightarrow \hat{\mu}_{ML}(y) = \frac{1}{n}\sum_{k=1}^{n} s_k y_k \Big/ \overline{s^2}$$

Compute

$$\frac{\partial^2}{\partial\mu^2}\left\{\text{above}\right\} = \frac{-n\overline{s^2}}{\sigma^2} < 0$$