Friday April 9 Lecture 34

ML Estimation of Exponential Param: $\Gamma = \mathbb{R}^n$

$\Lambda = (0, \infty)$

$Y_1\ Y_2 \cdots Y_n$ iid and exponentially distributed.

$$f_\theta(y) = \prod_{k=1}^{n} f_\theta^*(y_k) \qquad f_\theta^*(y_k) = \begin{cases} \theta e^{-\theta y_k} & y_k \geq 0 \\ 0 & y_k < 0. \end{cases}$$

$$= \theta^n e^{-\theta n \bar{y}} \qquad \bar{y} = \frac{1}{n}\sum_{k=1}^{n} y_k$$

sample mean.

$$\frac{\partial}{\partial\theta} \log f_\theta(y) = \frac{\partial}{\partial\theta}\left\{ n \log\theta - \theta n \bar{y} \right\}$$

$$= \frac{n}{\theta} - n\bar{y} \overset{\text{Set}}{=\!=} 0$$

solve for root $\theta \rightarrow \hat{\theta}_{ML}(y)$

$$\Longrightarrow \hat{\theta}_{ML}(y) = 1/\bar{y}$$

Π

$$\frac{\partial^2}{\partial\theta^2} \log f_\theta(y) = -n/\theta^2 < 0$$ $\Longrightarrow$ solution to the likelihood eqn is the unique & maximizer of $\log f_\theta(y)$.

<u>Properties of this Estimator</u>

$$E_\theta\{Y_k\} = \int^\infty y_k \theta e^{-\theta y_k} \, dy_k = \frac{1}{\theta}.$$

$\Longrightarrow$ Also $E_\theta\{\overset{\circ}{\bar{Y}}\} = 1/\theta \longrightarrow$ So seems reasonable to estimate θ by $1/\bar{Y}$

Go back and look at various defs. of convergence for rvs.

To indicate that sample mean has a dep. on n we change notation ...

$$\bar{Y}_n = \frac{1}{n} \sum_{k=1}^{n} Y_k$$

From the LLN $\bar{Y}_n \longrightarrow 1/\theta$ as $n \to \infty$ with prob. one (also "in probability") all wrt $P_\theta(\cdot)$ distribution.

Recall a simple real # seq. property:

b_n a seq. of real # st. $b_n \to b$ and st. $b \neq 0$ $\Longrightarrow$ $b_n^{-1} \longrightarrow b^{-1}$

$$\therefore \; P_\theta \left\{ \lim_{n \to \infty} \bar{Y}_n^{-1} = \theta \right\}$$

$\therefore$ the seq. of MLEs for θ converges wp1 to the true param.

Say that the MLE is (strongly) consistent.

Fisher's Information

$$I_\theta = -E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log f_\theta(Y)\right\} = n/\theta^2$$

$$\Longrightarrow \text{CRLB} = I_\theta^{-1} = \theta^2/n$$

Also note that $\frac{\partial}{\partial\theta}\log f_\theta(y)$ <u>does</u> <u>not</u> have the for required for equality in CRLB.

<u>Claim</u> Sample mean $\bar{Y}_n$ has a pdf

$$f_{n,\theta}(\bar{y}) = \begin{cases} \dfrac{(\bar{y}/n)^{n-1}}{n!\,\theta^n}\, e^{-\bar{y}/n\theta} & \bar{y} > 0 \\ 0 & \bar{y} < 0 \end{cases}$$

$$E_\theta\left\{\hat{\theta}_{ML}(Y)\right\} = E_\theta\left\{\frac{1}{\bar{Y}_n}\right\} = \int_0^\infty \frac{1}{\bar{y}}\,\frac{(\bar{y}/n)^{n-1}}{n!\,\theta^n}\, e^{-\bar{y}/n\theta}\, d\bar{y}$$

$$= \frac{n\theta}{n-1} \quad \text{for } n > 1.$$

Can also compute variance

$$\mathrm{Var}_\theta\left\{\hat{\theta}_{ML}(Y)\right\} = \frac{\theta^2 n^2}{(n-1)^2(n-2)} \quad \text{for } n > 2$$

Although MLE for finite n is biased, it is asymptotically unbiased

$$\lim_{n\to\infty} \frac{n\theta}{n-1} = \theta$$

In addition

$$\mathrm{Var}_\theta\left\{\hat{\theta}_{ML}(Y)\right\} I_\theta = \frac{\theta^n n^2}{(n-1)^2(n-2)} \frac{n}{\theta^2}$$

$$\longrightarrow 1 \quad \text{as } n \to \infty$$

ie var of MLE approaches CRLB as $n \to \infty$

$\Rightarrow$ said to be asymptotically efficient.

Relationship to MVUE From the fact. thm + exp. fam. thm see ...

$$\overline{Y}_n = \frac{1}{n} \sum_{k=1}^{n} Y_k$$

is a complete suff. stat. for θ. Furthermore,

$$\frac{n-1}{n} \hat{\theta}_{ML}(y) = \left[\frac{1}{n-1} \sum_{k=1}^{n} y_k \right]^{-1}$$

is an unbiased estimator for θ ... further, it is a function of $\overline{Y}_n$.

$$\hat{\theta}_{MV}(y) = \frac{n-1}{n} \hat{\theta}_{ML}(y)$$ this is the MVUE.

Can show

$$\mathrm{Var}_\theta \left\{ \hat{\theta}_{MV}(Y) \right\} = \frac{\theta^2}{n-2} \longrightarrow$$ this is strictly greater that the CRLB.

For the MLE

$$E_\theta\left\{\left(\hat{\theta}_{ML}(Y)-\theta\right)^2\right\} \triangleq MSE_\theta\left\{\hat{\theta}_{ML}(Y)\right\}$$

$$= Var_\theta\left\{\hat{\theta}_{ML}(Y)\right\} + B^2(\theta)$$

$$= \frac{\theta^2(n+2)}{(n-1)(n-2)} > MSE_\theta\left\{\hat{\theta}_{MV}(Y)\right\}$$

Would prefer MVUE to MLE here.

Note

$$\underbrace{B(\theta)}_{\text{bias.}} = E_\theta\left\{\hat{\theta}_{ML}(Y)\right\} - \theta$$